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Zigzag and foxtrot terraces for $\mathbb{Z}_n$

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Abstract

A terrace for $\mathbb{Z}_n$ (a $\mathbb{Z}_n$ terrace) is an arrangement $(a_1, a_2, \ldots, a_n)$ of the n elements of $\mathbb{Z}_n$ such that the sets of differences $a_{i+1} - a_i$ and $a_i - a_{i+1}$ $(i = 1, 2, \ldots, n-1)$ together contain each element of $\mathbb{Z}_n \setminus \{0\}$ exactly twice. Various general constructions are given for $\mathbb{Z}_n$ terraces of two special types: *zigzag terraces* and *foxtrot terraces*. Some special cases already appear in the literatures of recreational and combinatorial mathematics, of statistics, and of the map color theorem, though the term "terrace" is not used in these sources. Now some powerful generalisations of previous results are presented, along with some new constructions.

1 Introduction

Terraces are sequences of a type used for constructing designs with neighbour properties. A terrace for $\mathbb{Z}_n$ (briefly, a $\mathbb{Z}_n$ terrace) is an arrangement $(a_1, a_2, \ldots, a_n)$ of the n elements of $\mathbb{Z}_n$ such that the sets of differences $a_{i+1} - a_i$ and $a_i - a_{i+1}$ $(i = 1, 2, \ldots, n - 1)$ between them contain each element of $\mathbb{Z}_n \setminus \{0\}$ exactly twice. A $\mathbb{Z}_n$ terrace thus provides a partition of the edges of $2K_n$ into Hamiltonian paths, invariant under a group acting regularly.

The more general definition of a terrace for any finite group G was given by Bailey [8], but the present paper is concerned only with the special case where the group is $\mathbb{Z}_n$.

A terrace $(a_1, a_2, \ldots, a_n)$ for $\mathbb{Z}_n$ is *directed* if the set of differences $a_{i+1} - a_i$ $(i = 1, 2, \ldots, n - 1)$ contains each element of $\mathbb{Z}_n \setminus \{0\}$ exactly once, and is *narcissistic* [5] if

$$a_{i+1} - a_i \equiv a_{n-i+1} - a_{n-i} \qquad (\text{mod } n)$$

for $i = 1, 2, \ldots, n-1$.

If we write $b_i = a_{i+1} - a_i$ $(i = 1, 2, \ldots, n-1)$, the sequence $(b_1, b_2, \ldots, b_{n-1})$ for the terrace $(a_1, a_2, \ldots, a_n)$ is the terraces's 2-*sequencing*. If the terrace is directed, the 2-sequencing is called its *sequencing*. If a directed terrace has

$$(a_{i+1} - a_i) \equiv -(a_{n-i+1} - a_{n-i}) \pmod{n}$$

for $i = 1, 2, \ldots, n-1$, then its sequencing is said to be *symmetric* [9, p. 70], and it remains unchanged if the elements of the terrace are taken in reverse order.

Various methods of constructing terraces for $\mathbb{Z}_n$ are known. We now mention just four of them. When we present a terrace in a display, we adopt the practice, used in previous papers, of omitting brackets and commas.

First, Anderson and Preece [4, 5, 7] obtained *power-sequence terraces* that involve sequences of successive powers of elements of $\mathbb{Z}_n$. For example, the following power-sequence terrace for $\mathbb{Z}_{11}$ is obtained, as indicated, by evaluating a 3-part sequence modulo 11:

$$\begin{array}{ccccccccccc} 1 & 4 & 5 & 9 & 3 & 0 & 2 & 7 & 8 & 6 & 10 \\ \\ 4^0 & 4^1 & 4^2 & 4^3 & 4^4 & 0 & 2\cdot 9^0 & 2\cdot 9^1 & 2\cdot 9^2 & 2\cdot 9^3 & 2\cdot 9^4 \end{array} .$$

Second, a different Anderson [1, 2], following a different approach, and generalising work of Williams [21, p. 154], used the first $(n+1)/2$ triangular numbers to produce terraces for $\mathbb{Z}_n$ where n is a prime. For example, the narcissistic terrace

$$0 \quad 1 \quad 3 \quad 6 \quad 10 \quad 4 \quad 9 \quad 2 \quad 5 \quad 7 \quad 8$$

for $\mathbb{Z}_{11}$ is a *triangular numbers terrace* obtained by evaluating the following sequence modulo 11, where $\tau_j = j(j+1)/2$:

$$\tau_0 \quad \tau_1 \quad \tau_2 \quad \tau_3 \quad \tau_4 \quad \tau_5 \quad 2\tau_5 - \tau_4 \quad 2\tau_5 - \tau_3 \quad 2\tau_5 - \tau_2 \quad 2\tau_5 - \tau_1 \quad 2\tau_5 - \tau_0 \, .$$

Anderson's approach can be viewed as an enhancement of Lindner and Rodger's construction [12, Lemma 3.2] of m-sequences ($m = (n-1)/2$ with n prime).

Third, if $n = 2^r$ for some positive integer r, then the sequence

$$\tau_0 \quad \tau_1 \quad \ldots \quad \tau_{n-1} \, ,$$

when evaluated modulo n, is a directed $\mathbb{Z}_n$ terrace with a symmetric sequencing (Gilbert [11]).

Fourth, for $n = 2^i p$ where $i \geq 0$ and p is any prime satisfying $p \equiv 3 \pmod{4}$, there is the *alternating p-tuples construction* [9, p. 82] for directed $\mathbb{Z}_n$ terraces with symmetric sequencings. The successive entries in any such sequencing are the successive integers $1, 2, \ldots, n-1$ with signs assigned as follows: the first $(p-1)/2$ entries have $+$, then come p entries with $-$, then p entries with $+$, and so on, in runs of p entries, until finally the last $(p-1)/2$ entries have $+$. With $p = 3$ and $i = 2$, this construction gives the following directed terrace for $\mathbb{Z}_{12}$:

$$0 \quad 1 \quad 11 \quad 8 \quad 4 \quad 9 \quad 3 \quad 10 \quad 2 \quad 5 \quad 7 \quad 6 \, .$$

However, there is also a large class of constructions that produces $\mathbb{Z}_n$ terraces consisting solely of subsequences of the form

$$x \quad y \quad x+c \quad y-c \quad x+2c \quad y-2c \quad \ldots .$$

If, working modulo n, we represent the elements $0, 1, 2, \ldots, n-1, 0$, taken in order, by points equally spaced on the circumference of a circle, and join each pair of points that corresponds to 2 immediate neighbours in the above subsequence, we obtain a zigzag graph. Accordingly, we use the name *zigzag terraces* for $\mathbb{Z}_n$ terraces made up of subsequences of the form just given.

There are also $\mathbb{Z}_n$ terraces whose construction is based on sequences of 4-element components of the form

$$\underbrace{x \;\; y \;\; x+1 \;\; y+1}\;\underbrace{x+2 \;\; y-2 \;\; x+3 \;\; y-1}\;\underbrace{x+4 \;\; y-4 \;\; x+5 \;\; y-3}\ldots .$$

We adopt terminology from Ringel [19, p. 124] by referring to the terraces based on this 4-step pattern as *foxtrot terraces.*

In the following Sections of this paper, we give zigzag and foxtrot constructions that are successively more and more intricate. Some special cases already appear in the literatures of recreational and combinatorial mathematics, of statistics, and of the map color theorem, although the term "terrace" is not used in all of these sources. We present some powerful generalisations of previous results, and we give some new constructions. We believe that this paper provides the first systematic catalogue of zigzag and foxtrot terraces for $\mathbb{Z}_n$.

Checking that each of our constructions does indeed produce $\mathbb{Z}_n$ terraces is merely a matter of writing down the list of differences between successive entries, as in [19, pp. 124–125]. Accordingly, no proofs appear in the present paper.

2 Lucas-Walecki-Williams terraces

We start with the simplest class of zigzag terraces for $\mathbb{Z}_n$. These have been "discovered" so often that their name varies. However, the earliest source for the construction for even n seems to be a posthumous publication of Lucas [13, pp. 162–166], where credit is given to Lucas's colleague Walecki. In the statistical literature, the construction for odd n emerged independently in work by Williams [21]. Accordingly, as in some previous publications, we use the nomenclature *Lucas-Walecki-Williams terraces.*

Each of these terraces for $\mathbb{Z}_n$ (where n is odd or even) is of the form

$$0 \quad 1 \quad n-1 \quad 2 \quad n-2 \quad \ldots \quad \lceil n/2 \rceil .$$

Such a terrace is directed, with a symmetric sequencing, if n is even, and is narcissistic if n is odd. The diagram for each terrace consists of a single zigzag.

Example 2.1: The Lucas-Walecki-Williams terrace for $\mathbb{Z}_{10}$ is

$$0 \quad 1 \quad 9 \quad 2 \quad 8 \quad 3 \quad 7 \quad 4 \quad 6 \quad 5 .$$

3 Twizzler terraces

Let n be a composite number of the form $n = fg$ where each of f and g is a positive (but not necessarily prime) integer greater than 1. (The integers f and g may or may not be distinct.) Suppose that we split the negative of the Lucas-Walecki-Williams terrace for $\mathbb{Z}_n$ into f consecutive segments, each containing g elements. Ollis [15] observed that, if the ordering of elements within each segment is reversed (*i.e.* if the segment is *twizzled*), the new sequence thus obtained is still a $\mathbb{Z}_n$ terrace. We call this new terrace the *g-twizzler terrace* for $\mathbb{Z}_n$. Thus for $\mathbb{Z}_{15}$ the 3-twizzler terrace is

$$\underbrace{1\ \ 14\ \ 0}\ \ \underbrace{12\ \ 2\ \ 13}\ \ \underbrace{4\ \ 11\ \ 3}\ \ \underbrace{9\ \ 5\ \ 10}\ \ \underbrace{7\ \ 8\ \ 9}\,,$$

and the 5-twizzler terrace is

$$\underbrace{2\ \ 13\ \ 1\ \ 14\ \ 0}\ \ \underbrace{10\ \ 4\ \ 11\ \ 3\ \ 12}\ \ \underbrace{7\ \ 8\ \ 6\ \ 9\ \ 5}\,.$$

These examples illustrate why we chose to start from the *negative* of the Lucas-Walecki-Williams terrace; with this choice the "low" integers in each segment decrease from left to right and the "high" ones increase.

We now consider how *imperfect g-twizzler terraces* with segments of length g can be obtained for $\mathbb{Z}_n$ when n is not a multiple of g, say when $n = fg + \delta$ where $0 < \delta < g$. We can start each of these terraces with, as before, a succession of twizzled segments of length g, but there will be some "left over" elements that will have to be arranged in some other way. Indeed it is not always possible to have f successive twizzled segments of length g. Consider, for example, the following imperfect 4-twizzler terrace for $\mathbb{Z}_{21}$:

$$\underbrace{19\ \ 1\ \ 20\ \ 0}\ \ \underbrace{17\ \ 3\ \ 18\ \ 2}\ \ \underbrace{15\ \ 5\ \ 16\ \ 4}\ \ \underbrace{13\ \ 7\ \ 14\ \ 6}\ \ 11\ \ 10\ \ 8\ \ 12\ \ 9\,.$$

Here we are forced to have 5 "left over" elements, not 1, as

$$\underbrace{19\ \ 1\ \ 20\ \ 0}\ \ \underbrace{17\ \ 3\ \ 18\ \ 2}\ \ \underbrace{15\ \ 5\ \ 16\ \ 4}\ \ \underbrace{13\ \ 7\ \ 14\ \ 6}\ \ \underbrace{11\ \ 9\ \ 12\ \ 8}\ \ 10$$

is not a terrace for $\mathbb{Z}_{21}$.

Imperfect 3-twizzler $\mathbb{Z}_n$ terraces with the fewest possible "left over" elements are as follow:

$n \equiv 1 \pmod 6$:

$$\underbrace{1\quad n-1\quad 0}\quad \underbrace{n-3\quad 2\quad n-2}\quad \ldots\quad \underbrace{\tfrac{n-5}{2}\ \ \tfrac{n+5}{2}\ \ \tfrac{n-7}{2}}\ \ \tfrac{n+1}{2}\ \ \tfrac{n-1}{2}\ \ \tfrac{n+3}{2}\ \ \tfrac{n-3}{2}$$

$n \equiv 4 \pmod 6$:

$$\underbrace{1\quad n-1\quad 0}\quad \underbrace{n-3\quad 2\quad n-2}\quad \ldots\quad \underbrace{\tfrac{n+4}{2}\ \ \tfrac{n-6}{2}\ \ \tfrac{n+6}{2}}\ \ \tfrac{n-2}{2}\ \ \tfrac{n}{2}\ \ \tfrac{n-4}{2}\ \ \tfrac{n+2}{2}$$

$n \equiv 2 \pmod 6$:

$$\underbrace{1\quad n-1\quad 0}\quad \underbrace{n-3\quad 2\quad n-2}\quad \cdots\quad \underbrace{\tfrac{n+2}{2}\quad \tfrac{n-4}{2}\quad \tfrac{n+4}{2}}\quad \tfrac{n}{2}\quad \tfrac{n-2}{2}$$

$n \equiv 5 \pmod 6$:

$$\underbrace{1\quad n-1\quad 0}\quad \underbrace{n-3\quad 2\quad n-2}\quad \cdots\quad \underbrace{\tfrac{n-3}{2}\quad \tfrac{n+3}{2}\quad \tfrac{n-5}{2}}\quad \tfrac{n-1}{2}\quad \tfrac{n+1}{2}$$

Imperfect 4-twizzler $\mathbb{Z}_n$ terraces with the fewest possible "left over" elements are as follow:

$n \equiv 1 \pmod 4$:

$$\underbrace{n-2\quad 1\quad n-1\quad 0}\quad \cdots\quad \underbrace{\tfrac{n+5}{2}\quad \tfrac{n-7}{2}\quad \tfrac{n+7}{2}\quad \tfrac{n-9}{2}}\quad \tfrac{n+1}{2}\quad \tfrac{n-1}{2}\quad \tfrac{n-5}{2}\quad \tfrac{n+3}{2}\quad \tfrac{n-3}{2}$$

$n \equiv 2 \pmod 4$:

$$\underbrace{n-2\quad 1\quad n-1\quad 0}\quad \cdots\quad \underbrace{\tfrac{n+2}{2}\quad \tfrac{n-4}{2}\quad \tfrac{n+4}{2}\quad \tfrac{n-6}{2}}\quad \tfrac{n-2}{2}\quad \tfrac{n}{2}$$

$n \equiv 3 \pmod 4$:

$$\underbrace{n-2\quad 1\quad n-1\quad 0}\quad \cdots\quad \underbrace{\tfrac{n+3}{2}\quad \tfrac{n-5}{2}\quad \tfrac{n+5}{2}\quad \tfrac{n-7}{2}}\quad \tfrac{n-1}{2}\quad \tfrac{n+1}{2}\quad \tfrac{n-3}{2}$$

If we take the negative of a 4-twizzler or imperfect 4-twizler terrace for $\mathbb{Z}_n$, as just given, and subtract 1 (mod n) throughout, each twizzled segment is of the form

$$x\quad y\quad x-1\quad y+1$$

where x and $x-1$ are "low" integers, and y and $y-1$ are "high" integers. Such segments, and reverses of such segments, also arise in certain $\mathbb{Z}_n$ terraces, $n \equiv 5 \pmod{12}$, that are obtainable from from a cycle construction of Friedlander, Gordon and Miller [10, Theorem 5, p. 314]. With $n = 12s+5$ ($s \geq 1$), these terraces are of the form

$$0\quad \underbrace{\underbrace{2s+2\quad 4s+1\quad 2s+1\quad 4s+2}\ \underbrace{2s+4\quad 4s-1\quad 2s+3\quad 4s}\quad \cdots}_{(s+1)/2 \text{ segments}}$$

$$\underbrace{\cdots\quad \underbrace{12s+2\quad 6s+5\quad 12s+1\quad 6s+6}\ \underbrace{12s+4\quad 6s+3\quad 12s+3\quad 6s+4}}_{(3s+1)/2 \text{ segments (generated from the right)}}$$

$$\underbrace{\underbrace{2\quad 6s+1\quad 1\quad 6s+2}\ \underbrace{4\quad 6s-1\quad 3\quad 6s}\quad \cdots\quad \underbrace{2s\quad 4s+3\quad 2s-1\quad 4s+4}}_{s \text{ segments}}\ .$$

If the zero entry is removed from the start of the terrace, and the two ends of the resulting sequence are joined, we obtain a cycle as given in [10, p. 314]. If s is even, two half-segments are produced, adjacent to one another.

Example 3.1: For $n = 29$, we take $s = 2$ in the above construction to obtain the following $\mathbb{Z}_{29}$ terrace, containing half-segments:

$$0\ \underbrace{\underbrace{6\ \ 9\ \ 5\ \ 10}\ \underbrace{8\ \ 7}}$$

$$\underbrace{\underbrace{21\ \ 22}\ \underbrace{24\ \ 19\ \ 23\ \ 20}\ \underbrace{26\ \ 17\ \ 25\ \ 18}\ \underbrace{28\ \ 15\ \ 27\ \ 16}}$$

$$\underbrace{\underbrace{2\ \ 13\ \ 1\ \ 14}\ \underbrace{4\ \ 11\ \ 3\ \ 12}}\ .$$

4 Some terraces with 2 zigzags

We now present some $\mathbb{Z}_n$ terraces each consisting of two zigzag segments.

First, for n odd, we have two types of $\mathbb{Z}_n$ terrace that are very similar to one another; we call these types *Ringel-Owens terraces* and *hiccup terraces.* With $n = 2s + 1$, the Ringel-Owens terraces are of the following form, which provides the solution to Exercise 7.6.3 of Ringel [19, p. 129]:

$$\underbrace{0\ \ s+1\ \ 2s\ \ s+2\ \ 2s-1\ \ldots\ \lfloor(3s+2)/2\rfloor}_{(s+1)\text{ elements}}\ \underbrace{\lfloor(s+2)/2\rfloor\ \ldots\ s-1\ \ 2\ \ s\ \ 1}_{s\text{ elements}^\dagger}\ .$$

† (generated from the right)

If this is multiplied by 2, modulo n, and each element is increased by 1, we obtain the terrace in the form given by P. J. Owens (see [17, p. 146] and [18, p. 269]).

The hiccup terraces are similarly of the following form:

$$\underbrace{s+1\ \ 0\ \ s+2\ \ 2s\ \ s+3\ \ 2s-1\ \ldots\ \lfloor(3s+2)/2\rfloor}_{(s+1)\text{ elements}}\ \underbrace{\lfloor(s+2)/2\rfloor\ \ldots\ s-1\ \ 2\ \ s\ \ 1}_{s\text{ elements}}.$$

To see why we use the word "hiccup", multiply the reverse of this terrace by 2, modulo n, and subtract 1. We then obtain the following:

$n \equiv 1 \pmod 4$:

$$\underbrace{+1\ \ -2\ \ +3\ \ -4\ \ldots\ \ +(s-1)\ \ -s}_{s\text{ elements}}\ \underbrace{-(s+1)\ \ +(s+2)\ \ldots\ +(n-1)\ \ -n}_{s+1\text{ elements}}\ .$$

$n \equiv 3 \pmod 4$:

$$\underbrace{+1\ \ -2\ \ +3\ \ -4\ \ldots\ \ -(s-1)\ \ +s}_{s\text{ elements}}\ \underbrace{+(s+1)\ \ -(s+2)\ \ldots\ +(n-1)\ \ -n}_{s+1\text{ elements}}\ .$$

Thus the unsigned elements are the successive integers 1 to n, in order, and the signs are $+$ and $-$ alternately, save that a sign is repeated after the hiccup following the first s elements.

Example 4.1: The Ringel-Owens terrace for $\mathbb{Z}_{17}$ is

$$\underbrace{0\ \ 9\ \ 16\ \ 10\ \ 15\ \ 11\ \ 14\ \ 12\ \ 13}\ \ \underbrace{5\ \ 4\ \ 6\ \ 3\ \ 7\ \ 2\ \ 8\ \ 1}\ ,$$

and the hiccup terrace for $\mathbb{Z}_{17}$ is

$$\underbrace{9\ \ 0\ \ 10\ \ 16\ \ 11\ \ 15\ \ 12\ \ 14\ \ 13}\ \ \underbrace{5\ \ 4\ \ 6\ \ 3\ \ 7\ \ 2\ \ 8\ \ 1}\ .$$

Second, for $n \equiv 2 \pmod 4$, we again have two types of $\mathbb{Z}_n$ terrace that are very similar to one another. We name them after the authors of the original papers containing them: the *Ollis terraces* and the *Street terraces*. The Ollis terraces [14, p. 147] are narcissistic and, with $n = 2s$, are of the following form:

$$\underbrace{0\ \ s+1\ \ 2s-1\ \ s+2\ \ 2s-2\ \ldots\ (3s+1)/2}_{s \text{ elements}}\ \underbrace{(s+1)/2\ \ldots\ s-1\ \ 2\ \ s\ \ 1}_{s \text{ elements}}\ .$$

A translate of the Ollis terrace was used by Anderson & Preece [6, p. 57] to obtain an infinite series of what they called *logarithmic terraces* (see [6]). For ease of comparison with the above form for the Ollis terraces, we now give the Street terraces in a form that is a translate of the negative of the original [20, p. 93]:

$$\underbrace{0\ \ s+2\ \ 2s-1\ \ s+3\ \ 2s-2\ \ldots\ (3s+1)/2}_{(s-1) \text{ elements}}\ \underbrace{(s+3)/2\ \ldots\ s\ \ 2\ \ s+1\ \ 1}_{(s+1) \text{ elements}}\ .$$

Example 4.2: For $\mathbb{Z}_{10}$, the Ollis terrace is

$$\underbrace{0\ \ 6\ \ 9\ \ 7\ \ 8}\ \underbrace{3\ \ 4\ \ 2\ \ 5\ \ 1},$$

whereas the Street terrace is

$$\underbrace{0\ \ 7\ \ 9\ \ 8}\ \underbrace{4\ \ 3\ \ 5\ \ 2\ \ 6\ \ 1}.$$

5 Tripartite and sesquipartite terraces

For n odd, *tripartite terraces* for $\mathbb{Z}_n$ were given by Anderson & Preece [3], but they are also easily constructed for even n. These terraces are directed, with a symmetric sequencing, if n is even, and are narcissistic if n is odd. Tiresomely, the tripartite terraces have slightly different forms depending on whether $n \equiv 0, 1, \ldots, 5 \pmod 6$. Each terrace falls into three segments of equal, or nearly equal, lengths, and its diagram consists of 3 joined zigzag sections, one for each segment. If $n \equiv 0 \pmod 6$, the tripartite $\mathbb{Z}_n$ terrace (as given below) is merely a translate of the $(n/3)$-twizzler terrace for $\mathbb{Z}_n$. Likewise, if $n \equiv 3 \pmod 6$, the tripartite terrace is a translate of the negative of the $(n/3)$-twizzler terrace. We nevertheless give all 6 forms of the tripartite terraces, so that they may readily be compared.

If the first and third segments of a tripartite terrace are interchanged, then
- for $n \equiv 0 \pmod 3$ we obtain a translate of the negative of the Lucas-Walecki-Williams terrace as given above;
- for $n \equiv 1 \pmod 3$ we obtain a translate of the Lucas-Walecki-Williams terrace;
- for $n \equiv 2 \pmod 3$ we obtain a sequence that is not a terrace.

The forms of the tripartite terraces are as follows:

$n \equiv 0 \pmod 6$, with $n = 6q$:

$$\underbrace{0\ \ 2q-1\ \ 1\ \ 2q-2\ \ldots\ q}_{2q \text{ elements}}\ \underbrace{5q\ \ 3q-1\ \ 5q+1\ \ 3q-2\ \ldots\ 2q}_{2q \text{ elements}}\ \underbrace{4q\ \ 4q-1\ \ 4q+1\ \ 4q-2\ \ldots\ 3q}_{2q \text{ elements}}$$

$n \equiv 1 \pmod 6$, with $n = 6q+1$:

$$\underbrace{0\ \ 2q-1\ \ 1\ \ 2q-2\ \ldots\ q}_{2q \text{ elements}}\ \underbrace{3q\ \ 5q+1\ \ 3q-1\ \ 5q+2\ \ldots\ 2q}_{(2q+1) \text{ elements}}\ \underbrace{4q\ \ 4q+1\ \ 4q-1\ \ 4q+2\ \ldots\ 5q}_{2q \text{ elements}}$$

$n \equiv 2 \pmod 6$, with $n = 6q+2$:

$$\underbrace{0\ \ 2q-1\ \ 1\ \ 2q-2\ \ldots\ q}_{2q \text{ elements}}\ \underbrace{3q\ \ 5q+1\ \ 3q-1\ \ 5q+2\ \ldots\ 6q+1}_{(2q+2) \text{ elements}}$$

$$\underbrace{4q+1\ \ 4q\ \ 4q+2\ \ 4q-1\ \ldots\ 3q+1}_{2q \text{ elements}}$$

$n \equiv 3 \pmod 6$, with $n = 6q+3$:

$$\underbrace{0\ \ 2q\ \ 1\ \ 2q-1\ \ldots\ q}_{(2q+1) \text{ elements}}\ \underbrace{3q+1\ \ 5q+3\ \ 3q\ \ 5q+4\ \ldots\ 2q+1}_{(2q+1) \text{ elements}}$$

$$\underbrace{4q+2\ \ 4q+1\ \ 4q+3\ \ 4q\ \ldots\ 5q+2}_{(2q+1) \text{ elements}}$$

$n \equiv 4 \pmod 6$, with $n = 6q-2$:

$$\underbrace{0\ \ 2q-2\ \ 1\ \ 2q-3\ \ldots\ q-1}_{(2q-1) \text{ elements}}\ \underbrace{5q-2\ \ 3q-2\ \ 5q-1\ \ 3q-3\ \ldots\ 2q-1}_{2q \text{ elements}}$$

$$\underbrace{4q-2\ \ 4q-1\ \ 4q-3\ \ 4q\ \ldots\ 3q-1}_{(2q-1) \text{ elements}}$$

$n \equiv 5 \pmod 6$, with $n = 6q-1$:

$$\underbrace{0\ \ 2q-1\ \ 1\ \ 2q-2\ \ldots\ q}_{2q \text{ elements}}\ \underbrace{5q-1\ \ 3q-2\ \ 5q\ \ 3q-3\ \ldots\ 6q-2}_{(2q-1) \text{ elements}}$$

$$\underbrace{4q-2\ \ 4q-1\ \ 4q-3\ \ 4q\ \ldots\ 5q-2}_{2q \text{ elements}}$$

Example 5.1: The tripartite terrace for $\mathbb{Z}_{19}$ is

$$\underbrace{0\ \ 5\ \ 1\ \ 4\ \ 2\ \ 3}_{6 \text{ elements}}\ \underbrace{9\ \ 16\ \ 8\ \ 17\ \ 7\ \ 18\ \ 6}_{7 \text{ elements}}\ \underbrace{12\ \ 13\ \ 11\ \ 14\ \ 10\ \ 15}_{6 \text{ elements}}\ .$$

For even values of n, the symmetric sequencings of the tripartite $\mathbb{Z}_n$ terraces allow us to take the first half of any of these terraces modulo $n/2$ to obtain a terrace for $\mathbb{Z}_{n/2}$. In the terminology of [9, pp. 73–75], these terraces for $\mathbb{Z}_{n/2}$ are *half-projections* of the corresponding $\mathbb{Z}_n$ terraces, which conversely are *lifts* of the $\mathbb{Z}_{n/2}$ terraces. Ollis [16] has suggested the name *sesquipartite terraces* for the half-projections. Changing our notation so that n now denotes the half-length, the sesquipartite terraces for $\mathbb{Z}_n$ are of the following forms:

$n \equiv 0 \pmod 3$, with $n = 3q$:

$$\underbrace{0\ \ 2q-1\ \ 1\ \ 2q-2\ \dots\ q}_{2q \text{ elements}}\ \ \underbrace{2q\ \ 3q-1\ \ 2q+1\ \ 3q-2\ \dots\ \lceil (5q-1)/2 \rceil}_{q \text{ elements}}$$

$n \equiv 1 \pmod 3$, with $n = 3q+1$:

$$\underbrace{0\ \ 2q-1\ \ 1\ \ 2q-2\ \dots\ q}_{2q \text{ elements}}\ \ \underbrace{3q\ \ 2q\ \ 3q-1\ \ 2q+1\ \dots\ \lfloor 5q/2 \rfloor}_{(q+1) \text{ elements}}$$

$n \equiv 2 \pmod 3$, with $n = 3q-1$:

$$\underbrace{0\ \ 2q-2\ \ 1\ \ 2q-3\ \dots\ q-1}_{(2q-1) \text{ elements}}\ \ \underbrace{2q-1\ \ 3q-2\ \ 2q\ \ 3q-3\ \dots\ \lceil (5q-3)/2 \rceil}_{q \text{ elements}}$$

Example 5.2: The sesquipartite terrace for $\mathbb{Z}_{16}$ is

$$\underbrace{0\quad 9\quad 1\quad 8\quad 2\quad 7\quad 3\quad 6\quad 4\quad 5}\quad \underbrace{15\quad 10\quad 14\quad 11\quad 13\quad 12}\ .$$

6 Fractured Ringel-Owens terraces

Suppose that we take the $\mathbb{Z}_{17}$ Ringel-Owens terrace as above. If we remove the entry 0 from the beginning and then join the ends of what remains, to form a circular arrangement instead of a linear one, the differences between successive elements are still such as are required for a terrace for $\mathbb{Z}_{17}$. If we now break the circular arrangement immediately before the element 3, and place the missing element 0 in front of the 3 to obtain

$$0\quad \underbrace{3\quad 7\quad 2\quad 8\quad 1\quad \overbrace{9}}\overbrace{\quad 16\quad 10\quad 15\quad 11\quad 14\quad 12\quad 13}\quad \underbrace{5\quad 4\quad 6}\,,$$

we again have a terrace for $\mathbb{Z}_{17}$. We denominate $\mathbb{Z}_n$ terraces obtained in this sort of way *fractured Ringel-Owens terraces.* They are obtainable for all odd values of n as follows. Each such terrace has, apart from the zero at the start, three zigzag segments, two of which overlap in a single element. The lengths of the three zigzags are roughly in the ratios 3 : 2 : 1.

The forms of the terraces are as follows:

$n \equiv 1 \pmod 6$, with $n = 6q+1$:

$$0\ \ \underbrace{5q+1\ \ 4q\ \ 5q+2\ \ 4q-1\ \ \dots\ \ 3q+1}_{2q \text{ elements}}\ \overbrace{\ 1\ \ 3q\ \ 2\ \ 3q-1\ \ \dots\ \ \lfloor (3q+2)/2 \rfloor}^{(3q+1) \text{ elements}}$$

$$\underbrace{\lfloor (9q+2)/2 \rfloor\ \ \dots\ \ 5q-1\ \ 4q+2\ \ 5q\ \ 4q+1}_{q \text{ elements}}$$

$n \equiv 3 \pmod 6$, with $n = 6q + 3$:

$$0\ \ \underbrace{q+1\ \ 2q+1\ \ q+2\ \ 2q\ \ \ldots\ \ \lceil(3q+2)/2\rceil}_{(q+1)\text{ elements}}$$

$$\underbrace{\lceil(9q+2)/2\rceil\ \ \ldots\ \ 6q+1\ \ 3q+3\ \ 6q+2\ \ \rlap{$\overbrace{3q+2\ \ 1\ \ 3q+1\ \ 2\ \ 3q\ \ \ldots\ \ 2q+2}^{(2q+1)\text{ elements}}$}3q+2}_{(3q+1)\text{ elements}}$$

$n \equiv 5 \pmod 6$, with $n = 6q - 1$:

$$0\ \ \underbrace{q\ \ 2q+1\ \ q-1\ \ 2q+2\ \ \ldots\ \ 1}_{2q\text{ elements}}\ \ \overbrace{3q\ \ 6q-2\ \ 3q+1\ \ 6q-3\ \ \ldots\ \ \lceil(9q-2)/2\rceil}^{(3q-1)\text{ elements}}$$

$$\underbrace{\lceil 3q/2\rceil\ \ \ldots\ \ q+2\ \ 2q-1\ \ q+1\ \ 2q}_{q\text{ elements}}$$

or

$$0\ \ \underbrace{5q-1\ \ 4q\ \ 5q-2\ \ 4q+1\ \ \ldots\ \ \lceil(9q-2)/2\rceil}_{q\text{ elements}}$$

$$\underbrace{\lceil 3q/2\rceil\ \ \ldots\ \ 2\ \ 3q-1\ \ 1\ \ \rlap{$\overbrace{3q\ \ 6q-2\ \ 3q+1\ \ 6q-3\ \ \ldots\ \ 4q-1}^{(2q-1)\text{ elements}}$}3q}_{3q\text{ elements}}$$

7 Foxtrot terraces

This type of terrace for $\mathbb{Z}_n$ was given by Ringel [19, p. 124] for $n \equiv 1 \pmod 4$ but his construction can readily be adapted for all n. Ringel's account predates the first use of the term "terrace", but he used the nomenclature "foxtrot" because the construction of the *foxtrot terraces* is based on sequences of 4-step (4-element) components of the form

$$\underbrace{x\ \ y\ \ x+1\ \ y+1}\ \ \underbrace{x+2\ \ y-2\ \ x+3\ \ y-1}\ \ \underbrace{x+4\ \ y-4\ \ x+5\ \ y-3}\ldots$$

where y is such that all the elements of the entire sequence are distinct, modulo n, or similarly of the form

$$\underbrace{v\ \ w\ \ v+1\ \ w+1}\ \ \underbrace{v-2\ \ w+2\ \ v-1\ \ w+3}\ \ \underbrace{v-4\ \ w+4\ \ v-3\ \ w+5}\ldots\ .$$

(The second form is equivalent to the negative of the reverse of the first.) Again we have to give six cases separately: firstly two for odd n, then four for even n. (Not all the minor changes from one case to another are obvious.) For the sake of mathematical tidyness, we give the first case in a slightly different form from that of Ringel.

In each case, the terrace starts with two or three elements that do not obey the foxtrot pattern; five of the six cases likewise have one or two elements appended on the end. For the two cases with $n \equiv 2 \pmod 4$, the foxtrot pattern is broken by two elements at or near the middle of the terrace.

$n \equiv 1 \pmod 4$, with $n = 2s + 1$ $(n \geq 9)$:

$$0\quad 2s-1\quad 2s$$

$$\overbrace{\underbrace{1\ \ 2s-3\ \ 2\ \ 2s-2}\ \ \underbrace{3\ \ 2s-5\ \ 4\ \ 2s-4}\ \ldots\ \underbrace{s-3\ \ s+1\ \ s-2\ \ s+2}}$$

$$s-1\quad s$$

$n \equiv 3 \pmod 4$, with $n = 2s + 1$ $(n \geq 7)$:

$$0\quad 1\quad 2$$

$$\overbrace{\underbrace{2s-1\ \ 3\ \ 2s\ \ 4}\ \ \underbrace{2s-3\ \ 5\ \ 2s-2\ \ 6}\ \ldots\ \underbrace{s+2\ \ s\ \ s+3\ \ s+1}}$$

$n \equiv 0 \pmod 8$, with $n = 4t$ $(n \geq 16)$:

$$0\quad 4t-1\quad 2t-1$$

$$\overbrace{\underbrace{2t-3\ \ 2t\ \ 2t-2\ \ 2t+1}\ \ \underbrace{2t-5\ \ 2t+2\ \ 2t-4\ \ 2t+3}\ \ldots\ \underbrace{t-1\ \ 3t-2\ \ t\ \ 3t-1}}^{2t \text{ elements}}$$

$$\overbrace{\underbrace{3t\ \ t-3\ \ 3t+1\ \ t-2}\ \ \underbrace{3t+2\ \ t-5\ \ 3t+3\ \ t-4}\ \ldots\ \underbrace{4t-4\ \ 1\ \ 4t-3\ \ 2}}^{(2t-4) \text{ elements}}$$

$$4t-2$$

$n \equiv 4 \pmod 8$, with $n = 4t$ $(n \geq 12)$:

$$0\quad 4t-1\quad 2t-1$$

$$\overbrace{\underbrace{2t-3\ \ 2t\ \ 2t-2\ \ 2t+1}\ \ \underbrace{2t-5\ \ 2t+2\ \ 2t-4\ \ 2t+3}\ \ldots\ \underbrace{t\ \ 3t-3\ \ t+1\ \ 3t-2}}^{(2t-2) \text{ elements}}$$

$$\overbrace{\underbrace{3t-1\ \ t-2\ \ 3t\ \ t-1}\ \ \underbrace{3t+1\ \ t-4\ \ 3t+2\ \ t-3}\ \ldots\ \underbrace{4t-4\ \ 1\ \ 4t-3\ \ 2}}^{(2t-2) \text{ elements}}$$

$$4t-2$$

$n \equiv 2 \pmod 8$, with $n = 4t + 2$ $(n \geq 18)$:

$$0\quad 2t+1$$

$$\overbrace{\underbrace{2t-1\ \ 2t+2\ \ 2t\ \ 2t+3}\ \ \underbrace{2t-3\ \ 2t+4\ \ 2t-2\ \ 2t+5}\ \ldots\ \underbrace{t+1\ \ 3t\ \ t+2\ \ 3t+1}}^{2t \text{ elements}}$$

$$t-1\quad t$$

$$\overbrace{\underbrace{3t+2\ \ t-3\ \ 3t+3\ \ t-2}\ \ \underbrace{3t+4\ \ t-5\ \ 3t+5\ \ t-4}\ \ldots\ \underbrace{4t-2\ \ 1\ \ 4t-1\ \ 2}}^{(2t-4) \text{ elements}}$$

$$4t\quad 4t+1$$

$n \equiv 6 \pmod 8$, with $n = 4t-2$ $(n \geq 14)$:

$$0 \;\; 2t-1$$

$$\overbrace{\underbrace{2t-3 \;\; 2t \;\; 2t-2 \;\; 2t+1} \;\; \underbrace{2t-5 \;\; 2t+2 \;\; 2t-4 \;\; 2t+3} \;\; \ldots \;\; \underbrace{t+1 \;\; 3t-4 \;\; t+2 \;\; 3t-3}}^{(2t-4)\text{ elements}}$$

$$t-1 \;\; t$$

$$\overbrace{\underbrace{3t-2 \;\; t-3 \;\; 3t-1 \;\; t-2} \;\; \underbrace{3t \;\; t-5 \;\; 3t+1 \;\; t-4} \;\; \ldots \;\; \underbrace{4t-6 \;\; 1 \;\; 4t-5 \;\; 2}}^{(2t-4)\text{ elements}}$$

$$4t-4 \;\; 4t-3$$

Example 7.1: The foxtrot terrace for $\mathbb{Z}_{19}$ is

$$0 \;\; 1 \;\; 2 \;\; \underbrace{17 \;\; 3 \;\; 18 \;\; 4} \;\; \underbrace{15 \;\; 5 \;\; 16 \;\; 6} \;\; \underbrace{13 \;\; 7 \;\; 14 \;\; 8} \;\; \underbrace{11 \;\; 9 \;\; 12 \;\; 10} \;.$$

Example 7.2: The foxtrot terrace for $\mathbb{Z}_{22}$ is

$$0 \;\; 11 \;\; \underbrace{9 \;\; 12 \;\; 10 \;\; 13} \;\; \underbrace{7 \;\; 14 \;\; 8 \;\; 15} \;\; 5 \;\; 6 \;\; \underbrace{16 \;\; 3 \;\; 17 \;\; 4} \;\; \underbrace{18 \;\; 1 \;\; 19 \;\; 2} \;\; 20 \;\; 21 \;.$$

8 One-plus-three terraces

These terraces for $\mathbb{Z}_n$, where n is even, can be said to fall into four zigzag segments, of which the second and third overlap in a single element, and the third and fourth overlap in a single element. Thus the 2-sequencing falls into five segments, the second being a single element corresponding to the difference across the break between the second and first segments of the terrace. Because the first of the four zigzags is distinctive, in not overlapping another zigzag, we refer to the terraces as being *one-plus-three terraces* for $\mathbb{Z}_n$. The constructions are believed to be new to the literature. To permit printing on a page of standard width, we split the third segment of each terrace, but indicate its start and end with $^{\hookrightarrow}$ and $^{\hookleftarrow}$ respectively.

$n \equiv 0 \pmod 8$, with $n = 4t$ (t even):

$$\underbrace{0 \;\; 2 \;\; 4t-2 \;\; 4 \;\; \ldots \;\; 3t+2 \;\; t}_{t\text{ elements}}$$

$$\underbrace{t+1 \;\; t+3 \;\; t-1 \;\; t+5 \;\; \ldots \;\; 2t+1 \;{}^{\hookrightarrow}1}_{(t+1)\text{ elements}} \;\; 2t+2 \;\; 4t-1 \;\; 2t+4$$

$$4t-3 \;\; \ldots \;\; 3t \;\; \underbrace{3t+1^{\hookleftarrow} \;\; t+2 \;\; 3t-1 \;\; t+4 \;\; \ldots \;\; 2t+3 \;\; 2t}_{t\text{ elements}}$$

$n \equiv 4 \pmod 8$, with $n = 4t$ (t odd):

$$\underbrace{0 \;\; 2t-3 \;\; 2 \;\; 2t-5 \;\; \ldots \;\; t \;\; t-1}_{t \text{ elements}}$$

$$\underbrace{3t-2 \;\; 3t \;\; 3t-4 \;\; 3t+2 \;\; \ldots \;\; 2t-1 \,{}^{\hookrightarrow} 4t-1}_{(t+1) \text{ elements}} \;\; 4t-2 \;\; 1 \;\; 4t-4$$

$$3 \;\; \ldots \;\; t-2 \;\; \underbrace{3t-1^{\hookleftarrow} \; t+1 \;\; 3t-3 \;\; t+3 \;\; \ldots \;\; 2t-2 \;\; 2t}_{t \text{ elements}}$$

$n \equiv 2 \pmod 8$, with $n = 4t+2$ (t even):

$$\underbrace{0 \;\; 2t-2 \;\; 2 \;\; 2t-4 \;\; \ldots \;\; t-2 \;\; t}_{t \text{ elements}}$$

$$\underbrace{3t \;\; 3t+1 \;\; 3t-2 \;\; 3t+3 \;\; \ldots \;\; 2t \,{}^{\hookrightarrow} 4t+1}_{(t+2) \text{ elements}} \;\; 4t \;\; 1 \;\; 4t-2$$

$$3 \;\; \ldots \;\; 3t+2 \;\; \underbrace{t-1^{\hookleftarrow} \; 3t-1 \;\; t+1 \;\; 3t-3 \;\; \ldots \;\; 2t+1 \;\; 2t-1}_{(t+1) \text{ elements}}$$

$n \equiv 6 \pmod 8$, with $n = 4t-2$ (t even):

$$\underbrace{0 \;\; 2t-4 \;\; 2 \;\; 2t-6 \;\; \ldots \;\; t \;\; t-2}_{(t-1) \text{ elements}}$$

$$\underbrace{3t-4 \;\; t-1 \;\; 3t-6 \;\; t+1 \;\; \ldots \;\; 2t-2 \,{}^{\hookrightarrow} 2t-3}_{t \text{ elements}} \;\; 4t-4 \;\; 2t-1 \;\; 4t-6$$

$$2t+1 \;\; \ldots \;\; 3t-2 \;\; \underbrace{3t-3^{\hookleftarrow} \; t-3 \;\; 3t-1 \;\; t-5 \;\; \ldots \;\; 1 \;\; 4t-5 \;\; 4t-3}_{t \text{ elements}}$$

Example 8.1: The one-plus-three terrace for $\mathbb{Z}_{20}$ is

$$\underbrace{0 \;\; 7 \;\; 2 \;\; 5 \;\; 4} \;\; \underbrace{13 \;\; 15 \;\; 11 \;\; 17 \;\; 9 \;\; 19} \;\; \overbrace{\;\; 18 \;\; 1 \;\; 16 \;\; 3 \;\;} \;\; \underbrace{14 \;\; 6 \;\; 12 \;\; 8 \;\; 10} \; .$$

9 Tetrazetal terraces

We now present some simple narcissistic $\mathbb{Z}_n$ terraces that each have four zigzag segments. For notational convenience, we take $a_{(n+1)/2} = 0$ in each of the terraces $(a_1, a_2, \ldots, a_n)$, so that, working modulo n, we have

$$a_{n-i+1} \equiv -a_i \;, \quad i = 1, 2, \ldots, (n-1)/2 \;.$$

We thus need specify only the first $(n+1)/2$ of the elements in the terrace.

First, we present what we call *tetrazetal terraces*, which are available when $n \equiv 3 \pmod 6$. The first element in each successive zigzag segment is also the last element of the previous zigzag segment. With $n = 6t+3$, the first $(n+1)/2$ elements of a tetrazetal terrace are as follows:

$$\underbrace{\lfloor (7t+4)/2 \rfloor \ \ \ldots\ \ 3t+3\ \ 4t+1\ \ 3t+2\ \ 4t+2}_{(t+1)\text{ elements}}\!\!\!\!\!\!\!\!\!\!\!\!\!\overbrace{\ \ t\ \ 4t+3\ \ t-1\ \ \ldots\ \ 0}^{(2t+2)\text{ elements}}\ .$$

Second, we have what we call *double hiccup terraces*, which are available when $n \equiv 3 \pmod 4$ [not mod 6]. These, like the hiccup terraces given above, have no overlaps between their zigzag segments, nor do they have overlaps between the zero in the middle and the segments on either side of it. With $n = 4r+3$, the first $(n+1)/2$ elements of a double hiccup terrace are as follows:

$$\underbrace{2r+2\ \ 4r+2\ \ 2r+3\ \ 4r+1\ \ \ldots}_{r\text{ elements}}\ \ \underbrace{\ldots\ \ r-1\ \ r+2\ \ r\ \ r+1}_{(r+1)\text{ elements}}\ \ 0\ .$$

The reason for the nomenclature "double hiccup" is readily seen if we multiply the first $2r+1$ elements of the terrace by 2, modulo n. We then have

$$\underbrace{+1\ \ -2\ \ +3\ \ -4\ \ \ldots}_{r\text{ elements}}\ \ \underbrace{\ldots\ \ +(2r-2)\ \ -(2r-1)\ \ +2r\ \ -(2r+1)}_{(r+1)\text{ elements}}\ \ 0\ ,$$

where the successive unsigned elements are $1, 2, \ldots, (n-1)/2$, in order, and the signs are allocated, with hiccup, as in §4 above.

Example 9.1: The tetrazetal terrace for $\mathbb{Z}_{15}$ is

$$\underbrace{9\ \ 8\ \ 10}\!\!\!\!\overbrace{\ \ 2\ \ 11\ \ 1\ \ 12\ \ 0}\!\!\underbrace{\ \ 3\ \ 14\ \ 4\ \ 13\ \ 5}\!\!\overbrace{\ \ 7\ \ 6}\ ,$$

whereas the double hiccup terrace for $\mathbb{Z}_{15}$ is

$$\underbrace{8\ \ 14\ \ 9}\ \ \underbrace{2\ \ 5\ \ 3\ \ 4}\ \ 0\ \ \underbrace{11\ \ 12\ \ 10\ \ 13}\ \ \underbrace{6\ \ 1\ \ 7}\ .$$

10 Pentazetal terraces

We now proceed to a more general family of terraces for $\mathbb{Z}_n$ where n is any odd integer. The specifications of these *pentazetal terraces* involve a variable parameter k, which is a further odd integer. We write $n = 2s+1$ and $k = 2t+1$. As is suggested by the adjective *pentazetal*, the diagram for each terrace falls into 5 zigzag segments. Again, the first vertex in each successive segment is the last vertex of the previous segment. The number of elements in the five successive segments of the 2-sequencing are respectively

$$t\ ,\qquad s-t\ ,\qquad 2t+1\ ,\qquad s-3t-1\ ,\qquad t\ .$$

Clearly we must have $s-3t-1\geq 0$. If $s-3t=1$ we have $n=3k$, and the pentazetal terraces degenerate into the tetrazetal terraces of §9 above.

If we take $k=5$, the pentazetal terraces differ only trivially from the *five chord terraces* of Ringel [19, pp. 124–125], where the first small segment is not a zigzag.

The pentazetal terraces come in two forms, depending on the parity of $s-t$ (equivalently $s-3t$), *i.e.* on the parity of $(n-k)/2$. For notational convenience we now take the $(t+1)^{\text{th}}$ element in each terrace to be 1, even though this means that, for $n=3k$, we now obtain translates of the tetrazetal terraces as presented above. We use the notation $^{\hookrightarrow}$ and $^{\hookleftarrow}$ as for the one-plus-three terraces.

$n-k\equiv 0 \pmod 4$, *i.e.* $s-t$ even:

$$\underbrace{-\lfloor(t-1)/2\rfloor\ \ \dots\ \ -(t-2)\ \ \ 0\ \ \ -(t-1)\ {}^{\hookrightarrow}1}_{(t+1)\text{ elements (generated from the right)}}$$

$$s+1\quad 2\quad s\quad 3\ \ \dots\ \ (s-t)/2\quad (s+t+4)/2$$

$$\underbrace{(s-t+2)/2^{\hookleftarrow}\ (3s-t+2)/2\ \ (s-t+4)/2\ \ (3s-t)/2\ \ \dots\ \ (s+t+2)/2\ {}^{\hookrightarrow}(3s-3t+2)/2}_{2t+2\text{ elements}}$$

$$(3s-t+4)/2\quad (3s-3t)/2\quad (3s-t+6)/2\quad (3s-3t-2)/2\ \ \dots\ \ 2s-2t\quad s+2$$

$$\underbrace{2s-2t+1^{\hookleftarrow}\ 2s-t+1\ \ 2s-2t+2\ \ 2s-t\ \ \dots\lceil(4s-3t+2)/2\rceil}_{(t+1)\text{ elements}}$$

$n-k\equiv 2 \pmod 4$, *i.e.* $s-t$ odd:

$$\underbrace{-\lfloor(t-1)/2\rfloor\ \ \dots\ \ -(t-2)\ \ \ 0\ \ \ -(t-1)\ {}^{\hookrightarrow}1}_{(t+1)\text{ elements (generated from the right)}}$$

$$s+1\quad 2\quad s\quad 3\ \ \dots\ \ (s+t+5)/2\quad (s-t+1)/2$$

$$\underbrace{(s+t+3)/2^{\hookleftarrow}(3s-3t+3)/2\ (s+t+1)/2\ (3s-3t+5)/2\dots(s-t+3)/2\ {}^{\hookrightarrow}(3s-t+3)/2}_{2t+2\text{ elements}}$$

$$(3s-3t+1)/2\quad (3s-t+5)/2\quad (3s-3t-1)/2\quad (3s-t+7)/2\ \ \dots\ \ 2s-2t\quad s+2$$

$$\underbrace{2s-2t+1^{\hookleftarrow}\ 2s-t+1\ \ 2s-2t+2\ \ 2s-t\ \ \dots\lceil(4s-3t+2)/2\rceil}_{(t+1)\text{ elements}}$$

Example 10.1: For $(n,k)=(21,5)$ we have $(s,t)=(10,2)$ and we obtain the following pentazetal terrace for $\mathbb{Z}_{21}$:

$$\underbrace{0\ \ 20\ \ 1}\ \ \overbrace{11\ \ 2\ \ 10\ \ 3\ \ 9\ \ 4\ \ 8}^{\text{odd no. of terms}}\ \ \underbrace{5\ \ 15\ \ 6\ \ 14\ \ 7\ \ 13}\ \ \overbrace{16\ \ 12}\ \ \underbrace{17\ \ 19\ \ 18}\ .$$

Example 10.2: For $(n, k) = (19, 5)$ we have $(s, t) = (9, 2)$ and we obtain the following pentazetal terrace for $\mathbb{Z}_{19}$:

$$\underbrace{0\ \ 18\ \ 1}\ \ \overbrace{10\ \ 2\ \ 9\ \ 3\ \ 8\ \ 4}^{\text{even no. of terms}}\ \ \underbrace{7\ \ 12\ \ 6\ \ 13\ \ 5\ \ 14}\ \ \overbrace{11}\ \ \underbrace{15\ \ 17\ \ 16}\ .$$

11 A family of hybrid zigzag/foxtrot terraces

For $n \equiv 0 \pmod{16}$, a family of $\mathbb{Z}_n$ terraces exists where, say, the first half of each terrace has two zigzag segments, and the second has $n/8$ foxtrot segments. However, the steps in the foxtrot half of the terrace are now of double length, so the pattern is now

$$\underbrace{x \quad y \quad x+2 \quad y+2} \quad \underbrace{x+4 \quad y-4 \quad x+6 \quad y-2} \quad \ldots\ .$$

Correspondingly, the zigzags are of the form

$$x \quad y \quad x+2 \quad y-2 \quad x+4 \quad y-4 \quad \ldots\ .$$

With $n = 4t$, each of the *hybrid zigzag/foxtrot terraces* for $\mathbb{Z}_n$ is of the form

$$\underbrace{0\ \ 2t-2\ \ 2\ \ 2t-4\ \ \ldots\ \ t-2\ \ t}\ \ \underbrace{1\ \ 4t-1\ \ 3\ \ 4t-3\ \ \ldots\ \ t-1\ \ 3t+1}$$
$$\underbrace{t+1\ \ 4t-4\ \ t+3\ \ 4t-2}\ \ \underbrace{t+5\ \ 4t-8\ \ t+7\ \ 4t-6}\ \ \ldots\ \ \underbrace{3t-3\ \ 2t\ \ 3t-1\ \ 2t+2}\ .$$

Example 11.1: The hybrid zigzag/foxtrot terrace for $\mathbb{Z}_{16}$ is

$$\underbrace{0 \quad 6 \quad 2 \quad 4} \quad \underbrace{1 \quad 15 \quad 3 \quad 13} \quad \underbrace{5 \quad 12 \quad 7 \quad 14} \quad \underbrace{9 \quad 8 \quad 11 \quad 10}\ .$$

12 More zigzags

Consider the Lucas-Walecki-Williams terrace

$$0 \quad 1 \quad 10 \quad 2 \quad 9 \quad 3 \quad 8 \quad 4 \quad 7 \quad 5 \quad 6$$

for $\mathbb{Z}_{11}$. If we split this after either of the entries 9 and 3, and interchange the two sequences that lie before and after the split, we still have a terrace for $\mathbb{Z}_{11}$, but now it has two distinct zigzag segments. This approach can be used generally to generate $\mathbb{Z}_n$ terraces with more zigzags, and hardly merits further attention, save to point out that, if n is even, and the difference between the two end elements of the original terrace is $n/2$, then only one splitting position will be found.

But consider splitting the above $\mathbb{Z}_{11}$ terrace after the element 8. If the whole sequence before the split is now reversed and re-attached to the front of the sequence that lies after the split, we again have a $\mathbb{Z}_{11}$ terrace with two distinct zigzags instead of one. This approach can be applied to any Lucas-Walecki-Williams $\mathbb{Z}_n$ terrace with $n \equiv 0$ or 2 (modulo 3).

In general, a further $\mathbb{Z}_n$ terrace may *perhaps* be obtainable from a given one by reversing the sequence *before* a split or by reversing the sequence *after* a split. Either way, there *may* be a choice of splitting position. For example, consider the $\mathbb{Z}_{17}$ fractured Ringel-Owens terrace given at the start of §4 above:

$$0 \quad 3 \quad 7 \quad 2 \quad 8 \quad 1^{\downarrow}\, 9 \quad 16 \quad 10 \quad 15 \quad 11^{\downarrow}\, 14 \quad 12 \quad 13 \quad 5_{\uparrow}\, 4 \quad 6\,,$$

Here, a further $\mathbb{Z}_{17}$ terrace may be obtained by splitting at either of the $^{\downarrow}$ positions and reversing the *before* sequence, or by splitting at the $_{\uparrow}$ position and reversing the *after* sequence. Indeed, as both $^{\downarrow}$ positions precede the $_{\uparrow}$ position, we can split both at a $^{\downarrow}$ position and at the $_{\uparrow}$ position, reversing both the sequence before the former position and the sequence after the latter.

The possibilities of the splitting and reversing approach are so great that we cannot explore them fully. Nevertheless, one special case perhaps now merits attention.

Suppose that we split and reverse at the $^{\downarrow}$ position in the left-hand side of the $\mathbb{Z}_{45}$ tetrazetal terrace from §8 above:

$$26 \quad 27 \quad 25 \quad 28 \quad 24 \quad 29 \quad 23 \quad 30 \quad 7 \quad 31^{\downarrow}\, 6 \quad 32 \quad 5 \quad \ldots \quad 37 \quad 0\,;$$

we then obtain

$$31 \quad 7 \quad 30 \quad 23 \quad 29 \quad 24 \quad 28 \quad 25 \quad 27 \quad 26 \quad 6 \quad 32 \quad 5 \quad \ldots \quad 37 \quad 0\,.$$

Accordingly, if we also make the corresponding reversal in the right-hand side, we obtain a $\mathbb{Z}_{45}$ terrace with six zigzag segments, of which the first and second overlap in a single element, as do the third and fourth, and also the fifth and sixth. Similar conversion of tetrazetal $\mathbb{Z}_n$ terraces to 6-zigzags terraces is possible for any n satisfying $n \equiv 9$, 15, 21 or 28 (mod 36). For $n \equiv 9$ and 15 (mod 36), the first and sixth zigzags are the shortest; for $n \equiv 21$ and 27, the shortest are the third and fourth.

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