A census of the orientable biembeddings of Steiner triple systems of order 15

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Abstract

A complete census is given of the orientable biembeddings of Steiner triple systems of order 15. There are 80 Steiner triple systems of order 15 and these generate a total of 9 530 orientable biembeddings.

1   Introduction

This paper is concerned with biembeddings of Steiner triple systems. A *Steiner triple system* of order n, STS(n), is a pair $(V, \mathcal{B})$, where V is a set of n *points* and $\mathcal{B}$ is a collection of *triples*, also called *blocks*, taken from V and such that every pair of distinct points from V appears in precisely one block. Such systems exist if and only if $n \equiv 1$ or $3 \pmod 6$ [9]. Biembeddings of such systems in orientable and nonorientable

surfaces arise as follows. Consider a triangular embedding M of the complete graph K_n in the orientable surface S_g, the sphere with g handles, or in the nonorientable surface N_γ, the sphere with γ crosscaps. If the faces of M can be properly 2-coloured then the faces in each colour class form an STS(n). Euler's formula gives g (respectively γ) for triangular embeddings of K_n in an orientable (respectively nonorientable) surface, namely $g = (n-3)(n-4)/12$ $(\gamma = (n-3)(n-4)/6)$. Face 2-colourability requires n to be odd. It easily follows that a necessary condition for an orientable biembedding is that $n \equiv 3$ or $7 \pmod{12}$, and a necessary condition for a nonorientable biembedding is $n \equiv 1$ or $3 \pmod{6}$. These necessary conditions are also sufficient for the existence of such biembeddings, except in the case $n = 7$, where there is no biembedding of STS(7)s in the surface N_2 [11, 12, 6].

Two STS(n)s, A and B, are said to be *biembeddable* in some surface if there exists a face 2-colourable triangular embedding of K_n in that surface in which the Steiner triple systems arising from the two colour classes are isomorphic copies of A and B. The issue which then naturally arises is to determine which pairs of STS(n)s are biembeddable. For $v = 3$ there is a trivial and unique embedding of a triangle in the sphere, and this provides a biembedding of STS(3)s. The STS(7) is unique and there is precisely one biembedding of the system, this with an isomorphic copy of itself, in the torus S_1. Here and subsequently when enumerating, we refer to the number of isomorphism classes. For the STS(9), which is also unique, there is again precisely one biembedding, again with an isomorphic copy of itself, in the nonorientable surface N_5 [1, 5]. There are two STS(13)s, one is cyclic and the other is not. We will refer to these here as C and N respectively. There are 615 biembeddings of C with C, 8 539 biembeddings of C with N, and 29 454 biembeddings of N with N [7], all of these being in the nonorientable surface N_{15}.

Biembeddings of STS(15)s lie in either the orientable surface S_{11} or the nonorientable surface N_{22}. There are 80 isomorphism classes of STS(15)s and we follow the standard numbering of these given in [10] which also gives the orders of their automorphism groups. It was shown in [3] that every pair, including isomorphic pairs, has a biembedding in a nonorientable surface, although a complete enumeration of all nonorientable biembeddings of STS(15)s is probably beyond current computational capabilities.

It was proved in [2] that at least one pair, namely $\{\#1, \#2\}$ in the standard numbering, has no biembedding in an orientable surface. In [4] a search was made to find an orientable *self-embedding*, that is a biembedding of two isomorphic copies of the same system, for each of the 80 STS(15)s. The biembedding was assumed to have an involutory automorphism, with a single fixed point, that reversed the colour classes. Self-embeddings of this type were found for 78 of the 80 systems. It was also shown in [4] that there is no orientable self-embedding of system $\#2$, whether of this type or any other type. The other exception was system $\#79$. In [8] it was shown that this latter system also has no orientable self-embedding. In the same paper we gave all orientable biembeddings in which one of the two systems is one of $\#1$, $\#2$, $\#76$, $\#79$, $\#80$, these systems being of particular interest for reasons given in that paper. In the current paper we complete the determination of all the orientable biembeddings of the 80 Steiner triple systems of order 15.

2   Method

The biembeddings were found by one of the two computer programs used in [8]. This program takes two systems A and B from the listing of [10]. The representation of system B is not altered, but permutations are in turn applied to the vertices of system A. For each permutation, two sets of triangles result, those from A and those from B. To form a surface embedding we have, for example, the trivial test that these sets of triangles must be disjoint. During the examination of possible permutations, the automorphism groups of the two systems are exploited. If we examine all permutations π that take a point a of A to a point b of B, then we do not need to consider further any other permutation π' that takes a point in the orbit of a under $Aut(A)$ to a point in the orbit of b under $Aut(B)$.

The permutations were constructed as follows. One vertex, say b_1, is chosen. Assume that the triples of B containing b_1 are $b_1b_2b_3$, $b_1b_4b_5$, ..., $b_1b_{14}b_{15}$. Then in any biembedding, in the rotation around b_1 there are pairs b_2b_3, b_4b_5, ..., $b_{14}b_{15}$. Since these pairs may be reversed and mutually interchanged, there are $6! \cdot 2^6 = 46\,080$ possible rotations at b_1, and these are considered in turn. In any biembedding of A with B, one vertex of A is mapped to b_1, and some other vertex is mapped to b_2. But as we already have the rotation at b_1, the image of the third vertex in the triple from A containing the inverse image of the pair b_1b_2 is determined. Similarly, the image of a fourth vertex determines that of a fifth vertex, and so on. In this way, only $15 \cdot 14 \cdot 12 \cdot \ldots \cdot 2 = 9\,676\,800$ permutations are constructed for each possible rotation at b_1. So, all together, for every pair of systems we examine $(46\,080) \cdot (9\,676\,800) = 445\,906\,944\,000$ possible permutations instead of $15! = 1\,307\,674\,368\,000$, that is approximately one-third. But in fact, by using the orbits of the automorphism groups of A and B as described above, a much smaller number is examined. To confirm that a biembedding is formed, we simply check that the rotation at each vertex is a single cycle of length 14. To test for orientability, we fix the orientation of the triangle $b_1b_2b_3$. This implies an orientation for all the other triangles. Since every triangle is in three rotations, after assigning it an orientation, we check the two further occurrences.

The program described above was used in [8] and the results there were verified by a second independent program. In the current paper, as an additional check on the accuracy of the results, we computed both the biembeddings of A with B and of B with A for each pair of systems A and B, and we verified that the results were identical. Furthermore the second program from the earlier paper was used to confirm all the self-embeddings.

The programs were run on a Linux cluster, with each of the 80 systems being allocated to a separate processor. Determination of all the biembeddings of each system took between two and ten weeks; the systems having the smallest automorphism groups generally taking the longest times.

3   Results

Up to isomorphism, there are $9\,530$ orientable biembeddings of Steiner triple systems of order 15. Table 1 summarizes these biembeddings in the form of an 80×80 array. The entry in the (i, j) cell gives the number of orientable biembeddings of system $\#i$ with system $\#j$. A dot (.) indicates that no biembedding exists and the 26 letters a to z represent the numbers from 10 to 35 inclusive. The letter l (representing 21 biembeddings), which could be confused with the number 1, only appears in the $(10, 10)$ cell. The symbol $+$ denotes an entry greater than 35. In fact $+$ only appears on the diagonal and these entries are described in more detail in Table 2. Altogether in Table 1 there are $(80 \times 81)/2 = 3\,240$ cells (i, j) with $1 \leq i \leq j \leq 80$, representing distinct pairs of systems. Of these, $1\,150$ admit no biembedding while each of the remaining $2\,090$ admits at least one biembedding.

For reasons of space, it is not feasible to list all the biembeddings. However, these are available as a file from http://mcs.open.ac.uk/mjg47/emb15.txt. Each biembedding is described by a line of the file containing the system numbers i and j, the order of the automorphism group of the biembedding and followed by a permutation p. To generate the biembedding, p is applied to the standard form of system i given in [10] to produce a copy $p(i)$. The triples of the systems $p(i)$ and j are then taken as triangles and sewn together along common edges. As an example, the first line of the file is:

$$1\ 1\ 10\ 1\ 8\ 10\ 3\ 5\ 2\ 12\ 7\ 4\ 6\ 9\ 11\ 14\ 15\ 13$$

Here $i = j = 1$, the biembedding has 10 automorphisms, and the permutation p is taken as $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 8 & 10 & 3 & 5 & 2 & 12 & 7 & 4 & 6 & 9 & 11 & 14 & 15 & 13 \end{pmatrix}$.

The diagonal entries of Table 1 represent $1\,587$ self-embeddings. There is a general tendency for the diagonal entry to be the greatest entry in the corresponding row (and column). Table 2 lists the diagonal entries from Table 1 and also gives the row (and column) sums which represent the total number of biembeddings of each system. In Table 2, the first column in each section gives the system number i, the second gives the number of self-embeddings s, and the third gives the total number of biembeddings t.

```
1.........  .........  .........  .........  .........  .........  .........  .........
.........  1..1.....  ......1...  .........  .........  .........  .........  .........
..3......  1.2.1....  .....1.1..  .11.....1.  .........  .1........  ....1.....  .........
...7...111 11..1..1..  ...131.5..  ..31.1.121 2...213121 311.3.12..  ..1.112251 35...11....
....1...1.  .........1  ..1.12....  .........1 12..2.1...  ...2..1...  .......11.  .........
.....9...2 1.........  ....112...  ..11...1.. .11.11.1.1 ..1..1....  ......11.1 11..1.31...
......2...  .........  .........  .........  ..1......  1.........  .........  .........
...1...a.1 241.1.23.2  ..34123113 .4411..542 11131..212 4.1253463.  ..12.12335 6a1.1111..
..11...g3 41.32....2  .483621313 154221.666 2224412841 6333521211 .213544423 52.2113...
...1.2.131 422.4.1.1.  235472.8..  1136.3.458 3..1211975 5464396515 .211463467 32.1.141..

.111.1.244 e2.211.2.3 1324342513 1365.1.446 12124661.2 5635421112 .111444432 15..3112..
...1..412 2d..2...22 1157133522 2224.3.154 22..153133 55324523.1 ...422223 21222.1...
..2...1.2 ..c......1 ..2.31312. ..111.1111 1.....211. 1.4..111.3 ...12...11 .3..1..1..
.1.....3. 2..4...1.. ...312.... ...1...1.1 .1...11..1 .3..13...1 ......1... ...1......
..11...124 12..91.1.2 .32324121. 1235.2.371 ...2212312 43133223.. ...1.143.3. 12113.1...
......... 1...11.... ......... ......... ......... .........1 ......... ........1.
.......2.1 ......3..1 ...1..2.. .1.....121 .1.....111 1.1..1...1 ....1.1..1 .........
...1..2.. 2..11..a.1 .11.5231.. 111.212.34 2111211452 .33311.313 ..1.3.21.2 31.1.1.11.
.........1 .2......61 1...1141.1 ......11. 221.1..3.. 2..21..1.. ......... .2.1......
....1..22. 321.2.111b 1151344212 ..432..145 31.164.353 445222.123 1.2.315473 31........

.........2 11......11 12262.3.12 c221.1.122 11.2113514 .51332.311 ..1.1.1311 11.....1..
..1....451 32..2.11.. 169466fb23 2x1533.a65 4342556565 57a9899656 .12.7959ac 57324.66..
..13.1.443 621.3..1.4 43459887.2 21v831.84c 121234757c 3b88b88a16 12119a8858 79242132..
...1.1.126 54115....3 618cab4812 158u41.a79 9511c55743 66b8d67254 .414c9b545 5f122.311.
......12. ..1...2.2 12162132.. .334d.1415 3..1123122 3343221.15 ..11.35.221 .22.1.....
...1...13 13..2..1.. ...32211..1 1311.4.322 121.21.113 41.1322... ....222132 111......
......... ...1...2.. 1.2....... ...1.11.. 21..1.1... 111.1..... ...2.1.... ..1.......
...1.1.564 41113.1.11 15g677554. 1a8a431v26 4231e94799 6986baa915 161.678cba 9g3141....
..121..465 451.7.2314 5.88756964 264712.2+9 6512995a66 5ea3a95749 2433998898 a4134121..
...1...268 64111.14.5 4246cb5b23 25c952.69v 67119b6689 87b76c775b .8.7368655 866452331.

...21..123 121...223 3598b66326 1419312466 v343aaa28a 87768896.9 .13275b849 4a2214211.
....21.12. 22.1..1121 51478412.1 1325.21257 3g2.811414 3653334312 ..2125a136 371.1.11..
....1112. 1......11. ..21.21.11 .411.1.311 42c11.111. 1121..1.23 111.23.321 11..2.5...
......341 2...2..1.1 .1343551.. 22211..121 3.1j462373 3233365122 1311324554 31.1..2...
...221.142 41..2..216 2187695c23 153c121e99 a814+88465 7785499956 .232c73666 741312.2..
...1.1..11 65.11..1.4 4166a98b21 154521.99b a1.68x8e58 5945896c27 13419a7636 86613211..
...31...21 63212..1.. 7297e9b722 36753.1456 a11288q891 776182371b ..5289b63a 39131113..
...1.1.289 111.3.1433 329b7a8735 555711.7a6 24134e8z67 a97595a458 .4135aa98d 7b45212...
...2..147 .31.1.15.5 318794aa35 167421.968 81176596r9 aa39499579 .233294a45 79333133..
...1.1.215 23.12.12.3 43d5658434 45c323.969 a4.358179v 8d9c68a922 .221b59899 78114.21..

...3..1465 551.4.1.24 32b9a75a15 .536341658 8313757aa8 vd958a4526 .13.7ad797 a7113.52.1
..11....34 65.33..3.4 1278586834 57b63119e7 76127979ad d+aa4a4a34 .1319d6c25 68.24.12.1
...1.1.136 334.1.13.5 23ab762835 1a8b4.18ab 7523846739 9as6d9a419 .535a49866 3312635...
...2..234 52..3..322 35c9b9a842 398831.637 631355159c 5a6r3776.5 .2.5675c85 a5233145..
...3...553 44.13..112 .387b8b314 38bd231ba6 83.3488946 84d3w29538 1212475b78 862222212.
.....1.329 25132.11.2 ..ae7ae651 298622.a9c 83.6992598 aa972oe838 ...5777d3d 62132122..
...11..416 121.2..... 22a6646628 .98712.a57 9415963a9a 44a79eq538 .141444ba5 56133322..
...2..625 131.3..311 12bbb76712 36a2...977 63.19c7459 5a46585u13 141.447g63 452.21231.
......311 1....1.1.2 ..5141233. 15151..145 .122521572 231.333192 .111236442 372...1...
........15 2131..13.3 .36d697c25 16645..59b 923267b892 649588832u .125843789 951.4113..

......... .........1 ..13..11.. ..1....12. ..11.1.... ....1..1.. 11...2..2. 2....1....
........22 1......... 1.3231.232 .1241..648 1.1323.422 11522.1411 1512123121 .2..1.11..
...1...111 1...1..1.2 1.3.4424.. 12111..13. 3211345132 333.1.4112 .181553421 .3.1..2...
......231 1.11..... 3.133332.. ..14....37 21.1212331 .155251.15 .21d114243 12.21.1...
..11....54 442.1.13.3 2128863425 179c322693 7223c9852b 79a6474428 .151+68757 4522..311.
.1.1..146 42.14....1 22ad8256.4 .9a952.796 55327a9a95 ad47774434 22516+8765 34314132..
...2.1.243 42..3.12.5 45848b6752 158b.21888 ba.437ba49 d695574763 .33488m557 86135331..
..211.344 42.2...1.4 .596e62623 398521.c86 81356669a8 7c8cbdbg47 .142775p73 751.3222..
..51..326 321.3...7 3266879a22 1a5423.b95 4325633849 926873a648 22245657t8 35113133..
...1.1.537 231...12.3 21793a8731 1c8512.a85 961466ad59 75658d5329 .11375738s 971321.2..

...3.1.653 12..1..3.3 23855693.3 1575.1.9a8 4313783777 a63a865439 2..1438739 v5243211..
..5.1.a22 513.2..121 .197557b31 179f21.g46 a711469b98 7835626575 .232546557 5u.11242..
.......1. .2..1.... .14121131. .32121.316 21..161431 1.12211221 ...231111 2.6121....
........21 .2..1..11. .312241... .242..1134 2..1313531 1223233... ..12213.13 41161.1...
...1.1.11. 32113..... .12126.... .4221..445 112.131234 34632232.4 .1.1.45332 3121t.51..
...1..111 1......1.. 3111111..1 ..1....112 4...22111. ..312131.1 1....13211 221..b....
.....3.134 11..1..... 1165132112 .633...23 2152.11232 5154222211 .12133323. 14.15.h21.
....1.1.1 2.1..1.... 2.12411121 1621....13 11..213.31 22.51223.3 .1..121232 12..1.28..
......... .........1 ..11..21.. ...1....1 ......... ....2..1.. ......... .........
......... ......... ......... ......... ......... 11........ ......... .........1
```

Table 1. The number of biembeddings of the 80 STS(15)s.

i	s	t	i	s	t	i	s	t	i	s	t
1	1	1	21	14	129	41	31	352	61	1	25
2	0	4	22	13	133	42	16	194	62	5	108
3	3	14	23	35	422	43	12	85	63	8	116
4	7	79	24	57	413	44	19	162	64	13	121
5	1	19	25	31	413	45	39	369	65	43	347
6	9	37	26	40	370	46	33	365	66	41	366
7	2	4	27	46	361	47	26	352	67	22	370
8	10	143	28	34	371	48	35	401	68	25	381
9	16	196	29	16	122	49	27	362	69	29	338
10	21	227	30	19	165	50	31	369	70	28	352
11	14	189	31	12	98	51	31	387	71	31	318
12	13	157	32	33	358	52	40	404	72	30	349
13	12	63	33	31	354	53	28	385	73	6	90
14	4	37	34	30	373	54	27	375	74	6	100
15	9	122	35	13	119	55	32	371	75	29	156
16	1	5	36	4	79	56	24	387	76	11	69
17	3	26	37	1	22	57	26	345	77	17	147
18	10	101	38	31	395	58	30	330	78	8	95
19	6	39	39	43	404	59	9	143	79	0	14
20	11	151	40	31	412	60	30	343	80	1	3

Table 2. Numbers of self-embeddings and biembeddings.

Overall, the value of t varies from 1 to 422. Examining the results in more detail, we observe that the systems having the largest automorphism groups have, in general, the smallest total number of biembeddings t. However, if we exclude the exceptional projective system #1, the value of the product of t with the order of the automorphism group varies from 264 to 1344, a factor of approximately 5. For the 36 systems with the trivial automorphism group, this product ($= t$, the total number of biembeddings) lies in the narrow range from 318 to 422.

Most of the 9 530 biembeddings have a trivial automorphism group and no biembedding has an automorphism group of order greater than 10. We include as automorphisms of a biembedding mappings that reverse the orientation and, for self-embeddings, those that reverse the colour classes. Table 3 analyzes the biembeddings by the order of their automorphism group Γ.

$|\Gamma|$	1	2	3	4	5	6	7	8	9	10
# biembeddings	8 097	1 371	22	0	2	32	0	0	0	6

Table 3. Automorphism group orders.

The six biembeddings having $|\Gamma| = 10$ comprise one self-embedding of system #1, four self-embeddings of system #76, and one self-embedding of system #80. These are all described in [8].

Acknowledgement

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