AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **46** (2010), Pages 241–261

Super edge-graceful labelings of complete bipartite graphs

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Abstract

Let $[n]^*$ denote the set of integers $\{-\frac{n-1}{2}, \ldots, \frac{n-1}{2}\}$ if n is odd, and $\{-\frac{n}{2}, \ldots, \frac{n}{2}\} \setminus \{0\}$ if n is even. A super edge-graceful labeling f of a graph G of order p and size q is a bijection $f : E(G) \rightarrow [q]^*$, such that the induced vertex labeling f^* given by $f^*(u) = \sum_{uv\epsilon\ E(G)} f(uv)$ is a bijection $f^* : V(G) \rightarrow [p]^*$. A graph is super edge-graceful if it has a super edge-graceful labeling. We show by construction that all complete bipartite graphs are super edge-graceful except for $K_{2,2}$, $K_{2,3}$, and $K_{1,n}$ if n is odd.

* Research supported by NSF Grant DMS0648919, University of West Georgia

1 Introduction

In this paper we consider only simple, finite, undirected graphs. We define the set of integers $[n]^*$ to be $\{-\frac{n-1}{2}, \dots, \frac{n-1}{2}\}$ if n is odd, and $\{-\frac{n}{2}, \dots, \frac{n}{2}\} \setminus \{0\}$ if n is even. Notice that the cardinality of $[n]^*$ is n, and $[n]^*$ contains 0 if and only if n is odd. A graph of order p and size q is said to be *super edge-graceful* (SEG) if there is a bijection $f : E(G) \to [q]^*$, such that the induced vertex labeling f^* given by $f^*(u) = \sum_{uv\epsilon\ E(G)} f(uv)$ is a bijection $f^* : V(G) \to [p]^*$.

A graph of order p and size q is *edge-graceful* [2] if the edges can be labeled by $1, 2, \dots, q$ such that the vertex sums are distinct (mod p). A necessary condition for a graph with p vertices and q edges to be edge-graceful is that $q(q+1) \equiv \frac{p(p+1)}{2}$ (mod p).

Super edge-graceful labelings (SEGL) were first considered by Mitchem and Simoson [5] who showed that super edge-graceful trees are edge-graceful. In particular, Mitchem and Simoson noticed that if G is a super-edge graceful graph and $p|q$, if q is odd, or $p|q+1$, if q is even, then G is edge-graceful. Some families of graphs have been shown to be super-edge graceful by explicit labelings. It is known that, for example, paths of all orders except 2 and 4 and cycles of all orders except 4 and 6 are super edge-graceful [1], as are trees of odd order with three even vertices [4] and complete graphs of all orders except 1, 2 and 4 [3].

The main result of this paper is that all complete bipartite graphs are super edge-graceful except $K_{2,2}$, $K_{2,3}$, and $K_{1,n}$ if n is odd. First we introduce a notation we will use to display edge-labelings.

Notation. A super edge-graceful labeling for the complete bipartite graph $K_{m,n}$ may be written as an $m \times n$ array containing the edge labels $[q]^*$ with the property that the set of row sums and column sums of this array is the set $[p]^*$. The m row sums correspond to the induced vertex labels for the vertices in the partite set of size m, while the n column sums correspond to the induced vertex labels for the vertices in the partite set of size n. As an example, consider the following super edge-graceful labeling of $K_{2,4}$. Notice that the column sums are 2, 1, -2, and -1, and the row sums are 3 and -3.

$$\begin{bmatrix} 4 & 2 & 1 & -4 \\ -2 & -1 & -3 & 3 \end{bmatrix}$$

Figure 1: An SEGL for $K_{2,4}$

Throughout this paper, $\ell(v)$ denotes the induced label of the vertex v.

To begin, we consider a trivial case.

Theorem 1. *$K_{1,n}$ is super edge-graceful if and only if n is even.*

Proof. Consider a bijective labeling f of $E(K_{1,n})$ with the labels $[n]^*$. The sum of the elements of $[n]^*$ is always 0, so the induced label for the vertex of degree n will necessarily be 0. The set of induced labels for the vertices of degree 1 is clearly the same as the edge label set $[n]^*$. So $K_{1,n}$ is super-edge graceful if and only if $[n+1]^* = [n]^* \cup \{0\}$, which is true if and only if n is even. □

2 SEGL of $K_{m,n}$, m and n even

In this section we prove that if $m \geq 2$ and $n \geq 4$ are even, then there exists an SEGL of $K_{m,n}$. First we show that $K_{2,n}$ for $n \geq 4$ has an SEGL. Neither $K_{2,2}$ nor $K_{2,3}$ has an SEGL, which can be verified easily. Throughout this section we denote the partite sets of $K_{2,n}$ by $\{u, v\}$ and $\{w_1, w_2, \ldots, w_n\}$.

Lemma 2. *Let $n \equiv 0, 4$ (mod 6), $n \geq 4$. Then there exists an SEGL of $K_{2,n}$.*

Proof. Let $n \equiv 0, 4 \pmod 6$, with $n \geq 6$. (Recall that we already dealt with the case $n = 4$, see Figure 1.) Define an array $A_{2,n} = [a_{j,k}]$ by: $a_{1,k} = k$ for $1 \leq k \leq n$ and

$$a_{2,k} = \begin{cases} -2k & \text{if } 1 \leq k \leq n/2 \\ n+1-2k & \text{if } n/2+1 \leq k \leq n. \end{cases}$$

Label the edge uw_k with $a_{1,k}$ and the edge vw_k with $a_{2,k}$ for $1 \leq k \leq n$. Then

$$\ell(w_k) = \begin{cases} -k & \text{if } 1 \leq k \leq n/2 \\ n+1-k & \text{if } n/2+1 \leq k \leq n. \end{cases}$$

Hence, $\{\ell(w_k) \mid 1 \leq k \leq n\} = [n+2]^* \setminus \{\frac{n}{2}+1, -(\frac{n}{2}+1)\}$. We also have $\ell(u) = n(n+1)/2$ and $\ell(v) = -n(n+1)/2$. This means that, if we can change $\{\ell(u), \ell(v)\}$ to $\{\frac{n}{2}+1, -(\frac{n}{2}+1)\}$ without changing the other vertex labels, the labeling will become super edge-graceful. We will do this by interchanging the elements of columns in our array.

Let $s(k) = a_{1,k} - a_{2,k}$ for each $1 \leq k \leq n$. Notice that $s(k) = 3k$ if $k \leq n/2$ and $s(k) = 3k - n - 1$ if $k > n/2$. If we interchange the elements of column k, $\ell(u)$ decreases by $s(k)$ and $\ell(v)$ increases by $s(k)$. So, we can make the labeling super edge-graceful if we can find a set of integers $I \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in I} s(i) = \frac{n(n+1)}{2} + \frac{n+2}{2} = \frac{n^2+2n+2}{2}.$$

(We could also use $\sum_{i \in I} s(i) = \frac{n(n+1)}{2} - \frac{n+2}{2} = \frac{n^2-2}{2}$, but we will not need to until we handle the $n \equiv 2 \pmod 6$ cases in the next lemma.)

Below, we list the appropriate set I depending on the congruence class of n (mod 12), and verify that I satisfies the condition.

Case 1: $n \equiv 0$ (mod 12). Let

$$I_1 = \left\{1 + 2i \;\middle|\; 0 \leq i \leq \frac{n}{12} + 1\right\} \text{ and } I_2 = \left\{\frac{n}{6} + 2i \;\middle|\; 2 \leq i \leq \frac{5n}{12}, i \neq \frac{n}{6} + 1\right\}.$$

Then

$$\begin{aligned}\sum_{i\in I_1} s(i) &= \sum_{k=0}^{(n/12)+1} s(1+2k) = \sum_{k=0}^{(n/12)+1} 3+6k \\ &= 3\left(\frac{n}{12}+2\right)+6\left(\frac{\left(\frac{n}{12}+1\right)\left(\frac{n}{12}+2\right)}{2}\right) \\ &= \frac{n}{4}+6+\frac{n^2}{48}+\frac{3n}{4}+6 = \frac{n^2}{48}+n+12\end{aligned}$$

and

$$\begin{aligned}\sum_{i\in I_2} s(i) &= \sum_{k=2}^{n/6} s\left(\frac{n}{6}+2k\right)+\sum_{k=(n/6)+2}^{5n/12} s\left(\frac{n}{6}+2k\right) \\ &= \sum_{k=0}^{(n/6)-2} s\left(\frac{n}{6}+2k+4\right)+\sum_{k=0}^{(n/4)-2} s\left(\frac{n}{2}+2k+4\right) \\ &= \sum_{k=0}^{(n/6)-2} \left(\frac{n}{2}+6k+12\right)+\sum_{k=0}^{(n/4)-2} \left(\frac{n}{2}+6k+11\right) \\ &= \frac{23n^2}{48}-11.\end{aligned}$$

So if we set $I = I_1 \cup I_2$ (note that I_1 contains only odd numbers and I_2 contains only even numbers, so they are disjoint), we have

$$\sum_{i\in I} s(i) = \frac{n^2}{48}+n+12+\frac{23n^2}{48}-11 = \frac{n^2}{2}+n+1 = \frac{n^2+2n+2}{2}$$

as desired.

Case 2: $n \equiv 4$ **(mod 12).** Let

$$\begin{aligned}I_1 &= \left\{2i \;\middle|\; 1 \le i \le \frac{n+2}{6}\right\} \\ I_2 &= \left\{\frac{n+5}{3}+2i \;\middle|\; 0 \le i \le \frac{n-16}{12}\right\} \\ I_3 &= \left\{\frac{n}{2}+2i \;\middle|\; 1 \le i \le \frac{n}{4}\right\}.\end{aligned}$$

Then

$$\sum_{i\in I_1} s(i) = \sum_{k=1}^{(n+2)/6} s(2k) = \sum_{k=1}^{(n+2)/6} 6k = \frac{(n+2)(n+8)}{12} = \frac{n^2+10n+16}{12},$$

$$\sum_{i\in I_2} s(i) = \sum_{k=0}^{(n-16)/12} s\left(\frac{n+5}{3}+2k\right) = \sum_{k=0}^{(n-16)/12} (n+5)+6k$$
$$= (n+5)\left(\frac{n-4}{12}\right) + \left(\frac{(n-16)(n-4)}{48}\right) = \frac{5n^2-16n-16}{48},$$

and

$$\sum_{i\in I_3} s(i) = \sum_{k=1}^{n/4} s\left(\frac{n}{2}+2k\right) = \sum_{k=1}^{n/4} \frac{n}{2}+6k-1$$
$$= \left(\frac{n-2}{2}\right)\left(\frac{n}{4}\right) + \frac{3n(n+4)}{16} = \frac{5n^2+8n}{16}.$$

Since I_1, I_2, and I_3 are pairwise disjoint, if we set $I = I_1 \cup I_2 \cup I_3$, we have

$$\sum_{i\in I} s(i) = \frac{4n^2+40n+64}{48} + \frac{5n^2-16n-16}{48} + \frac{15n^2+24n}{48}$$
$$= \frac{24n^2+48n+48}{48} = \frac{n^2+2n+2}{2}$$

as desired.

Case 3: $n \equiv 6$ **(mod 12).** Let

$$I_1 = \left\{1+2i \;\middle|\; 0 \le i \le \frac{n-6}{4}\right\} \text{ and } I_2 = \left\{\frac{n+2}{2}+2i \;\middle|\; 0 \le i \le \frac{n-2}{4}\right\}.$$

Then

$$\sum_{i\in I_1} s(i) = \sum_{k=0}^{(n-6)/4} s(1+2k) = \sum_{k=0}^{(n-6)/4} 3+6k$$
$$= 3\left(\frac{n-2}{4}\right) + \frac{3(n-6)(n-2)}{16} = \frac{3n^2-12n+12}{16}$$

and

$$\sum_{i\in I_2} s(i) = \sum_{k=0}^{(n-2)/4} s\left(\frac{n+2}{2}+2k\right) = \sum_{k=0}^{(n-2)/4} \frac{n+4}{2}+6k$$
$$= \frac{n+4}{2}\left(\frac{n+2}{4}\right) + \frac{3(n-2)(n+2)}{16} = \frac{5n^2+12n+4}{16}.$$

So we can take

$$I = \left\{\frac{n}{3}\right\} \cup I_1 \cup I_2$$

and get

$$\sum_{i \in I} s(i) = s\left(\frac{n}{3}\right) + \frac{3n^2 - 12n + 12}{16} + \frac{5n^2 + 12n + 4}{16} = \frac{n^2 + 2n + 2}{2},$$

as desired.

Case 4: $n \equiv 10$ **(mod 12).** Let I_1 and I_2 be defined as in Case 3. This time, we let

$$I = \left\{\frac{2n+1}{3}\right\} \cup I_1 \cup I_2,$$

and we get

$$\sum_{i \in I} s(i) = s\left(\frac{2n+1}{3}\right) + \frac{n^2+2}{2} = n + \frac{n^2+2}{2} = \frac{n^2 + 2n + 2}{2}$$

as desired.

So in all cases needed, we have constructed a set I of columns to exchange that lets us modify the array to create an SEGL. □

Lemma 3. *Let* $n \equiv 2$ *(mod 6),* $n \geq 8$*. Then there exists an SEGL of* $K_{2,n}$*.*

Proof. Define an array $A_{2,n} = [a_{j,k}]$ by: $a_{1,k} = k$ for $1 \leq k \leq n$, $a_{2,n-1} = -n$, $a_{2,n} = -(n-1)$ and

$$a_{2,k} = \begin{cases} -(n/2) - 3i & \text{if } k = 6i+1 \\ -(n/2) - 3i - 2 & \text{if } k = 6i+2 \\ -(n/2) - 3i - 1 & \text{if } k = 6i+3 \\ -3i - 2 & \text{if } k = 6i+4 \\ -3i - 1 & \text{if } k = 6i+5 \\ -3i - 3 & \text{if } k = 6i+6, \end{cases}$$

where $0 \leq i \leq (n-8)/6$. As an example, Figure 2 displays the array $A_{2,20}$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ -10 & -12 & -11 & -2 & -1 & -3 & -13 & -15 & -14 & -5 & -4 & -6 & -16 & -18 & -17 & -8 & -7 & -9 & -20 & -19 \end{bmatrix}$$

Figure 2: Array $A_{2,20}$

$$\begin{bmatrix} 1 & 2 & 3 & -2 & -1 & -3 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & -18 & -17 & -8 & -7 & -9 & -20 & 20 \\ -10 & -12 & -11 & 4 & 5 & 6 & -13 & -15 & -14 & -5 & -4 & -6 & -16 & 14 & 15 & 16 & 17 & 18 & 19 & -19 \end{bmatrix}$$

Figure 3: An SEGL for $K_{2,20}$

Label the edge uw_k with $a_{1,k}$ and the edge vw_k with $a_{2,k}$ for $1 \leq k \leq n$. Then $\ell(w_{n-1}) = -1$ and $\ell(w_n) = 1$, $\ell(u) = n(n+1)/2$ and $\ell(v) = -n(n+1)/2$, and

$$\ell(w_k) = \begin{cases} -(n/2) + 3i + 1 & \text{if } k = 6i+1 \\ -(n/2) + 3i & \text{if } k = 6i+2 \\ -(n/2) + 3i + 2 & \text{if } k = 6i+3 \\ 3i+2 & \text{if } k = 6i+4 \\ 3i+4 & \text{if } k = 6i+5 \\ 3i+3 & \text{if } k = 6i+6, \end{cases}$$

for each $0 \leq i \leq (n-8)/6$. Hence,

$$\{\ell(w_k) \mid 1 \leq k \leq n\} = [n+2]^* \setminus \left\{ \frac{n}{2} + 1, -\left(\frac{n}{2} + 1\right) \right\}.$$

We employ the same strategy as in the previous lemma, that is, to modify $A_{2,n}$ in order that u and v are labeled with the missing two values and the $\ell(w_k)$ are unchanged. As before, if we interchange the two elements in column k of $A_{2,n}$, then $\ell(u)$ decreases ($\ell(v)$ increases) by $s(k)$, where

$$s(k) = a_{1,k} - a_{2,k} = \begin{cases} (n/2) + 9i + 1 & \text{if } k = 6i+1 \\ (n/2) + 9i + 4 & \text{if } k = 6i+2 \\ (n/2) + 9i + 4 & \text{if } k = 6i+3 \\ 9i+6 & \text{if } k = 6i+4 \\ 9i+6 & \text{if } k = 6i+5 \\ 9i+9 & \text{if } k = 6i+6, \end{cases}$$

$0 \leq i \leq (n-8)/6$, and $s(n-1) = s(n) = 2n-1$. We can make the labeling super edge-graceful if we can find a set of integers $I \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in I} s(i) = \frac{(n)(n+1)}{2} - \frac{n+2}{2} = \frac{n^2-2}{2}.$$

Case 1: $(n-8)/6$ **is even.** For $n = 8$, $I = \{2, 3, 5, 6\}$ satisfies the condition; for $n = 20$, $I = \{4, 5, 6, 14, 15, 16, 17, 18, 19\}$ satisfies the condition (see Figure 3), and for $n = 32$, $I = \{4, 5, 6, 20, 21, 22, \dots, 30\}$ satisfies the condition. If $n \geq 44$, let

$$\begin{aligned} I_1 &= \left\{ 6i+1, 6i+2, 6i+3 \;\middle|\; \frac{n-8}{12} \leq i \leq \frac{n-8}{6} \right\} \\ I_2 &= \left\{ 6i+4, 6i+5, 6i+6 \;\middle|\; 0 \leq i \leq \frac{n-32}{12} \right\} \setminus \{4, 5, 10\} \\ I_3 &= \{2, 3, n-1\} \\ I &= I_1 \cup I_2 \cup I_3. \end{aligned}$$

Then

$$\begin{aligned} \sum_{k \in I_1} s(k) &= \sum_{i=(n-8)/12}^{(n-8)/6} (s(6i+1) + s(6i+2) + s(6i+3)) \\ &= \sum_{i=(n-8)/12}^{(n-8)/6} \left(\frac{n}{2} + 9i + 1 + \frac{n}{2} + 9i + 4 + \frac{n}{2} + 9i + 4 \right) \\ &= \left(\frac{3n}{2} + 9 \right) \left(\frac{n+4}{12} \right) + 27 \left(\frac{n^2 - 10n + 16}{72} - \frac{n^2 - 28n + 160}{288} \right) \\ &= \frac{13n^2 + 4n - 192}{32}, \end{aligned}$$

$$\begin{aligned} \sum_{k \in I_2} s(k) &= \sum_{i=0}^{(n-32)/12} (s(6i+4) + s(6i+5) + s(6i+6)) - s(4) - s(5) - s(10) \\ &= \sum_{i=0}^{(n-32)/12} (9i + 6 + 9i + 6 + 9i + 9) - 6 - 6 - 15 \\ &= 21 \left(\frac{n-20}{12} \right) + 27 \left(\frac{n^2 - 52n + 640}{288} \right) - 27 \\ &= \frac{3n^2 - 100n - 64}{32}, \end{aligned}$$

$$\sum_{k \in I_3} s(k) = s(2) + s(3) + s(n-1) = \left(\frac{n}{2} + 4 \right) + \left(\frac{n}{2} + 4 \right) + (2n - 1) = 3n + 7,$$

and

$$\sum_{k \in I} s(k) = \sum_{k \in I_1} s(k) + \sum_{k \in I_2} s(k) + \sum_{k \in I_3} s(k) = \frac{n^2 - 2}{2},$$

as desired.

Case 2: $(n-8)/6$ is odd. For $n = 14$, $I = \{2, 3, 10, 11, 12, 13\}$ gives an SEGL of $K_{2,14}$. If $n \geq 26$, let

$$\begin{aligned}
I_1 &= \left\{ 6i+1, 6i+2, 6i+3 \;\middle|\; \frac{n-2}{12} \leq i \leq \frac{n-8}{6} \right\} \\
I_2 &= \left\{ 6i+4, 6i+5, 6i+6 \;\middle|\; 0 \leq i \leq \frac{n-26}{12} \right\} \\
I_3 &= \left\{ 3, \frac{n-22}{2}, n-1 \right\} \\
I &= I_1 \cup I_2 \cup I_3.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{k \in I_1} s(k) &= \sum_{i=(n-2)/12}^{(n-8)/6} (s(6i+1) + s(6i+2) + s(6i+3)) \\
&= \sum_{i=(n-2)/12}^{(n-8)/6} \left(\frac{3n}{2} + 9 + 27i \right) = \frac{13n^2 - 56n + 60}{32},
\end{aligned}$$

$$\begin{aligned}
\sum_{k \in I_2} s(k) &= \sum_{i=0}^{(n-26)/12} (s(6i+4) + s(6i+5) + s(6i+6)) \\
&= \sum_{i=0}^{(n-26)/12} (27i + 21) = \frac{3n^2 - 64n + 308}{32},
\end{aligned}$$

$$\sum_{k \in I_3} s(k) = \left(\frac{n}{2} + 4 \right) + \left(\frac{5n-62}{4} \right) + (2n-1) = \frac{15n-50}{4},$$

and

$$\sum_{k \in I} s(k) = \sum_{k \in I_1} s(k) + \sum_{k \in I_2} s(k) + \sum_{k \in I_3} s(k) = \frac{n^2-2}{2},$$

as desired. □

Theorem 4. *Let m and n be even and $m \leq n-2$. Let there exist an SEGL of $K_{m,n}$ such that, for some $x \geq 0$, two vertices in the partite set with n vertices have induced vertex labels x and $x+1$. Then there exists an SEGL of $K_{m+2,n}$ such that two vertices in the partite set with n vertices have induced vertex labels x and $x+1$.*

Proof. Assume $W = \{w_1, w_2, \ldots, w_n\}$ are the vertices in the partite set with n vertices. In addition, assume the labels for vertices w_1 and w_2 are x and $x+1$, respectively. Add two new vertices u and v to the partite set with m vertices and join these two to every vertex in W to obtain a $K_{m+2,n}$. The labels we need to add to the new edges are $\{\pm(\frac{mn}{2}+1), \pm(\frac{mn}{2}+2), \ldots, \pm(\frac{mn}{2}+n)\}$. Label the edge uw_i with $(-1)^i(\frac{mn}{2}+i)$ and the edge vw_i with $(-1)^{i+1}(\frac{mn}{2}+i)$ for $i \in \{1,2,3,\ldots,n\}$. Note that with this labeling the vertex labels of $K_{m,n}$ do not change and $\ell(u) = n/2$ and $\ell(v) = -n/2$. To obtain an SEGL, we need to make $\{\ell(u), \ell(v)\} = \{\pm(\frac{m+n+2}{2})\}$.

Case 1: $(m+2)/2$ is even. If $-(\frac{mn}{2}+\frac{m+2}{4}+1)$ appears at an edge at u, we swap the edge labels $\frac{mn}{2}+2$ and $-(\frac{mn}{2}+\frac{m+2}{4}+1)$ with the edge labels $-(\frac{mn}{2}+2)$ and $(\frac{mn}{2}+\frac{m+2}{4}+1)$, respectively. Now $\ell(u) = \frac{m+n+2}{2}-2$ and $\ell(v) = -\frac{m+n+2}{2}+2$. Swap the edge labels $-(\frac{mn}{2}+n-1)$ and $\frac{mn}{2}+n-2$ with the edge labels $\frac{mn}{2}+n-1$ and $-(\frac{mn}{2}+n-2)$, respectively. Then $\ell(u) = \frac{m+n+2}{2}$ and $\ell(v) = -\frac{m+n+2}{2}$ and the labels of the other vertices of $K_{m+2,n}$ remain unchanged. Hence, the resulting labeling is an SEGL of $K_{m+2,n}$.

If $-(\frac{mn}{2}+\frac{m+2}{4}+2)$ appears at an edge at u, we swap the edge labels $\frac{mn}{2}+2$ and $-(\frac{mn}{2}+\frac{m+2}{4}+2)$ with the edge labels $-(\frac{mn}{2}+2)$ and $(\frac{mn}{2}+\frac{m+2}{4}+2)$, respectively. Then $\ell(u) = \frac{m+n+2}{2}$ and $\ell(v) = -\frac{m+n+2}{2}$ as required.

Case 2: $(m+2)/2$ is odd. When $(m,n) = (4,6)$ we label the edge uw_j with $a_{1,j}$ and the edge vw_j with $a_{2,j}$ for $j = 1,2,\ldots,6$, where $a_{i,j}$ is the entry in row i and column j of the following array.

$$\begin{bmatrix} -16 & 16 & -13 & -14 & 15 & 18 \\ 17 & -17 & 13 & 14 & -15 & -18 \end{bmatrix}$$

The resulting labeling is an SEGL of $K_{6,6}$. Note that $\ell(w_1) = x+1$ and $\ell(w_2) = x$ in $K_{6,6}$. Now assume $n \geq 8$. Swap the edge label $\frac{mn}{2}+2$ at u with the edge label $\frac{mn}{2}+1$ at v. Then $\ell(w_1) = x+1$, $\ell(w_2) = x$, $\ell(u) = n/2-1$ and $\ell(v) = -n/2+1$. If $(m,n) = (4,8)$, we swap the edge labels $\{20,-21,22,-23\}$ with $\{-20,21,-22,23\}$. Then $\ell(u) = 7$ and $\ell(v) = -7$, as required. Hence, we may assume $n \geq 10$.

If $-(\frac{mn}{2}+\frac{m+4}{4}+4)$ appears at an edge at u, we swap the edge labels $\frac{mn}{2}+4$ and $-(\frac{mn}{2}+\frac{m+4}{4}+4)$ with $-(\frac{mn}{2}+4)$ and $(\frac{mn}{2}+\frac{m+4}{4}+4)$, respectively. Then $\ell(u) = \frac{m+n+2}{2}$ and $\ell(v) = -\frac{m+n+2}{2}$, as required.

If $-(\frac{mn}{2}+\frac{m+4}{4}+5)$ appears at an edge at u, we swap the edge labels $\frac{mn}{2}+4$ and $-(\frac{mn}{2}+\frac{m+4}{4}+5)$ with $-(\frac{mn}{2}+4)$ and $(\frac{mn}{2}+\frac{m+4}{4}+5)$, respectively. Then $\ell(u) = \frac{m+n+2}{2}+2$ and $\ell(v) = -\frac{m+n+2}{2}-2$. Swap the edge labels $\frac{mn}{2}+n$ and $-(\frac{mn}{2}+n-1)$ with $-(\frac{mn}{2}+n)$ and $\frac{mn}{2}+n-1$, respectively. Then $\ell(u) = \frac{m+n+2}{2}$ and $\ell(v) = -\frac{m+n+2}{2}$, as required.

□

Theorem 5. *Let $m \geq 2$ and $n \geq 4$ be even and $m \leq n$. Then there exists an SEGL of $K_{m,n}$.*

Proof. First note that each SEGL of $K_{2,n}$ obtained in Lemmas 2 and 3 has this property that vertex labels 1 and 2 appear at two vertices in the partite set of size n. Now the result follows by Theorem 4. □

3 SEGL of $K_{m,n}$, m and n odd

When $(n-m)/2$ is odd our construction for an SEGL of $K_{m,n}$ is based on an SEGL of $K_{3,n}$. When $(n-m)/2$ is even our construction for an SEGL of $K_{m,n}$ is based on an SEGL of $K_{5,n}$.

Lemma 6. *$K_{3,n}$ is super edge-graceful for all odd integers $n \geq 3$.*

Proof. For an SEGL of $K_{3,3}$ see Figure 4. Let $n > 3$. We define a $3 \times n$ array $A = [a_{i,j}]$ as follows. We fill the first three rows and three columns of A with an SEGL of $K_{3,3}$. Define

$$a_{i,j} = \begin{cases} 3+k & \text{if } i=1,\ j=2k \\ -(3+k) & \text{if } i=1,\ j=2k+1 \\ (n-1)/2+2k & \text{if } i=2,\ j=2k \\ -[(n-1)/2+2k] & \text{if } i=2,\ j=2k+1 \\ -[(n+1)/2+2k] & \text{if } i=3,\ j=2k \\ (n+1)/2+2k & \text{if } i=3,\ j=2k+1, \end{cases}$$

where $2 \leq k \leq (n-1)/2$. It is easy to see that the obtained $3 \times n$ array is an SEGL of $K_{3,n}$. The first three rows in Figure 4 yield an SEGL of $K_{3,11}$. □

Theorem 7. *Let $m \geq 3$ and $n \geq 3$ be odd. Assume $n \geq m$ and $(n-m)/2$ is odd. Then $K_{m,n}$ is super edge-graceful.*

Proof. By Lemma 6, there is an SEGL of $K_{3,n}$. We define an $m \times n$ array $A = [a_{i,j}]$ as follows. We fill the first three rows of A with an SEGL of $K_{3,n}$. Define

$$a_{i,j} = \begin{cases} (3n-3)/2+k & \text{if } i=2k, j=1 \\ -[(3n-3)/2+k] & \text{if } i=2k+1, j=1 \\ (3n+m-6)/2+k & \text{if } i=2k, j=2 \\ -[(3n+m-6)/2+k] & \text{if } i=2k+1, j=2 \\ -[(4n+m-7)/2+k] & \text{if } i=2k, j=3 \\ (4n+m-7)/2+k & \text{if } i=2k+1, j=3, \end{cases}$$

where $2 \leq k \leq (m-1)/2$. So far, we have filled the first three rows and three columns of A. The remaining edge labels are

$$\begin{aligned} L_1 &= \{\pm(\tfrac{3n+2m-7}{2}+p) \mid 1 \leq p \leq \tfrac{n-m+2}{2}\} \\ L_2 &= \{\pm(2n+m-4+p) \mid 1 \leq p \leq \tfrac{mn+7}{2}-(2n+m)\}. \end{aligned}$$

Note that since m, n and $(n-m)/2$ are odd, $L_1 \cup L_2$ can be partitioned into $\frac{(m-3)(n-3)}{4}$ 4-subsets of the form $\{\pm\ell, \pm(\ell+1)\}$ for some positive integer ℓ. Now partition the empty cells of A into $\frac{(m-3)(n-3)}{4}$, 2×2 sub-arrays and fill each 2×2 sub-array with a 4-subset in such a way that the column sums are zero and the row sum for the first row is -1 and for the second row is 1 in each sub-array. Note that the column sums of A are the same as the column sums of the $3 \times n$ array corresponds to the starting SEGL of $K_{3,n}$. Moreover, the row sum for row $2k$ is

$$\frac{3n-3}{2}+k+\frac{3n+m-6}{2}+k-[\frac{4n+m-7}{2}+k]-\frac{n-3}{2}=\frac{n+1}{2}+k,$$

where $2 \leq k \leq (m-1)/2$. Similarly, the row sum for row $2k+1$ is $-[(n+1)/2+k]$. Hence, A defines an SEGL of $K_{m,n}$. Figure 4 displays an SEGL of $K_{9,11}$. □

$$\left[\begin{array}{rrr|rrrrrrrr}
0 & -4 & 3 & 5 & -5 & 6 & -6 & 7 & -7 & 8 & -8 \\
4 & 2 & -3 & 9 & -9 & 11 & -11 & 13 & -13 & 15 & -15 \\
-2 & -1 & 1 & -10 & 10 & -12 & 12 & -14 & 14 & -16 & 16 \\
\hline
17 & 20 & -25 & 23 & -24 & 28 & -29 & 30 & -31 & 32 & -33 \\
-17 & -20 & 25 & -23 & 24 & -28 & 29 & -30 & 31 & -32 & 33 \\
18 & 21 & -26 & 34 & -35 & 36 & -37 & 38 & -39 & 40 & -41 \\
-18 & -21 & 26 & -34 & 35 & -36 & 37 & -38 & 39 & -40 & 41 \\
19 & 22 & -27 & 42 & -43 & 44 & -45 & 46 & -47 & 48 & -49 \\
-19 & -22 & 27 & -42 & 43 & -44 & 45 & -46 & 47 & -48 & 49
\end{array}\right]$$

Figure 4: SEGLs for $K_{3,3}$, $K_{3,11}$ and $K_{9,11}$

Lemma 8. *$K_{5,n}$ is super edge-graceful for all odd integers $n \geq 3$.*

Proof. $K_{5,3}$ is super edge-graceful by Lemma 6 and Figure 5 displays an SEGL of $K_{5,5}$. Let $n \geq 7$. If $n \equiv 3 \pmod 4$, then the result follows by Theorem 7. Let $n \equiv 1 \pmod 4$. We define a $5 \times n$ array $A = [a_{i,j}]$ as follows. We fill the first five rows and columns with an SEGL of $K_{5,5}$. Define $a_{1,2k} = 10 + k$ and $a_{1,2k+1} = -(10 + k)$ for $3 \leq k \leq (n-1)/2$. Assume $n - 5 = 12q + r$, where $0 \leq r < 12$. Note that since $n \equiv 1 \pmod 4$, $r \equiv 0 \pmod 4$. For $0 \leq s \leq q - 1$ define

$$L_s = \{(n+15)/2 + k + 12s, -[(n+15)/2 + k + 12s] \mid 3 \leq k \leq 8\},$$

and

$$L_q = \{(n+15)/2 + k + 12q, -[(n+15)/2 + k + 12q] \mid 3 \leq k \leq r/2 + 2\}.$$

We use the elements of L_s to fill the empty cells in the second row of A. We put positive numbers in even columns and negative numbers in odd columns of A. Now define $a_{3,2k} = -(a_{2,2k} + 6)$ and $a_{3,2k+1} = -a_{2,2k+1} + 6$ for $3 \leq k \leq (n-1)/2$.

Since $r \equiv 0 \pmod 4$, the remaining $2(n-5)$ edge labels can be partitioned into 4-subsets of the form $\{\pm\ell, \pm(\ell+1)\}$ for some positive integer ℓ. Now partition the empty cells of A into 2×2 sub-arrays and fill each 2×2 sub-array with a 4-subset in such a way that the row sums are zero and the column sum for the first column is -1 and for the second column is 1 in each sub-array. It is easy to see that the obtained $5 \times n$ array A is an SEGL of $K_{5,n}$. The first five rows in Figure 5 is an SEGL of $K_{5,13}$. □

Theorem 9. *Let $m \geq 3$ and $n \geq 3$ be odd. Assume $n \geq m$ and $(n-m)/2$ is even. Then $K_{m,n}$ is super edge-graceful.*

Proof. By Lemmas 6 and 8 the theorem is true for $(m,n) \in \{(3,3),(3,7),(5,5)\}$. Now assume $m \geq 7$ and $n \geq 7$. We construct an SEGL of $K_{m,n}$ as follows. Let A be

$$\left[\begin{array}{rrrrr|rrrrrrrr}
0 & -12 & 11 & -10 & 9 & 13 & -13 & 14 & -14 & 15 & -15 & 16 & -16 \\
12 & -8 & -7 & 6 & 1 & 17 & -17 & 18 & -18 & 19 & -19 & 20 & -20 \\
-11 & 8 & 4 & 3 & -2 & -23 & 23 & -24 & 24 & -25 & 25 & -26 & 26 \\
10 & 5 & -9 & -5 & -4 & 21 & -21 & 27 & -27 & 29 & -29 & 31 & -31 \\
-6 & 2 & -3 & 7 & -1 & -22 & 22 & -28 & 28 & -30 & 30 & -32 & 32 \\
\hline
33 & 35 & -53 & 37 & -38 & 41 & -42 & 45 & -46 & 49 & -50 & 55 & -56 \\
-33 & -35 & 53 & -37 & 38 & -41 & 42 & -45 & 46 & -49 & 50 & -55 & 56 \\
34 & 36 & -54 & 39 & -40 & 43 & -44 & 47 & -48 & 51 & -52 & 57 & -58 \\
-34 & -36 & 54 & -39 & 40 & -43 & 44 & -47 & 48 & -51 & 52 & -57 & 58
\end{array}\right]$$

Figure 5: SEGLs for $K_{5,5}$, $K_{5,13}$ and $K_{9,13}$

an $m \times n$ array. Fill the first five rows of A with an SEGL of $K_{5,n}$. Define

$$a_{i,j} = \begin{cases} (5n-5)/2+k & \text{if } \; i=2k,\ j=1 \\ -[(5n-5)/2+k] & \text{if } \; i=2k+1,\ j=1 \\ (5n+m-10)/2+k & \text{if } \; i=2k,\ j=2 \\ -[(5n+m-10)/2+k] & \text{if } \; i=2k+1,\ j=2 \\ (8n+m-13)/2+k & \text{if } \; i=2k,\ j=3 \\ -[(8n+m-13)/2+k] & \text{if } \; i=2k+1,\ j=3, \end{cases}$$

where $3 \le k \le (m-1)/2$. So far, we have filled the first five rows and the first three columns of A. The remaining edge labels are

$$\begin{aligned} L_1 &= \{\pm(\tfrac{5n+2m-11}{2}+p) \mid 1 \le p \le \tfrac{3n-m+2}{2}\} \\ L_2 &= \{\pm(4n+m-7+p) \mid 1 \le p \le \tfrac{mn+13}{2}-(4n+m)\}. \end{aligned}$$

Note that since m and n are odd and $(n-m)/2$ is even, $L_1 \cup L_2$ can be partitioned into $\frac{(m-5)(n-3)}{4}$, 4-subsets of the form $\{\pm\ell, \pm(\ell+1)\}$ for some positive integer ℓ. We fill the remaining empty cells of A with these 4-subsets as before. The resulting array is an SEFL of $K_{m,n}$. Figure 5 displays an SEGL of $K_{9,13}$. □

By Theorems 7 and 9 we have the main result of this section.

Theorem 10. *Let $m \ge 3$ and $n \ge 3$ be odd. Then $K_{m,n}$ is super edge-graceful.*

4 SEGL of $K_{m,n}$, m even and n odd

First consider Figure 6, a super edge-graceful labeling of $K_{2,9}$.

$$\begin{bmatrix} 2 & 4 & 6 & 8 & -1 & -2 & -3 & -4 & -5 \\ -6 & -7 & -8 & -9 & 1 & 3 & 5 & 7 & 9 \end{bmatrix}$$

Figure 6: An SEGL for $K_{2,9}$

In the following proof, we will use the structure of this labeling as a starting point for constructing super edge-graceful labelings of other graphs.

Theorem 11. *$K_{2,n}$ is super edge-graceful for odd $n > 3$.*

Proof. For $n = 5$ we use the following labeling for $K_{2,5}$.

$$\begin{bmatrix} 1 & -3 & -2 & 2 & 4 \\ -4 & 3 & 5 & -1 & -5 \end{bmatrix}$$

Let graph $G = K_{2,n}$ for some odd integer $n \geq 7$. Clearly, $|V(G)| = n + 2$ and $|E(G)| = 2n$. Let r_1 and r_2 be the vertices of degree n, and let $c_1, c_2, \ldots, c_n$ be the vertices of degree 2. Let $e_{a,b}$ be the edge $r_a c_b$.

Define a bijection $f : E(G) \leftrightarrow [2n]^*$ as follows:

$$f(e_{1,b}) = \begin{cases} 2b & 1 \leq b \leq \dfrac{n-1}{2} \\ \dfrac{n-1}{2} - b & \dfrac{n+1}{2} \leq b \leq n \end{cases}$$

$$f(e_{2,b}) = \begin{cases} -\dfrac{n+1}{2} - b & 1 \leq b \leq \dfrac{n-1}{2} \\ 2b - n & \dfrac{n+1}{2} \leq b \leq n. \end{cases}$$

This gives us the following matrix of edge labels:

$$\begin{bmatrix} 2 & 4 & \cdots & (n-1) & -1 & -2 & \cdots & -\dfrac{n+1}{2} \\ -\dfrac{n+3}{2} & -\dfrac{n+5}{2} & \cdots & -n & 1 & 3 & \cdots & n \end{bmatrix}$$

Let $f^* : V(G) \to \mathbb{Z}$ be the vertex labeling induced by f. The sums of the rows and columns in the matrix give us the vertex labels from f^*. So

$$\begin{aligned} f^*(c_i) &= \begin{cases} (2i) + (-\frac{n+1}{2} - i) & 1 \leq i \leq \frac{n-1}{2} \\ (\frac{n-1}{2} - i) + (2i - n) & \frac{n+1}{2} \leq i \leq n \end{cases} \\ &= \begin{cases} i - \frac{n+1}{2} & 1 \leq i \leq \frac{n-1}{2} \\ i - \frac{n+1}{2} & \frac{n+1}{2} \leq i \leq n \end{cases} \\ &= i - \frac{n+1}{2}. \end{aligned}$$

Thus the range of f^* over all the c_i is $[n]^*$. For f to be a super edge-graceful labeling, we need f^* to be a bijection from $V(G)$ to $[n+2]^*$, that is, we need $f^*(r_1) = \pm\frac{n+1}{2}$ and $f^*(r_2) = -f^*(r_1)$.

Notice that $\sum_{e \in E(G)} f(e) = 0$, simply by the requirement that f be a bijection onto $[2n]^*$, which means that we always have $f^*(r_2) = -f^*(r_1)$ regardless of how f is defined, so we need consider only r_1.

By the definition of f,

$$f^*(r_1) = \sum_{i=1}^{n} f(e_{1,i}) = \sum_{i=1}^{\frac{n-1}{2}}(2i) + \sum_{i=\frac{n+1}{2}}^{n} (\frac{n-1}{2} - i)$$
$$= 2\sum_{i=1}^{\frac{n-1}{2}} i - \sum_{i=\frac{n+1}{2}}^{n} i + \sum_{i=\frac{n+1}{2}}^{n} \frac{n-1}{2}$$

So we have a super edge-graceful labeling if and only if $\frac{n^2-4n-5}{8} = \pm\frac{n+1}{2}$, which is only true when n is 9 (or ± 1).

But even though f is not a super edge-graceful labeling, we can use it to construct one. Let $S \subseteq \{1, \ldots, n\}$ and define another bijection $f' : E(G) \leftrightarrow [2n]^*$ by

$$f'(e_{a,b}) = \begin{cases} f(e_{3-a,b}) & b \in S \\ f(e_{a,b}) & b \notin S. \end{cases}$$

Essentially, f' exchanges the labels within some of the columns (in the matrix representation of G) to get the desired sum for the rows without changing the set of sums of columns ($[n]^*$). That is, if we choose S correctly and let f'^* be the vertex labeling induced by f', then we will have a super edge-graceful labeling.

What we need to do, then, is show that S can be chosen to achieve the desired result. For each $i \in \{1, \ldots, n\}$, let

$$s_i = f(e_{2,i}) - f(e_{1,i}).$$

This is the change in $f'^*(r_1)$ when we add i to S. Since $f^*(r_1) = \frac{n^2-4n-5}{8}$ and we need $f'^*(r_1) = \pm\frac{n+1}{2}$, we look for S such that

$$\sum_{i \in S} s_i = \frac{n+1}{2} - \frac{n^2-4n-5}{8} = -\frac{n^2-9}{8} + n.$$

We can find such an S by using a two-stage construction: By looking at the cases $n \equiv 1$ modulo 4 and $n \equiv 3$ modulo 4, we can show the existence of S_1 so that

$$\sum_{i \in S_1} s_i = -\frac{n^2-9}{8}.$$

Then, by looking at the congruence class of n modulo 6, we can find S_2 such that

$$\sum_{i \in S_2} s_i = n.$$

If S_1 and S_2 are disjoint, then taking $S = S_1 \cup S_2$ gives us the desired result.

Stage 1: Finding S_1

Case 1: $n \equiv 3 \pmod 4$.

Let $1 \leq i \leq \frac{n-1}{2}$ and $j = i + \frac{n-1}{2}$. Then

$$\begin{aligned} s_i + s_j &= f(e_{2,i}) - f(e_{1,i}) + f(e_{2,j}) - f(e_{1,j}) \\ &= (-\frac{n+1}{2} - i) - (2i) + (2j - n) - (\frac{n-1}{2} - j) \\ &= -2n - 3i + 3j = -\frac{n+3}{2}. \end{aligned}$$

Choose any $\frac{n-3}{4}$ values of i within the given range, and include these and the corresponding values of j in S_1. Then

$$\sum_{k \in S_1} s_k = (\frac{n-3}{4})(-\frac{n+3}{2}) = -\frac{n^2-9}{8}$$

as desired.

Notice that we have flexibility in which elements are chosen for S_1: as long as we take pairs of elements $\frac{n-1}{2}$ apart, the change in $f'^*(r_1)$ will be the same, and we need only take the correct number of such pairs. This property will also hold in Case 2. The second-stage cases will require us to take specific elements for S_2. To show the existence of disjoint S_1 and S_2, it will be sufficient to show that we have not blocked too many pairs by taking one or both of their elements in S_2.

Case 2: $n \equiv 1 \pmod 4$.

Notice that in Case 1, n is never an element of S_1: the maximum value of j is $n-1$. In this case, require as before that $1 \leq i \leq \frac{n-1}{2}$, but define $j = i + \frac{n+1}{2}$. Then

$$s_i + s_j = -2n - 3i + 3j = -2n - 3i + 3(i + \frac{n+1}{2}) = -\frac{n-3}{2}.$$

This time we choose $\frac{n+3}{4}$ pairs of values of i and corresponding j to include in S_1, and we also get the desired result of

$$\sum_{k \in S_1} s_k = (\frac{n+3}{4})(-\frac{n-3}{2}) = -\frac{n^2-9}{8}.$$

Stage 2: Finding S_2

Case 1: $n \equiv 5 \pmod 6$.

Let $i = \frac{5n-1}{6}$. This is an integer greater than $\frac{n-1}{2}$, so

$$s_i = f(e_{2,i}) - f(e_{1,i}) = (2i - n) - (\frac{n-1}{2} - i) = 3i - \frac{3n-1}{2} = n,$$

and we have the desired result of $\sum_{k \in S_2} s_k = n$ simply by setting $S_2 = \{i\}$.

Since $|S_2| = 1$, only one of the $\frac{n-1}{2}$ pairs available for S_1 is blocked, so we have that disjoint S_1 and S_2 exist as long as $\frac{n-3}{2} \geq \frac{n+3}{4}$ if $n \equiv 1 \pmod 4$ or $\frac{n-3}{2} \geq \frac{n-3}{4}$ if $n \equiv 3 \pmod 4$. This is true for all n under consideration in this case.

Case 2: $n \equiv 1 \pmod 6$.

Let $i = \frac{5n-5}{6}$ and $j = \frac{n+1}{2}$. Both of these are integers greater than $\frac{n-1}{2}$ (and they are distinct) for all n in our domain, so we have

$$\begin{aligned} s_i &= f(e_{2,i}) - f(e_{1,i}) \\ &= (2i-n) - (\frac{n-1}{2} - i) \\ &= 3i - \frac{3n-1}{2} = n-2 \end{aligned} \qquad \begin{aligned} s_j &= f(e_{2,j}) - f(e_{1,j}) \\ &= (2j-n) - (\frac{n-1}{2} - j) \\ &= 3j - \frac{3n-1}{2} = 2 \end{aligned}$$

This time we take $S_2 = \{i, j\}$ and get

$$\sum_{k \in S_2} s_k = s_i + s_j = (n-2) + 2 = n$$

as desired.

For the disjointness requirement, we have $|S_2| = 2$, so we need $\frac{n-5}{2} \geq \frac{n+3}{4}$ if $n \equiv 1 \pmod 4$ or $\frac{n-5}{2} \geq \frac{n-3}{4}$ if $n \equiv 3 \pmod 4$. (Actually, if $n \equiv 1 \pmod 4$, then we need to ensure even less space, because in Case 2, $\frac{n+1}{2}$ is never included in S_1, and so its inclusion in S_2 will not block any pairs.) This is satisfied by all n under this case.

Case 3: $n \equiv 3 \pmod 6$.

Let $i_1 = \frac{n+1}{2}$, $i_2 = \frac{2n-3}{3}$, and $i_3 = \frac{2n}{3}$. Again we have distinct integers in the upper half of the domain of f for all the n we are considering in this case, except for $n = 9$, where $i_1 = i_2$. But for $n = 9$, f is already a super edge-graceful labeling, so we do not need to worry about constructing f'. For all the other n, we have the following:

$$\begin{aligned} s_{i_1} &= 3i - \frac{3n-1}{2} = 3(\frac{n+1}{2}) - \frac{3n-1}{2} = 2 \\ s_{i_2} &= 3i - \frac{3n-1}{2} = 3(\frac{2n-3}{3}) - \frac{3n-1}{2} = \frac{n-5}{2} \\ s_{i_3} &= 3i - \frac{3n-1}{2} = 3(\frac{2n}{3}) - \frac{3n-1}{2} = \frac{n+1}{2}. \end{aligned}$$

With $S_2 = \{i_1, i_2, i_3\}$, we have

$$\sum_{k \in S_2} s_k = s_{i_1} + s_{i_2} + s_{i_3} = 2 + \frac{n-5}{2} + \frac{n+1}{2} = n$$

as desired.

Again the disjointness requirement is easily satisfied. We need $\frac{n-7}{2} \geq \frac{n+3}{4}$ if $n \equiv 1 \pmod 4$ or $\frac{n-7}{2} \geq \frac{n-3}{4}$ if $n \equiv 3 \pmod 4$, since $|S_2| = 3$, which is true for all the n we are considering.

Note that the discussion in each case of the requirement that there be enough room left for S_1 and S_2 to be disjoint was important because for $n \leq 5$, the set $\{1, \ldots, n\}$ is too small for this to happen. That is, the construction used cannot give a super edge-graceful labeling of $K_{2,3}$ and $K_{2,5}$. In fact, $K_{2,3}$ is not super edge-graceful (we showed earlier that $K_{2,5}$ is super edge-graceful, and $K_{2,1}$ is trivially super edge-graceful).

Since in every case of odd $n \geq 7$ we can construct $S = S_1 \cup S_2$ with

$$\sum_{i \in S} s_i = -\frac{n^2-9}{8} + n = \frac{n+1}{2} - \frac{n^2-4n-5}{8},$$

we have

$$f'^*(r_1) = \frac{n+1}{2},$$

and therefore f' is a super edge-graceful labeling of G, so we are done. □

We can use this result to construct super edge-graceful labelings of larger complete bipartite graphs, specifically, any with one even partite set and one odd partite set except $K_{2,3}$.

Theorem 12. *$K_{m,n}$ is super edge-graceful for all positive even integers m and positive odd integers n, except for $K_{2,3}$.*

Proof. Let graph $G = K_{m,n}$ for some positive even integer m and positive odd integer $n \neq 3$. Let $r_1, r_2, \ldots, r_m$ be the vertices of degree n, let $c_1, c_2, \ldots, c_n$ be the vertices of degree m, and let $e_{a,b}$ be the edge $r_a c_b$.

Let H be the induced subgraph of G with $V(H) = \{r_1, r_2, c_1, c_2, \ldots, c_n\}$. Then $H = K_{2,n}$ is super edge-graceful by Theorem 11. Let $f : E(H) \leftrightarrow [2n]^*$ be any super edge-graceful labeling of H, with $f^* : V(H) \leftrightarrow [n+2]^*$ its induced vertex labeling.

We can extend f to a super edge-graceful labeling of G. The edge labels already used in $f(H)$ are $[2n]^*$, and overall, we need to use the labels $[mn]^*$. So the set of edge labels not yet used is

$$\left\{-\frac{mn}{2}, \cdots, -(n+1), (n+1), \cdots, \frac{mn}{2}\right\}.$$

Similarly, the vertex labels already used are $[n+2]^*$, and we need to use $[m+n]^*$, so the set of vertex labels that still need to be induced is

$$\left\{-\frac{m+n-1}{2}, \cdots, -\frac{n+3}{2}, \frac{n+3}{2}, \cdots, \frac{m+n-1}{2}\right\}.$$

Our extension will not change any of the edge or vertex labels in H, which means we must have

$$\sum_{i=3}^{m} f(e_{i,j}) = 0$$

for each $1 \leq j \leq n$. We will accomplish this by setting for each $2 \leq i \leq m/2$, $1 \leq j \leq n$,

$$f(e_{2i,j}) = -f(e_{2i-1,j}).$$

So the problem is reduced to setting the edge labels for the odd rows, using only one label of each absolute value in $\{(n+1), \cdots, \frac{mn}{2}\}$, so that the induced vertex labels of the rows are $\{\frac{n+3}{2}, \cdots, \frac{m+n-1}{2}\}$.

We do this by first setting

$$f(e_{2i-1,1}) = n - 1 + i$$

for each $2 \leq i \leq m/2$, essentially filling in the smallest labels down the first column, and then labeling the rest of the edges in pairs of consecutive integers. We do not need any particular arrangement for the remaining edges, just the property that

$$f(e_{2i-1,2j+1}) = -(f(e_{2i-1,2j}) + 1)$$

for each $1 \leq j \leq (n-1)/2$.

We will presently show why these requirements are always sufficient to achieve a super edge-graceful labeling, but first consider a nice large example:

$$\begin{bmatrix} -7 & -8 & 6 & 8 & 10 & 1 & 3 & -3 & 7 & -5 & -6 \\ 2 & 4 & -9 & -10 & -11 & -1 & -2 & 5 & -4 & 9 & 11 \\ 12 & 15 & -16 & 17 & -18 & 19 & -20 & 21 & -22 & 23 & -24 \\ -12 & -15 & 16 & -17 & 18 & -19 & 20 & -21 & 22 & -23 & 24 \\ 13 & 25 & -26 & 27 & -28 & 29 & -30 & 31 & -32 & 33 & -34 \\ -13 & -25 & 26 & -27 & 28 & -29 & 30 & -31 & 32 & -33 & 34 \\ 14 & 35 & -36 & 37 & -38 & 39 & -40 & 41 & -42 & 43 & -44 \\ -14 & -35 & 36 & -37 & 38 & -39 & 40 & -41 & 42 & -43 & 44 \end{bmatrix}$$

This super edge-graceful labeling of $K_{8,11}$ was constructed, for the top two rows, using the method of Theorem 11, and for the rest, with the just-described method. It is illustrative of the pattern our construction of f creates. Notice how the pairing of edge labels starting at $e_{2,3}$ and continuing rightward has the effect that each pair adds -1 to the row total (in odd-numbered rows; it adds $+1$ in the inversely-labeled even-numbered rows). We could permute those pairs in any way (as long as we move them with their inverses), without affecting this property.

In general, there are clearly $\frac{n-1}{2}$ such pairs in each row, so for each (odd) row we get

$$\begin{aligned} f^*(r_{2i-1}) &= f(e_{2i-1,1}) + \sum_{j=1}^{\frac{n-1}{2}} (f(e_{2i-1,2j}) + f(e_{2i-1,2j+1})) \\ &= n - 1 + i - \frac{n-1}{2} = i + \frac{n-1}{2}. \end{aligned}$$

Since i ranges from 2 to $m/2$, this gives us the set

$$f^*(\{r_{2i-1}\}) = \left\{\frac{n+3}{2}, \cdots, \frac{m+n-1}{2}\right\}.$$

And with the even rows giving us the negatives of this, we have the desired result.

Finally, we still need to take care of the case of $n = 3$. Since $K_{2,3}$ is not super edge-graceful, we will need to modify the extension method to get a super edge-graceful labeling of $K_{m,3}$. We do most of the steps as before, but we begin with the following labeling of the top two rows:

$$\begin{bmatrix} -1 & 2 & -3 \\ 1 & -2 & 3 \end{bmatrix}$$

Notice that all the column sums are 0, while the row sums are ± 2. Applying the rest of the edge labels as before leaves us with the column sums still 0, and without vertices labeled 1 or -1. We can fix this by considering the last two rows. Put the smallest pair of edge labels in the penultimate row with $n - 1 + \frac{m}{2} = 2 + \frac{m}{2}$, and of course put their inverses with them in the final row. Then these rows look like this:

$$\begin{bmatrix} \dfrac{m+4}{2} & \dfrac{m+6}{2} & -\dfrac{m+8}{2} \\ -\dfrac{m+4}{2} & -\dfrac{m+6}{2} & \dfrac{m+8}{2}. \end{bmatrix}$$

Now we break the inverses-in-matching-rows rule, and rearrange the last row to look like this:

$$\begin{bmatrix} \dfrac{m+4}{2} & \dfrac{m+6}{2} & -\dfrac{m+8}{2} \\ -\dfrac{m+6}{2} & -\dfrac{m+4}{2} & \dfrac{m+8}{2}. \end{bmatrix}$$

We have not changed the row sums, but the column sums now form the set $\{-1, +1, 0\}$ instead of three 0s, as we needed. Below is an SEGL of $K_{6,3}$ obtained using above construction.

$$\begin{bmatrix} -1 & 2 & -3 \\ 1 & -2 & 3 \\ 4 & 8 & -9 \\ -4 & -8 & 9 \\ 5 & 6 & -7 \\ -6 & -5 & 7 \end{bmatrix}$$

We have super edge-graceful labelings for every $K_{m,n}$ with even m, odd n except $K_{2,3}$, as desired. □

Acknowledgements

The authors wish to thank the referee for his/her helpful comments.

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(Received 28 Mar 2009)