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On independent domination and size

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Abstract

For a simple graph G, the *independent domination number* $i(G)$ is defined to be the minimum cardinality among all maximal independent sets of vertices of G. We establish upper bounds for the independent domination number of $K_{1,k+1}$-free graphs, as functions of the order, size and k. Also we present a lower bound for the size of connected graphs with given order and value of independent domination number. All results are best possible, and we cite classes of extremal graphs.

1 Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$, size $|E| = m$, maximum degree Δ and minimum degree δ. A subset I of V is a *dominating set* if every vertex of $V - I$ has at least one neighbour in I, and the *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. An *independent set* is a set of pairwise non-adjacent vertices of G. The *independent domination number* $i(G)$ is defined to be the minimum cardinality among all maximal independent sets of G. An independent set is maximal if and only if it is dominating, so $i(G)$ is also the minimum cardinality of an independent dominating set in G.

The main aim of this paper is to present two separate extremal theorems relating the independent domination number and size of a graph. The graph-theoretical invariant $i(G)$ has been studied extensively in the literature, with numerous papers focussed upon finding upper bounds, as functions of n and other parameters. For example, in [1] Bollobás and Cockayne proved that all connected graphs satisfy $i(G) \leq n - \gamma(G) + 1 - \lceil (n - \gamma(G))/\gamma(G) \rceil$. Favaron [3] observed that maximising the right-hand side of this inequality (by setting $\gamma(G) = \sqrt{n}$) gives the bound $i(G) \leq n - 2\sqrt{n} + 2$,

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with the graph formed by joining $\sqrt{n}-1$ pendant vertices to each vertex of a $K_{\sqrt{n}}$ being uniquely extremal.

Recently, analogous upper bounds have been derived for the independent domination number of graphs with certain forbidden induced subgraphs, primarily as functions of n and δ; such results include those of the present author and Wang for triangle-free graphs [4, 10] and graphs of higher girth [5, 6]. A graph G is said to be $K_{1,k+1}$-*free* if it does not contain an induced subgraph isomorphic to the star $K_{1,k+1}$. In the spirit of these investigations, in Section 2 we provide sharp upper bounds for the independent domination number of $K_{1,k+1}$-free graphs, as functions of the order, size and k. In particular, in Theorem 3 we extend work of Ryjáček and Schiermeyer [9] on the *independence number* $\beta(G)$ (the maximum cardinality of an independent set of G).

Dankelmann et al. [2] determined the maximum size of a graph of given order and value of independent domination number. Section 3 is a natural counterpart to their research, as we establish a best possible lower bound for the size of connected graphs G of order n, for any given value of $i(G)$. It transpires that the relationship between $i(G)$ and $\chi(G)$, the chromatic number of G, is central to the proof of our principal result (Theorem 5).

In what follows, the *open neighbourhood* in G of a vertex $v \in V$ will be denoted by $\Gamma(v) = \{u \in V : uv \in E\}$. For disjoint vertex sets $X, Y \subset V$, write $e(X)$ for the number of edges with both endvertices in X, and $e(X, Y)$ for the number of edges with one endvertex in each of X and Y. We abbreviate $i(G)$ to i, $\beta(G)$ to β, and assume that G is an extremal graph containing a minimum maximal independent set I.

2 The independent domination number of $K_{1,k+1}$-free graphs

In order for our analysis of this problem to make sense, it is implicit in our arguments that G is $K_{1,k+1}$-free but not $K_{1,k}$-free. Therefore, as every vertex of the k-set of a $K_{1,k}$ has degree at least δ in G, we must have $k \leq n - \delta$.

First we dispose of some trivial cases: if $k = 0$ then $G \cong nK_1$ and $i = \beta = n$, whilst if $k = 1$ then $G \cong K_n$ and $i = \beta = 1$. However, in general these two parameters do not have equal values. In addition, the case $k > n/2$ can be solved by using the elementary result that $i \leq n - \Delta$. Since $\Delta \geq k$ then $i \leq n - k$, with the upper bound attained by $K_{k,n-k}$ (for which $k = n - \delta$) and numerous other graphs.

In [9], Ryjáček and Schiermeyer established the following upper bound for the independence number of $K_{1,k+1}$-free graphs, as a function of the order, size and k.

Proposition 1 (Ryjáček and Schiermeyer [9]). *Let G be a $K_{1,k+1}$-free graph of order n and size m; then*

$$\beta \leq \frac{1}{2}\left(2n + 2k - 1 - \sqrt{8m + (2k-1)^2}\right),$$

and this bound is sharp.

For details of the extremal graphs, the reader is referred to [9]. Prior to proving an analogous result for the independent domination number, we require a preliminary lemma. It applies for graphs in general, and yields important structural information about any extremal graph.

Lemma 2 *Each vertex $x \in V - I$ appears in an independent set of order at least $|\Gamma(x) \cap I|$ in $V - I$.*

Proof. For any $x \in V - I$, form the sets $W = \Gamma(x) \cap I$ and $X = \{v \in V - I : \Gamma(v) \cap I \subseteq W\}$. Let R be a maximal independent set of $G[X]$ containing x. Then $R \cup (I - W)$ is maximal independent for G, so $|R| + (i - |W|) \geq i$, implying $|R| \geq |W|$. Thus R is the required independent set of $V - I$. □

Theorem 3 *Let G be a $K_{1,k+1}$-free graph of order n and size m; then*

$$i \leq \frac{1}{2}\left(2n + k - \sqrt{8m + k^2}\right),$$

and this bound is sharp for $2 \leq k \leq n/2$.

Proof. Every vertex of $V - I$ has degree at most $\Delta \leq n - i$ in G. Furthermore, since G is $K_{1,k+1}$-free, every vertex of $V - I$ has at most k neighbours in I, so $e(V - I, I) \leq (n - i)k$. We conclude that

$$\begin{aligned} m &= e(V - I) + e(V - I, I) \\ &\leq \frac{1}{2}\left[(n - i)^2 - e(V - I, I)\right] + e(V - I, I) \\ &\leq \frac{1}{2}(n - i)(n - i + k), \end{aligned}$$

from which we obtain

$$i^2 - (2n + k)i + n^2 + nk - 2m \geq 0.$$

This quadratic inequality has solution

$$i \leq \frac{1}{2}\left(2n + k - \sqrt{8m + k^2}\right).$$

An examination of our proof shows that in any extremal graph each vertex of $V - I$ has k neighbours in I. Consequently, by Lemma 2, each vertex of $V - I$ appears in an independent set of $V - I$ of order at least k, so we must have $n \geq 2k$.

Define the integers $q \geq 2$ and r by $n = qk + r$, $0 \leq r < k$. We construct a class of extremal graphs G as follows. Take an independent set I of $k + r$ vertices and a complete $(q - 1)$-partite graph with vertex classes all of order k. Then join every vertex of the $(q - 1)$-partite graph to k vertices of I so that any two vertices in the same partite set have the same neighbourhood in I. (A particular graph of this type is the union of r isolated vertices and a complete q-partite graph with vertex classes all of order k.) Then G has order n, independent domination number $i^* = k + r$ and size $m = \frac{1}{2}qk(n-r-k) = \frac{1}{2}(n-i^*+k)(n-i^*)$. Substituting for m in the upper bound above, we have

$$\begin{aligned} i &\le \frac{1}{2}\left(2n+k-\sqrt{4(n-i^*+k)(n-i^*)+k^2}\right) \\ &= \frac{1}{2}\left(2n+k-\sqrt{(2n-2i^*+k)^2}\right) = i^*. \end{aligned}$$

Thus Theorem 3 is sharp for $2 \le k \le n/2$. □

It remains an interesting problem to determine similar upper bounds on i for graphs G with other forbidden induced subgraphs.

3 The minimum size of connected graphs with given independent domination number

In view of our discussion at the outset, we know that if G is a connected graph of order n then the value of i is a positive integer bounded above by $n - 2\sqrt{n} + 2$.

First we dispose of the case $i \le n/2$. Clearly this inequality is satisfied by all bipartite (2-chromatic) graphs and thus in particular by all trees, so we deduce that $m \ge n-1$. The *doublestar* $S_{n,j}$ is formed by taking a K_2 and then attaching $j \le (n-2)/2$ pendant vertices to one vertex of the K_2 and $n-2-j$ to the other vertex. It is easily seen that for any given value of $i \le n/2$, the doublestar $S_{n,i-1}$ has independent domination number i. It would be an ambitious task to attempt a full classification of the extremal trees. However, all trees with $i = n/2$ were characterised by Ma and Chen in [7]; the reader is referred to that paper for details.

Henceforth we may assume $n/2 < i \le n-2\sqrt{n}+2$ and that G has chromatic number at least 3. If we replace $\gamma(G)$ with the chromatic number $\chi(G)$ in Bollobás and Cockayne's inequality from the Introduction, then we obtain another valid (and sharp) upper bound for the independent domination number, as established by MacGillivray and Seyffarth in [8]. For the purpose of this study we state their bound in a slightly different but equivalent form to that published originally.

Proposition 4 (MacGillivray and Seyffarth [8]). *If G is a connected graph of order n and chromatic number $\chi(G) = k$, then*

$$i(G) \le n-k+1-\left\lceil \frac{n-k}{k} \right\rceil.$$

For all integers $n \ge k \ge 2$, equality is achieved by the graph $H_{n,k}$, constructed as follows. Define the integers q and r by $n = qk + r$, $0 \le r < k$. Take a K_k and then attach $n-k$ pendant vertices so that r vertices of the K_k have q pendant vertices and the remaining $k-r$ vertices of the K_k have $q-1$ pendant vertices. Since an independent dominating set of $H_{n,k}$ contains at most one vertex of the K_k, then $i\,(H_{n,k}) = 1 + (k-1)(q-1) + (\max\{r,1\} - 1) = qk + \max\{r,1\} - k + 1 - q = n - k + 1 - \lceil (n-k)/k \rceil$. As expected, the bound is maximised when $k = \sqrt{n}$, so if n is a perfect square then $H_{n,\sqrt{n}}$ is precisely the extremal graph of Favaron cited at the beginning of the paper.

We are now in a position to prove our main result.

Theorem 5 *Let G be a connected graph of order n and independent domination number i. Let k be the largest integer in the range $3 \le k \le \sqrt{n}$ for which*

$$i > n - k + 2 - \left\lceil \frac{n-k+1}{k-1} \right\rceil ;$$

then G has chromatic number $\chi(G) \ge k$ and size

$$m \ge n - k + \binom{k}{2}.$$

Proof. Writing $\ell = k-1$, we have that ℓ is the largest integer in the range $2 \le \ell \le \sqrt{n} - 1$ for which $i > n - \ell + 1 - \lceil (n-\ell)/\ell \rceil$. Therefore, by Proposition 4, G has chromatic number at least $\ell + 1 = k \ge 3$. Then G necessarily contains a critical k-chromatic subgraph H, of order at least k, with the minimum degree therein being at least $k-1$. Now each component of $G - H$ has a spanning tree, and at least one edge connects that tree to H. Thus the size m of G is at least $|H|(k-1)/2 + n - |H|$. As a function of $|H|$, this expression has derivative $(k-3)/2 \ge 0$, and hence is minimised when $|H| = k$, which proves the theorem. □

For any given value of $i > n/2$, let k be the largest integer in the range $3 \le k \le \sqrt{n}$ for which $i > n - k + 2 - \lceil (n-k+1)/(k-1) \rceil$. Define the integer $s = n - k + 1 - \lceil (n-k)/k \rceil - i \ge 0$. We now construct classes of k-chromatic extremal graphs $F_{n,k,s}$ as follows. Take a K_k and then attach $n-k$ pendant vertices so that one vertex of the K_k has $\lceil (n-k)/k \rceil + s$ pendant vertices and each of the remaining $k-1$ vertices of the K_k has at most $\lceil (n-k)/k \rceil + s$ pendant vertices. Since an independent dominating set of such a graph contains at most one vertex of the K_k, then each member of $F_{n,k,s}$ has independent domination number $1 + [(n-k) - (\lceil (n-k)/k \rceil + s)] = n - k + 1 - \lceil (n-k)/k \rceil - s = i$, as required. Note that $H_{n,k} \in F_{n,k,0}$.

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