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About a Brooks-type theorem for improper colouring*

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Abstract

A graph is k-improperly ℓ-colourable if its vertices can be partitioned into ℓ parts such that each part induces a subgraph of maximum degree at most k. A result of Lovász states that for any graph G, such a partition exists if $\ell \geq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$. When $k = 0$, this bound can be reduced by

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Brooks' Theorem, unless G is complete or an odd cycle. We study the following question, which can be seen as a generalisation of the celebrated Brooks' Theorem to improper colouring: does there exist a polynomial-time algorithm that decides whether a graph G of maximum degree Δ has k-improper chromatic number at most $\lceil\frac{\Delta+1}{k+1}\rceil - 1$? We show that the answer is no, unless $\mathcal{P} = \mathcal{NP}$, when $\Delta = \ell(k+1)$, $k \geq 1$, and $\ell + \sqrt{\ell} \leq 2k+3$. We also show that, if G is planar, $k = 1$ or $k = 2$, $\Delta = 2k+2$, and $\ell = 2$, then the answer is still no, unless $\mathcal{P} = \mathcal{NP}$. These results answer some questions of Cowen, Goddard and Jesurum [*J. Graph Theory* 24(3) (1997), 205–219].

Introduction

An *ℓ-colouring* of a graph $G = (V, E)$ is a mapping $c : V \to \{1, 2, \ldots, \ell\}$. For any vertex $v \in V$, the *impropriety of v under c* is

$$\mathrm{im}_c(v) := |\{u \in V : uv \in E \text{ and } c(u) = c(v)\}| .$$

A colouring is *k-improper* provided that the impropriety of every vertex is at most k. A 0-improper colouring is *proper*. A graph is *k-improperly ℓ-colourable* if it admits a k-improper ℓ-colouring. The *k-improper chromatic number* is

$$c_k(G) := \min\{\ell : \text{ G is } k\text{-improperly } \ell\text{-colourable}\} .$$

In particular, $c_o(G)$ is the chromatic number $\chi(G)$ of the graph G. Since the early nineties, a lot of work has been devoted to various aspects of improper colourings, both from a purely theoretical point of view [7, 8, 14, 20, 21] and in relation with frequency assignment issues [12, 13, 16]. Let us note that improper colourings are also called in the literature *defective colourings*.

For all integers k and ℓ, let k-IMP ℓ-COL be the following problem:
INSTANCE: a graph G.
QUESTION: is G k-improperly ℓ-colourable?
Cowen *et al.* [8] showed that the problem k-IMP ℓ-COL is $\mathcal{NP}$-complete for all pairs (k, ℓ) of integers with $k \geq 1$ and $\ell \geq 2$. When $\ell \geq 3$, this is not very surprising since it is already hard to determine whether a given graph is properly 3-colourable. On the contrary, determining if a graph is 2-colourable, i.e. bipartite, can be done in polynomial-time, whereas it is $\mathcal{NP}$-complete to know if it is k-improper 2-colourable as soon as $k > 0$.

Of even more interest is the question of complexity of k-IMP ℓ-COL when restricted to graphs with maximum degree $(k+1)\ell$. Indeed, Lovász [17] proved that, for any graph G, it holds that $c_k(G) \leq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$, where $\Delta(G)$ is the maximum degree of G. When $k = 0$, this is the usual bound $\chi(G) \leq \Delta(G) + 1$. Brooks' Theorem [6] states that this upper bound can be decreased by one, provided that G is neither complete nor an odd cycle, which can be checked in polynomial-time.

Extensions of Brooks' Theorem have also been considered. A well-known conjecture of Borodin and Kostochka [5] states that every graph of maximum degree

$\Delta \geq 9$ and chromatic number at least Δ has a Δ-clique. Reed [19] proved that this is true when Δ is sufficiently large, thus settling a conjecture of Beutelspacher and Herring [4]. Further information about this problem can be found in the monograph of Jensen and Toft [15, Problem 4.8]. Generalisation of this problem has also been studied by Farzad, Molloy, and Reed [11] and Molloy and Reed [18]. In particular, it is proved [18] that determining whether a graph with large constant maximum degree Δ is $(\Delta - k)$-colourable can be done in linear time if $(k+1)(k+2) \leq \Delta$. This threshold is optimal by a result of Emden-Weinert, Hougardy, and Kreuter [10], since they proved that for any two constants Δ and $k \leq \Delta - 3$ such that $(k+1)(k+2) > \Delta$, determining whether a graph of maximum degree Δ is $(\Delta - k)$-colourable is $\mathcal{NP}$-complete.

It is natural to ask whether analogous results can be found for improper colouring. This first problem to grapple with is the existence, or not, of a Brooks-like theorem for improper colouring: does there exist a polynomial-time algorithm that decides whether a graph G of maximum degree Δ has k-improper chromatic number at most $\lceil \frac{\Delta+1}{k+1} \rceil - 1$? Proving that k-IMP ℓ-COL is $\mathcal{NP}$-complete when restricted to graphs with maximum degree $(k+1)\ell$ would provide a negative answer to this question unless $\mathcal{P} = \mathcal{NP}$. Cowen *et al.* [8] proved that k-IMP 2-COL is $\mathcal{NP}$-complete for the class of graphs with maximum degree $2(k+1)$, and asked what happens when $\ell \geq 3$. In this paper, we prove that k-IMP ℓ-COL restricted to graphs with maximum degree $(k+1)\ell$ is $\mathcal{NP}$-complete for all integers $k \geq 1$ and $\ell \in \{3, \dots, s\}$, where s is the biggest integer such that $s + \sqrt{s} \leq 2k + 3$ (Theorem 2). An intriguing question that remains unanswered is the complexity of this problem for larger values of ℓ.

Problem 1. What is the complexity of k-IMP ℓ-COL restricted to graphs with maximum degree $(k+1)\ell$ when $\ell + \sqrt{\ell} > 2k + 3$?

We conjecture that is always $\mathcal{NP}$-complete. As an evidence, we prove the $\mathcal{NP}$-completeness when $k = 1$ and $\ell = 4$ (Theorem 6).

In view of these negative results, one may ask what happens for planar graphs. It is known that every planar graph is 4-colourable [1, 3, 2], 2-improperly 3-colourable [9, 21], and Cowen *et al.* [8] proved that is is $\mathcal{NP}$-complete to know whether a planar graph is 1-improperly 3-colourable, but without any restriction on the maximum degree.

Cowen *et al.* [8] also proved that k-IMP 2-COL is $\mathcal{NP}$-complete for planar graphs, again without any restriction on the degree. In particular, they asked if 1-IMP 2-COL is still $\mathcal{NP}$-complete for planar graphs with maximum degree 4 — they could prove it only for maximum degree 5. In more general terms, we consider the following problem.

Problem 2. What is the complexity of k-IMP 2-COL restricted to planar graphs of maximum degree $2k + 2$?

We show in Section 2 that it is NP-complete when $k \in \{1, 2\}$. Note that for $k = 1$, it settles Cowen *et al.* [8] question. However, we conjecture that if k is sufficiently large then k-IMP 2-COL can be polynomially decided, as the answer is always affirmative.

Conjecture 1. *There exists an integer $k_0 \geq 3$ such that for any $k \geq k_0$, any planar graph with maximum degree at most $2k+2$ is k-improperly 2-colourable.*

We end the introduction with two definitions. Given an undirected graph G, an *orientation* of G is any directed graph obtained from G by assigning a unique direction to each edge. Then, a vertex u is *dominated* by a vertex v if there is an edge between v and u directed from v to u.

1 Complexity of k-IMP ℓ-COL for graphs with maximum degree $(k+1)\ell$

In this section, we study the complexity of k-IMP ℓ-COL restricted to graphs with maximum degree $(k+1)\ell$. The main result is the following theorem establishing the NP-completeness of the problem when $k \geq 1$ and $\ell + \sqrt{\ell} \leq 2k+3$.

Theorem 2. *Fix a positive integer k, and an integer $\ell \geq 3$ such that $\ell + \sqrt{\ell} \leq 2k+3$. The following problem is $\mathcal{N}P$-complete:*
INSTANCE: *a graph G with maximum degree at most $(k+1)\ell$.*
QUESTION: *is G k-improperly ℓ-colourable?*

Remark 3. The maximum value of ℓ is approximately $2k+4-\sqrt{2k+2}$.

To prove Theorem 2, we need some preliminaries. Let k and ℓ be two positive integers, and let $H(k,\ell)$ be the graph with vertex set $X \cup Y \cup \{z\}$ where $|X| = (k+1)(\ell-1)$ and $|Y| = (k+1)$ such that xy is an edge unless $x = z$ and $y \in Y$ (see Figure 1).

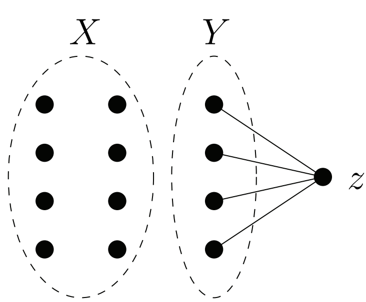

Figure 1: The complement of the graph $H(3,3)$.

Proposition 4. *The graph $H(k,\ell)$ is k-improperly ℓ-colourable, and in any k-improper ℓ-colouring of $H(k,\ell)$ the vertices of $Y \cup \{z\}$ are coloured the same.*

Proof. Since $H(k,\ell)$ has $(k+1)\ell+1$ vertices, at least one colour class must contain at least $k+2$ vertices. Observe that a vertex of X must be in a colour class containing at most $k+1$ vertices since it is connected to every other vertex. Hence, the colour class with $k+2$ vertices is $Y \cup \{z\}$. □

Lemma 5. *Let G be a graph with maximum degree at most $2k+2$. Then G has an orientation D such that every vertex has indegree and outdegree at most $k+1$.*

Proof. Since every graph with maximum degree at most $2k+2$ is a subgraph of a $(2k+2)$-regular graph, it suffices to prove the assertion for $(2k+2)$-regular graphs. Let G' be such a graph, then it admits an Eulerian cycle C. Let D be the orientation of G' such that (u,v) is an arc if and only if u precedes v in C. Then D has indegree and outdegree at most $k+1$. □

Proof of Theorem 2. Reduction to the following problem:
INSTANCE: a graph G with maximum degree at most $2k+2 \geq \ell+1 \geq 4$.
QUESTION: is G ℓ-colourable?
Thanks to the choice of ℓ, this problem is $\mathcal{NP}$-complete by the result of Emden-Weinert *et al.* [10] cited in the introduction.

Let $G=(V,E)$ be a graph of maximum degree at most $2k+2$, and let D be an orientation of G with in- and outdegree at most $k+1$. Such an orientation exists by Lemma 5.

Let G' be the graph constructed as follows: replace each vertex v of G by a copy $H(v)$ of $H(k,\ell)$; if v dominates u in D then connect $z(v)$ to an element of $Y(u)$, in such a way that every vertex of $Y(u)$ is connected to a single vertex not in $H(u)$. The maximum degree of G' is $(k+1)\ell$.

Let us now prove that G is ℓ-colourable if and only if G' is k-improperly ℓ-colourable. If G admits an ℓ-colouring c, then for any vertex v we assign the colour $c(v)$ to the vertices of $Y(v)\cup\{z(v)\}$ and the $\ell-1$ other colours to $\ell-1$ disjoint sets of $k+1$ vertices of $X(v)$. This yields a k-improper ℓ-colouring of G'.

Conversely, suppose that G' admits a k-improper ℓ-colouring c'. Let c be defined by $c(v) := c'(z(v))$. We prove now that c is a proper ℓ-colouring of G: let u and v be two neighbours. Without loss of generality, v is the predecessor of u in D. Thus, the vertex $z(v)$ is connected to an element $y(u)$ of $Y(u)$. Note that $c'(z(v)) \neq c'(z(u))$, otherwise by Proposition 4, all the vertices of $Y(u)\cup Y(v)\cup\{z(u),z(v)\}$ are coloured the same. Then $y(u)$ would have degree $k+1$ in this set which is impossible. This concludes the proof. □

We now extend this result to the case when $k=1$ and $\ell=4$.

Theorem 6. *The following problem is $\mathcal{NP}$-complete:*
INSTANCE: *a graph G with maximum degree at most* 8.
QUESTION: *is G 1-improperly 4-colourable?*

Let B be the graph with vertex set $\{a_1,a_2,a_3,b_1,b_2,b_3\}$, and xy is an edge except if there exists $i\in\{1,2,3\}$ such that $\{x,y\}=\{a_i,b_i\}$.

Let A be the graph with vertex set $\{x_1,x_2,y_1,y_2\}$ and with the two edges x_1x_2 and y_1y_2. For $i\in\{2,3,4\}$, let J_i be the union of a copy A_i of A and a copy B_i of B, to which we add every edge xy such that $(x,y)\in A_i\times B_i$. Let A' be the graph obtained from A by removing the edge y_1y_2. We define J_1 to be the union of a copy A_1 of A' and a copy B_1 of B. We let J_1' be a copy of J_1 (with A_1' and B_1' defined analogously).

Let $H := J_1' \cup \bigcup_{i=1}^4 J_i$, to which we add the following edges (see Figure 2):

$$\begin{array}{lll} y_1^{J_1}y_1^{J_2}, & y_1^{J_1}y_1^{J_3}, & y_2^{J_1}y_1^{J_4} \\ y_1^{J_1'}y_2^{J_2}, & y_1^{J_1'}y_2^{J_3}, & y_2^{J_1'}y_2^{J_4} \\ x_1^{J_2}x_2^{J_4}, & x_2^{J_2}x_1^{J_3}, & x_2^{J_3}x_1^{J_4} \end{array}$$

Proposition 7. *The graph H is 1-improperly 4-colourable, and for any 1-improper 4-colouring of H, the sets A_i, for $i \in \{1,2,3,4\}$, and $A_1 \cup A_1'$ are monochromatic.*

Proof. Consider a 1-improper 4-colouring of H. For every $i \in \{1,2,3,4\}$, the colour of each vertex not belonging to A_i is assigned at most twice. Therefore, all the vertices of A_i must be coloured the same. The same holds also for A_1'. Moreover, for every $j \in \{2,3,4\}$, the colour of the vertices of A_1' and of A_j must be different from the colour of the vertices of A_i for every $i \in \{1,2,3,4\} \setminus \{j\}$. Hence, A_1 and A_1' are coloured the same. □

Proof of Theorem 6. Reduction to the following problem, which is $\mathcal{NP}$-complete [10]:
INSTANCE: a graph G with degree at most 6.
QUESTION: is G 4-colourable?

Let $G = (V, E)$ be a graph of maximum degree 6. By Lemma 5, let D be an orientation of G with in- and outdegree at most $k + 1$.

Let G' be the graph obtained by replacing each vertex v of G by a copy $H(v)$ of H; we set $X(v) := \{x_1^{J_1}(v), x_2^{J_2}(v), x_1^{J_1'}(v)\}$ and $Y(v) := \{y_2^{J_1}(v), y_2^{J_1'}(v), x_2^{J_1'}(v)\}$; if v dominates u in D, then we connect an element of $Y(v)$ to an element of $X(u)$ in such a way that every vertex of $X(v) \cup Y(v)$ is connected to a single vertex not in $H(v)$, see Figure 2.

The maximum degree of G' is 8. Moreover G' is 1-improper 4-colourable if and only if G is 4-colourable. Indeed, consider any proper 4-colouring of G. For every vertex $v \in V$, assign to every vertex of $A_1(v) \cup A_1'(v)$ the colour of v. This partial colouring of G' can be extended to a 1-improper 4-colouring of G'. Conversely, if C' is a 1-improper 4-colouring of G', then by Proposition 7, for each $v \in V$ the vertices of $A_1(v)$ are monochromatic. For all $v \in V$, we define $C(v)$ to be the colour assigned to every vertex of $A_1(v)$. Then C is a proper 4-colouring of G: consider an edge uv of G, and say it is oriented from u to v in D. By Proposition 7, the vertex of $X(u)$ to which the corresponding edge of G' is linked has impropriety 1 in $H(u)$. Thus, the vertex of $Y(v)$ to which it is linked is coloured differently. Consequently, by the definition of C, we deduce that $C(u) \neq C(v)$, as desired. □

2 Planar graphs

In this section we show that 1-IMP 2-COL and 2-IMP 2-COL are $\mathcal{NP}$-complete when restricted to planar graphs of bounded maximum degree.

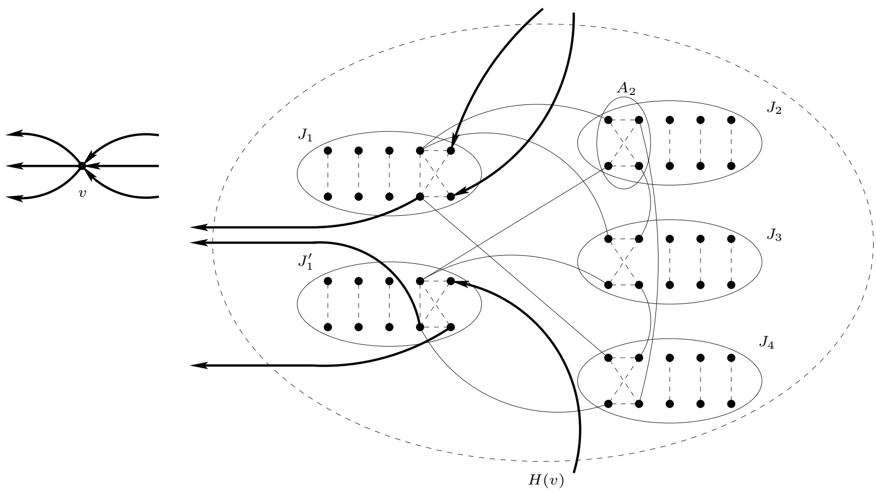

Figure 2: Replacing a vertex v of G by a copy $H(v)$ of H. For each subgraph J_i or J_1', dotted lines indicate missing edges.

Theorem 8. *The following problem is $\mathcal{N}P$-complete:*
INSTANCE: *a planar graph G with maximum degree* 4*.*
QUESTION: *is there a* 1*-improper* 2*-colouring of G ?*

Proof. This result is proved by slightly modifying the proof of Cowen *et al.* [8]. The only change is in the crossing gadget. The crossing gadget of Cowen *et al.* [8] has maximum degree 5, and they asked whether one with maximum degree 4 exists. We shall exhibit such a crossing gadget. We make the whole proof here for completeness.

The reduction is from 3-SAT. Let Φ be a 3-CNF. We shall construct, in polynomial time, a planar graph G_Φ of maximum degree 4 such that Φ is satisfiable if and only if G_Φ is 1-improperly 2-colourable. We use several gadgets with useful properties.

An *xy-regulator* is depicted in Figure 3. There is a unique 1-improper 2-colouring of this graph, in which x, u_1, u_2, and y form a colour-class and all have impropriety zero. Note also that both x and y have degree 1 within an xy-regulator. An *xy-inversor*, depicted in Figure 4, is a $K_{2,3}$, with x and y being any two vertices not in the same part of the bipartition. There is a unique 1-improper 2-colouring of this graph, in which x and y receive different colours, and both have impropriety zero. Note that one out of x and y has degree 2 in the inversor while the other has degree 3. The variable-gadget that represents the literals of a particular variable is constructed from the two aforementioned graphs as shown in Figure 5. We put one variable-gadget for each variable, with as many literals as needed. Each of the literals will be linked to exactly one clause.

Let $C_1, C_2, \ldots, C_m$ be the clauses of Φ. For each clause C_i, we put a copy G_i' of the graph G'. It is left to the reader to check out that the graph G', depicted in Figure 6, has the following property: in any 1-improper 2-colouring, z is coloured 1 if and only

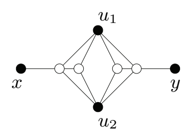

Figure 3: A regulator and its unique 1-improper 2-colouring.

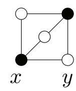

Figure 4: An inversor and its unique 1-improper 2-colouring.

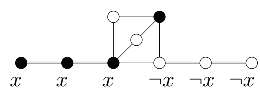

Figure 5: The vertex gadget. A double edge stands for a regulator.

if at least one of p, q, r is. Then we add the edges $z_i z_{i+1}$, for $i \in \{1, 2, \ldots, m-1\}$.

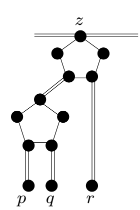

Figure 6: The clause gadget. A double edge stands for a regulator.

The obtained graph has maximum degree 4, and the vertex-gadgets and the clauses can be arranged so that the only edges that can cross are the ones joining vertex-gadgets to clauses. We uncross two crossing edges by using the crossing gadget CG depicted in Figure 7. The graph CG has maximum degree 4, and fulfils the following properties:

- **(i)** the graph CG is 1-improperly 2-colourable but not 1-improperly 1-colourable;
- **(ii)** in any 1-improper 2-colouring of CG, for $i \in \{1, 2\}$, the vertices a_i and b_i are coloured the same; and
- **(iii)** there exist two 1-improper 2-colourings c_1 and c_2 of CG such that $c_1(a_1) = c_1(b_1) = c_1(a_2) = c_1(b_2)$ and $c_2(a_1) = c_2(b_1) \neq c_2(a_2) = c_2(b_2)$.

Properties **(i)** and **(iii)** can be directly checked. For property **(ii)**, suppose that c is a 1-improper 2-colouring of CG. First, observe that, necessarily, three vertices among o_1, o_2, o_3, and o_4 are coloured the same, because the vertex o_0 is linked to all of them. Thus, the vertices o_2 and o_3 must be coloured differently. If $c(a_1) \neq c(b_1)$, then $c(o_4) \neq c(o_1)$ and by the preceding observation, the vertex o_0 has two neighbours of each colour, a contradiction. Now, suppose that $c(a_2) \neq c(b_2)$ and so $c(o_3) \neq c(o_7)$. By the construction, $c(o_7) = c(o_6) \neq c(o_5)$, because o_6 and o_7 are joined by a regulator and o_5 and o_6 are joined by an inversor. Hence $c(o_3) = c(o_5)$, and so $c(o_4) \neq c(o_3)$. Therefore, $c(o_4) = c(o_2) = c(o_1)$. This is not possible since $c(o_4) = c(o_7)$, so the vertex o_1 would have impropriety 2.

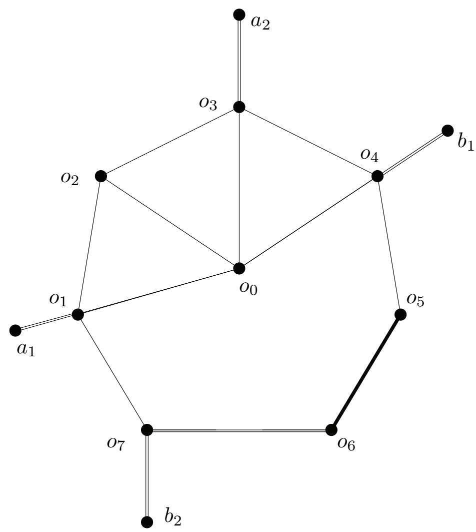

Figure 7: The crossing gadget CG. A double edge stands for a regulator, and the bold edge stands for an inversor.

It remains to show that Φ is satisfiable if and only if the obtained graph G_Φ, which is planar and of maximum degree 4, is 1-improperly 2-colourable. Consider a 1-improper 2-colouring of G_Φ. Without loss of generality, say that each clause — vertices z of the clause gadgets — is coloured 1. Then, at least one of the input vertices of each clause is coloured 1. Therefore, associating 1 with TRUE and 2 with FALSE yields a truth assignment for Φ. Conversely, starting from a truth assignment of Φ, one can derive a 1-improper 2-colouring of G_Φ as follows. Vertices corresponding to literals are coloured 1 if the corresponding literal is TRUE, and 2 otherwise. Thanks to the properties of the gadgets, such a partial colouring can be extended to a 1-improper 2-colouring of G_Φ. □

Theorem 9. *The following problem is $\mathcal{NP}$-complete:*
INSTANCE: *a planar graph G of maximum degree* 6.
QUESTION: *is there a* 2*-improper* 2*-colouring of G?*

Proof. We shall reduce the problem to 1-improper 2-colouring of planar graphs with maximum degree 4. Let G be a planar graph of maximum degree 4: we construct, in polynomial time, a planar graph $\hat{G}$ of maximum degree 6 that is 2-improperly 2-colourable if and only if G is 1-improperly 2-colourable. The graph H showed in Figure 8 fulfils the following properties:

(i) it is planar and has maximum degree 6;

(ii) in any 2-improper 2-colouring of H, the vertex v must have impropriety at least 1; and

(iii) there exists a 2-improper 2-colouring of H in which the vertex v has impropriety exactly 1.

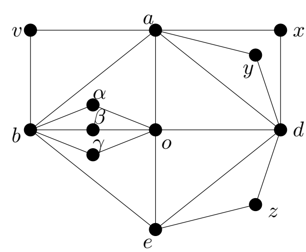

Figure 8: The graph H.

Property **(i)** can be directly checked. To prove **(ii)**, it is sufficient to show that no 2-improper 2-colouring of H such that a and b are coloured the same exists. So, suppose that $c(a) = c(b) = 1$.

If $c(o) = 1$, then d, y, and x must be coloured 2, so e and z both receive colour 1. Therefore e has impropriety 3, because of b, o, and z, a contradiction.

If $c(o) = 2$, then the three vertices α, β, γ cannot be coloured the same, otherwise b or o would have impropriety at least 3. Moreover, since $c(b) = c(a) = 1$, exactly two vertices among α, β, γ are coloured with colour 2. Hence, the vertices b and o both have impropriety 2 in the subgraph of G induced by the vertices $a, b, o, \alpha, \beta, \gamma$. But the vertex e does not belong to this subgraph, and is linked to both b and o, a contradiction. This proves **(ii)**.

Assigning 1 to $\{v, a, d, e, \alpha, \beta\}$ and 2 to $\{b, o, x, y, z, \gamma\}$, we obtain the colouring of **(iii)**.

To construct the graph $\hat{G}$, put a copy $H(x)$ of H for each vertex $x \in V(G)$. Then, for each edge $xy \in E(G)$, we put an edge between the vertex v of $H(x)$ and the vertex v of $H(y)$. Note that H has maximum degree 6 and v has degree 2 in H, so, as G has maximum degree 4, the graph $\hat{G}$ has maximum degree 6. Furthermore, the graph $\hat{G}$ is planar.

Now let c be a 1-improper 2-colouring of G. For any $x \in V(G)$, assign the colour $c(x)$ to the vertex v of $H(x)$, and then extend the colouring to each copy of H by property **(iii)**.

If c is a 2-improper 2-colouring of $\hat{G}$, then for each $x \in V(G)$, assign to x the colour of the vertex v of $H(x)$. The obtained 2-colouring of G is 1-improper because of property **(ii)** of H. □

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