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On α-resolvable directed cycle systems for cycle length 4*

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Abstract

A directed m-cycle system of order v with index λ, denoted m-$DCS(v, \lambda)$, is a collection of directed cycles of length m whose directed edges partition the directed edges of λDK_v. An m-$DCS(v, \lambda)$ is α-resolvable if its directed cycles can be partitioned into classes such that each point of the design occurs in precisely α cycles in each class. The necessary conditions for the existence of such a design are $m \mid \alpha v$ and $\alpha \mid \lambda(v-1)$. It is shown in this paper that these conditions are also sufficient when $m = 4$, except for the case $v = 4$, $\lambda \equiv 1 \pmod 2$.

1 Introduction

Let m, v, λ be positive integers and let X be a v-set. An *edge* of X is an unordered pair $\{x, y\}$, and a *directed edge* of X is an ordered pair (x, y), where x, y are distinct vertices of X. A *complete multigraph* of order v and index λ, denoted by λK_v, is a graph on X in which each pair of vertices x, y is joined by exactly λ edges $\{x, y\}$. A *directed complete multigraph* of order v and index λ, denoted by λDK_v, is a directed graph on X in which each pair of vertices x, y is joined by exactly λ directed edges (x, y) and λ directed edges (y, x). A *cycle* of length m is a sequence

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of m distinct vertices $u_1, u_2, \ldots, u_m$, denoted by $(u_1, u_2, \ldots, u_m)$, and its edge set is $\{\{u_i, u_{i+1}\} : i = 1, 2, \ldots, m-1\} \cup \{\{u_1, u_m\}\}$. A *directed cycle* of length m is a sequence of m distinct vertices $u_1, u_2, \ldots, u_m$, denoted by $\langle u_1, u_2, \ldots, u_m\rangle$, and its directed edge set is $\{(u_i, u_{i+1}) : i = 1, 2, \ldots, m-1\} \cup \{(u_m, u_1)\}$.

If the (directed) edges of a λK_v (λDK_v) can be decomposed into (directed) cycles of length m, then these (directed) cycles are called a *(directed) m-cycle system*, and denoted by m-$CS(v,\lambda)$ (m-$DCS(v,\lambda)$). An m-$CS(v,\lambda)$ (m-$DCS(v,\lambda)$) is said to be *α-resolvable* if its (directed) cycles can be partitioned into classes (called *α-resolution classes*) such that each point of the design occurs in precisely α cycles in each class. A 1-resolvable m-$CS(v,\lambda)$ (m-$DCS(v,\lambda)$) is simply called a *resolvable* m-$CS(v,\lambda)$ (m-$DCS(v,\lambda)$). When $m = 3$ or 4, the existence of α-resolvable m-$CS(v,\lambda)s$ has been solved completely.

Lemma 1.1 [2] *An α-resolvable 3-$CS(v,\lambda)$ exists if and only if*

$$\lambda(v-1) \equiv 0 \pmod 2, \quad \lambda v(v-1) \equiv 0 \pmod 6, \quad 3 \mid \alpha v, \quad \alpha \mid \lambda(v-1)/2,$$

and $(v,\alpha,\lambda) \notin \{(6,1,4i+2) : i \geq 0\}$.

Lemma 1.2 [3] *An α-resolvable 4-$CS(v,\lambda)$ exists if and only if*

$$4 \mid \lambda v(v-1)/2, \quad 2 \mid \lambda(v-1), \quad 4 \mid \alpha v, \quad \alpha \mid \lambda(v-1)/2.$$

The necessary conditions for the existence of an α-resolvable m-$DCS(v,\lambda)$ are:

$$m \mid \alpha v, \quad \alpha \mid \lambda(v-1) \tag{1}$$

The following result due to Adams and Bryant appears as Theorem 12.42 in [1].

Lemma 1.3 [1] *For $m \in \{3,4\}$, there exists a resolvable m-$DCS(v,\lambda)$ if and only if m divides v, $(m,v,\lambda) \notin \{(3,6,2i+1),(4,4,2i+1) : i \geq 0\}$.*

In addition, the existence of α-resolvable 3-$DCS(v,\lambda)s$ has also been solved.

Lemma 1.4 [4] *Let v, λ, α be positive integers and $v \geq 3, \alpha \geq 1, \lambda \geq 1$. There exists an α-resolvable 3-$DCS(v,\lambda)$ if and only if $3 \mid \lambda v(v-1)$, $3 \mid \alpha v$, $\alpha \mid \lambda(v-1)$, and $(v,\alpha,\lambda) \notin \{(6,5,1)\} \cup \{(6,1,2i+1) : i \geq 0\}$.*

In this paper, we investigate the existence of α-resolvable 4-$DCS(v,\lambda)s$. From the conditions (1), we can derive minimum values for α and λ, and call them α_0 and λ_0. Similar to Lemmas 2.1–2.3 in [5], we have some lemmas listed below.

Lemma 1.5 *If an α-resolvable m-$DCS(v,\lambda)$ exists, then $\alpha_0 \mid \alpha$, and $\lambda_0 \mid \lambda$.*

Lemma 1.6 *If an α-resolvable m-$DCS(v,\lambda)$ exists, then a $t\alpha$-resolvable m-$DCS(v, n\lambda)$ exists for any positive integers n, t with $t \mid \lambda(v-1)/\alpha$.*

Lemma 1.7 *If an α_0-resolvable m-$DCS(v,\lambda_0)$ exists, and α,λ satisfy conditions* (1)*, then an α-resolvable m-$DCS(v,\lambda)$ exists.*

Thus, in order to prove the necessary conditions (1) for the existence of α-resolvable m-$DCS(v,\lambda)$s are also sufficient, we only need to prove the existence of α_0-resolvable m-$DCS(v,\lambda_0)$s. For the relationship between an α-resolvable m-$CS(v,\lambda)$ and an α-resolvable m-$DCS(v,\lambda)$, we have the following lemma.

Lemma 1.8 *If an α-resolvable m-$CS(v,\lambda)$ exists, then an α-resolvable m-$DCS(v,\lambda)$ exists.*

Proof. Let X be a v-set, and $\mathcal{C}$ be the set of cycles of an α-resolvable m-$CS(v,\lambda)$ on X. We define

$$\mathcal{C}' = \bigcup_{(u_1,u_2,\ldots,u_m)\in\mathcal{C}} \{\langle u_1,u_2,\ldots,u_m\rangle, \langle u_1,u_m,\ldots,u_2\rangle\};$$

then $\mathcal{C}'$ forms a set of directed cycles of an α-resolvable m-$DCS(v,\lambda)$, and one α-resolution class of the α-resolvable m-$CS(v,\lambda)$ generates two α-resolution classes of the α-resolvable m-$DCS(v,\lambda)$. ∎

2 Direct constructions

In order to solve the existence of α-resolvable 4-$DCS(v,\lambda)$s, we need some definitions and remarks.

Let m,v be positive integers and ∞ an infinity point. Let Z_v be the residue ring of integers modulo v and let $Z_v^* = Z_v\backslash\{0\}$. Let $\mathcal{C}$ be a set of directed cycles of length m which are constructed on Z_v or $Z_v\cup\{\infty\}$. For each directed cycle $C=\langle c_1,c_2,\ldots,c_m\rangle$ and $j\in Z_v$, define $C+j$ to be $\langle c_1+j,c_2+j,\ldots,c_m+j\rangle$ where $\infty+j=\infty$ if $\infty\in C$. Let $\mathcal{C}+j=\{C+j : C\in\mathcal{C}\}$ for $j\in Z_v$. The differences of a directed cycle $C=\langle c_1,c_2,\ldots,c_m\rangle$ mean $\{c_{i+1}-c_i : i=1,2,\ldots,m-1\}\cup\{c_1-c_m\}$, where $\infty-j=\infty$, $j-\infty=-\infty$ for any $j\in Z_v$ and $\infty\in C$.

In what follows, we will get α-resolvable 4-$DCS(v,\lambda)$s through direct constructions. According to the conditions (1), α_0 and λ_0 are as follows.

$$\begin{cases} \alpha_0=1,\ \lambda_0=1,\ \ v\equiv 0 \pmod 4, \\ \alpha_0=4,\ \lambda_0=1,\ \ v\equiv 1 \pmod 4, \\ \alpha_0=2,\ \lambda_0=2,\ \ v\equiv 2 \pmod 4, \\ \alpha_0=4,\ \lambda_0=2,\ \ v\equiv 3 \pmod 4. \end{cases}$$

Lemma 2.1 *If $v\equiv 0 \pmod 4$ and $v\ge 8$, there exists a resolvable 4-$DCS(v,1)$. Furthermore, there exists a resolvable 4-$DCS(4,\lambda)$ for $\lambda\equiv 0 \pmod 2$ and there is no resolvable 4-$DCS(4,\lambda)$ for $\lambda\equiv 1 \pmod 2$.*

Proof. The result follows directly from Lemma 1.3. ∎

Lemma 2.2 *There exists a* 4*-resolvable* 4*-*$DCS(v,1)$ *for* $v \equiv 1 \pmod 4$.

Proof. (1) $v \equiv 1 \pmod 8$. The conclusion holds by Lemma 1.2 and Lemma 1.8.

(2) $v \equiv 5 \pmod 8$. Let the point set $X = Z_{8k+5}$, $k \geq 0$. A 4-resolvable 4-$DCS(v,1)$ contains $\frac{\lambda v(v-1)}{m} = (8k+5)(2k+1)$ directed cycles and $\frac{\lambda(v-1)}{\alpha} = 2k+1$ 4-resolution classes. Let $\mathcal{C}$ consist of the following $2k+1$ directed cycles:

Part 1: Construct k directed cycles:

$$\begin{array}{l}\langle\ 1,\ 0,\ 2,\ 4k+3\ \rangle,\\ \langle\ 3,\ 0,\ 4,\ 4k+3\ \rangle,\\ \langle\ 5,\ 0,\ 6,\ 4k+3\ \rangle,\\ \quad\vdots \qquad \vdots \qquad \vdots\\ \langle 2k-3,\ 0,\ 2k-2,\ 4k+3\ \rangle,\\ \langle 2k-1,\ 0,\ 2k,\ 4k+3\ \rangle.\end{array}$$

Part 2: Construct k directed cycles:

$$\begin{array}{l}\langle\ 2,\ 0,\ 1,\ 4k+3\ \rangle,\\ \langle\ 4,\ 0,\ 3,\ 4k+3\ \rangle,\\ \langle\ 6,\ 0,\ 5,\ 4k+3\ \rangle,\\ \quad\vdots \qquad \vdots \qquad \vdots\\ \langle 2k-2,\ 0,\ 2k-3,\ 4k+3\ \rangle,\\ \langle\ 2k,\ 0,\ 2k-1,\ 4k+3\ \rangle.\end{array}$$

Part 3: Construct one directed cycle:

$$\langle 2k+1,\ 0,\ 2k+2,\ 4k+3\ \rangle.$$

It is easy to check that the differences of all cycles of $\mathcal{C}$ give every value of Z_{8k+5}^* exactly once, which implies that $\{\mathcal{C}+i : i \in Z_{8k+5}\}$ forms a 4-$DCS(v,1)$. In addition, for every $C \in \mathcal{C}$, $\{C+i : i \in Z_{8k+5}\}$ is a 4-resolution class of the 4-$DCS(v,1)$, so we derive a 4-resolvable 4-$DCS(v,1)$. □

Lemma 2.3 *There exists a* 2*-resolvable* 4*-*$DCS(v,2)$ *for* $v \equiv 2 \pmod 4$.

Proof. Let the point set $X = Z_{4k+1} \cup \{\infty\}, k > 0$. A 2-resolvable 4-$DCS(v,2)$ contains $(4k+1)(2k+1)$ directed cycles and $4k+1$ 2-resolution classes. Let $\mathcal{C}$ consist of the following $2k+1$ directed cycles:

Part 1: Construct $2k-1$ directed cycles:

$$\begin{array}{l}\langle\ 0,\ 4k-3,\ 4k-1,\ 4k-2\ \rangle,\\ \langle\ 1,\ 4k-4,\ 4k,\ 4k-3\ \rangle,\\ \langle\ 2,\ 4k-5,\ 0,\ 4k-4\ \rangle,\\ \quad\vdots \qquad \vdots \qquad \vdots\\ \langle 2k-3,\ 2k,\ 2k-5,\ 2k+1\ \rangle,\\ \langle 2k-2,\ 2k-1,\ 2k-4,\ 2k\ \rangle.\end{array}$$

Part 2: Construct 2 directed cycles:

$$\langle\ \infty,\ 2k-2,\ 2k-1,\ 2k-3\ \rangle,$$

$$\langle\ \infty,\quad 4k-2,\quad 4k,\quad 4k-1\ \rangle.$$

The differences of all cycles of $\mathcal{C}$ give every value of $Z^*_{4k+1} \cup \{\pm\infty\}$ exactly twice, so $\{\mathcal{C}+i : i \in Z_{4k+1}\}$ forms a 4-$DCS(v,2)$. In addition, $\mathcal{C}$ is a 2-resolution class of the 4-$DCS(v,2)$ exactly, so $\mathcal{C}, \mathcal{C}+1, \ldots, \mathcal{C}+4k$ are all 2-resolution classes of the 4-$DCS(v,2)$. We get a 2-resolvable 4-$DCS(v,2)$. ∎

Lemma 2.4 *There exists a* 4*-resolvable* 4-$DCS(v,2)$ *for* $v \equiv 3 \pmod 4$.

Proof. (1) $v \equiv 3 \pmod 8$. Let the point set $X = Z_{8k+3}, k > 0$. A 4-resolvable 4-$DCS(v,2)$ contains $(8k+3)(4k+1)$ directed cycles and $4k+1$ 4-resolution classes. Let $\mathcal{C}$ consist of the following $4k+1$ directed cycles:

Part 1: Construct $2k$ directed cycles:

$$\begin{array}{cccccc}
\langle & 1, & 0, & 2, & 5 & \rangle, \\
\langle & 5, & 0, & 6, & 13 & \rangle, \\
\langle & 9, & 0, & 10, & 21 & \rangle, \\
& \vdots & & \vdots & \vdots & \\
\langle & 8k-7, & 0, & 8k-6, & 8k-14 & \rangle, \\
\langle & 8k-3, & 0, & 8k-2, & 8k-6 & \rangle.
\end{array}$$

Part 2: Construct $2k$ directed cycles:

$$\begin{array}{cccccc}
\langle & 3, & 0, & 4, & 9 & \rangle, \\
\langle & 7, & 0, & 8, & 17 & \rangle, \\
\langle & 11, & 0, & 12, & 25 & \rangle, \\
& \vdots & & \vdots & \vdots & \\
\langle & 8k-5, & 0, & 8k-4, & 8k-10 & \rangle, \\
\langle & 8k-1, & 0, & 8k, & 8k-2 & \rangle.
\end{array}$$

Part 3: Construct one directed cycle:

$$\langle 1,\ 0,\ 2,\ 3 \rangle.$$

It is easy to check that $\{\mathcal{C}+i : i \in Z_{8k+3}\}$ forms a 4-$DCS(v,2)$, and for every $C \in \mathcal{C}$, $\{C+i : i \in Z_{8k+3}\}$ is a 4-resolution class of the 4-$DCS(v,2)$ exactly. We get a 4-resolvable 4-$DCS(v,2)$.

(2) $v \equiv 7 \pmod 8$. Let the point set $X = Z_{8k+7}$, $k \geq 0$. A 4-resolvable 4-$DCS(v,2)$ contains $(8k+7)(4k+3)$ directed cycles and $4k+3$ 4-resolution classes. Let $\mathcal{C}$ consist of the following $4k+3$ directed cycles:

Part 1: Construct k directed cycles and repeat them twice:

$$\begin{array}{cccccc}
\langle & 1, & 0, & 2, & 4k+4 & \rangle, \\
\langle & 3, & 0, & 4, & 4k+4 & \rangle, \\
\langle & 5, & 0, & 6, & 4k+4 & \rangle, \\
& \vdots & & \vdots & \vdots & \\
\langle & 2k-3, & 0, & 2k-2, & 4k+4 & \rangle, \\
\langle & 2k-1, & 0, & 2k, & 4k+4 & \rangle.
\end{array}$$

Part 2: Construct k directed cycles and repeat them twice:

$$\begin{array}{llll}
\langle\ 2, & 0, & 1, & 4k+4\ \rangle, \\
\langle\ 4, & 0, & 3, & 4k+4\ \rangle, \\
\langle\ 6, & 0, & 5, & 4k+4\ \rangle, \\
\ \vdots & \vdots & \vdots & \\
\langle 2k-2, & 0, & 2k-3, & 4k+4\ \rangle, \\
\langle\ 2k, & 0, & 2k-1, & 4k+4\ \rangle.
\end{array}$$

Part 3: Construct 3 directed cycles:

$$\begin{array}{llll}
\langle 2k+1, & 0, & 2k+2, & 4k+4\ \rangle, \\
\langle 2k+1, & 0, & 2k+3, & 4k+4\ \rangle, \\
\langle 2k+2, & 0, & 2k+1, & 4k+4\ \rangle.
\end{array}$$

It is easy to check that $\{\mathcal{C}+i : i \in Z_{8k+7}\}$ forms a 4-$DCS(v,2)$. Furthermore, for every $C \in \mathcal{C}$, $\{C+i : i \in Z_{8k+7}\}$ is a 4-resolution class of the 4-$DCS(v,2)$. We derive a 4-resolvable 4-$DCS(v,2)$. ∎

3 Main result

Combining Lemmas 2.1–2.4, we obtain the main result:

Theorem 3.1 *There exists an α-resolvable 4-$DCS(v,\lambda)$ if and only if*

$$4 \mid \alpha v, \qquad \alpha \mid \lambda(v-1),$$

except for $v=4$, $\lambda \equiv 1 \pmod 2$.

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