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Double domination stable graphs upon edge removal

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Abstract

In a graph $G = (V(G), E(G))$, a vertex dominates itself and its neighbors. A subset S of $V(G)$ is a double dominating set if every vertex of $V(G)$ is dominated at least twice by the vertices of S. The double domination number of G is the minimum cardinality among all double dominating sets of G. We consider the effects of edge removal on the double domination number of a graph. We give a necessary and sufficient condition for a graph to have the property that the double domination number is unchanged upon the removal of any edge. Then we give a constructive characterization of trees having this property for every non-pendant edge.

1 Introduction

For many applications of graphs, such as network design, it is important to know the effect that a graph modification has on a graphical invariant. Much has been written about the graphs where a parameter (such as diameter or domination number) goes up or down whenever an arbitrary edge or vertex is removed or added. See Chapter 5 of [3] and Chapter 17 of [4] for surveys on the effects graph modifications have on the domination number. In this paper, we study the effects of edge removal on the double domination number of a graph. In particular, we consider graphs that are "stable" in the sense that the double domination number remains the same upon the removal of an arbitrary edge. We begin with some definitions.

In general we follow [1, 3]. Let $G = (V(G), E(G))$ be a graph. The *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and its *closed neighborhood* $N[v] = N(v) \cup \{v\}$. The *degree* of v, denoted by $d_G(v)$, is the size of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. An edge incident to a leaf is called a *pendant edge*. If D is a subset of $V(G)$, then the subgraph induced by D in G is denoted $G[D]$. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A *subdivided star* SS_t is a tree obtained from a star $K_{1,t}$ by replacing each edge uv of $K_{1,t}$ by a vertex w and edges uw and vw. For a vertex v in a rooted tree T, we denote by $C(v)$ and $D(v)$ the set of children and descendants, respectively, of v. The *maximal subtree* at v is the subtree of T induced by $D(v) \cup \{v\}$, and is denoted by T_v.

As introduced in [2], a subset S of $V(G)$ is a *double dominating set* (abbreviated DDS) of G if for every vertex $v \in V(G)$, $|N[v] \cap S| \geqslant 2$, that is, v is in S and has at least one neighbor in S or v is in $V(G) - S$ and has at least two neighbors in S. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality among all double dominating sets of G. If S is a DDS of G of size $\gamma_{\times 2}(G)$, then we call S a $\gamma_{\times 2}(G)$-*set*. Clearly, double domination is defined only for graphs without isolated vertices.

We first state some useful observations.

Observation 1 *Every DDS of a graph G contains all the leaves and support vertices of G.*

It is clear that removing an edge from a graph cannot decrease the double domination number, so we have the following.

Observation 2 *For a graph G and edge $uv \in E(G)$ such that G and $G - uv$ have no isolated vertex, $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G - uv)$.*

An edge $uv \in E(G)$ is a *critical edge* if $\gamma_{\times 2}(G - uv) > \gamma_{\times 2}(G)$, and a *stable edge* otherwise.

Definition 3 *A graph G is called $\gamma^-_{\times 2}$-stable if $\gamma_{\times 2}(G - e) = \gamma_{\times 2}(G)$ for all $e \in E(G)$, that is, every edge of G is stable.*

It follows from Definition 3 that if G is a $\gamma^-_{\times 2}$-stable graph, then the minimum degree of G, denoted $\delta(G)$, is at least two. It still may be important to know whether a graph has a certain measure of stability even if $\delta(G) = 1$, so we give the following definition. Let $F(G)$ be the set of non-pendant edges of a graph G.

Definition 4 *A graph G is called $\gamma^-_{\times 2}$-semi-stable if $\gamma_{\times 2}(G - e) = \gamma_{\times 2}(G)$ for all $e \in F(G)$, that is, every non-pendant edge of G is stable.*

Note that all $\gamma^-_{\times 2}$-stable graphs are $\gamma^-_{\times 2}$-semi-stable.

We begin with some examples in Section 2. Then in Section 3, we give a necessary and sufficient condition for $\gamma^-_{\times 2}$-stable graphs and provide a constructive characterization of $\gamma^-_{\times 2}$-semi-stable trees.

2 Examples

Note that non-trivial stars are vacuously $\gamma^-_{\times 2}$-semi-stable. We note also that the complete graph K_3 is not $\gamma^-_{\times 2}$-stable because removing any edge increases the double domination number. If $n \geq 4$, then there are two adjacent vertices that dominate $K_n - e$ for any edge $e \in E(K_n)$, so our next observation follows.

Observation 5 *The complete graph K_n, $n \geq 4$, is $\gamma^-_{\times 2}$-stable.*

Next we consider the complete bipartite graph $K_{r,s}$. As noted, the star $K_{1,s}$ is $\gamma^-_{\times 2}$-semi-stable. Consider $3 \leq r \leq s$. Then $\gamma_{\times 2}(K_{r,s}) = 4$, and removing any edge does not change the double domination number.

Observation 6 *The complete bipartite graph $K_{r,s}$, $3 \leq r \leq s$, is $\gamma^-_{\times 2}$-stable.*

Our final examples are paths P_n and cycles C_n. We first give the the double domination numbers of these graphs. The proofs are straightforward, so we omit them here.

Observation 7 *If $G = P_n$ with $n \geq 2$, then*

$$\gamma_{\times 2}(G) = \begin{cases} 2n/3 + 1 & \text{if } n \equiv 0 \,(mod\, 3) \\ 2\lceil n/3 \rceil & \text{otherwise.} \end{cases}$$

Observation 8 *If $G = C_n$ with $n \geq 3$, then*

$$\gamma_{\times 2}(G) = \begin{cases} 2\lfloor n/3 \rfloor + 1 & \text{if } n \equiv 1 \,(mod\, 3) \\ 2\lceil n/3 \rceil & \text{otherwise.} \end{cases}$$

Since removing an edge from the cycle C_n yields the path P_n, our next result follows directly from the Observations 7 and 8.

Proposition 9 *A cycle C_n is $\gamma^-_{\times 2}$-stable if and only if $n \equiv 2 \,(mod\, 3)$.*

Since for a nontrivial path P_n, $\delta(P_n) = 1$, no path is $\gamma^-_{\times 2}$-stable. We have seen that the stars P_2 and P_3 are $\gamma^-_{\times 2}$-semi-stable. Removing the center edge of a P_4 does not change the double domination number, so the P_4 is also $\gamma^-_{\times 2}$-semi-stable. Next we show that these are the only $\gamma^-_{\times 2}$-semi-stable paths.

Proposition 10 *A path P_n is $\gamma^-_{\times 2}$-semi-stable if ond only if $n \in \{2, 3, 4\}$.*

Proof. As noted, P_2, P_3, and P_4 are $\gamma^-_{\times 2}$-semi-stable. Assume that $n \geq 5$. To show that P_n for $n \geq 5$ is not $\gamma^-_{\times 2}$-semi-stable, it suffices to give a critical edge. Let e be the edge on the path such that $P_n - e = P_3 \cup P_{n-3}$. If $n \equiv 0\,(mod\,3)$, then Observation 7 implies that $\gamma_{\times 2}(P_n - e) = 3 + 2(n-3)/3 + 1 > 2n/3 + 1 = \gamma_{\times 2}(P_n)$. And for all other values of n, $\gamma_{\times 2}(P_n - e) = 3 + 2\lceil (n-3)/3 \rceil > 2\lceil n/3 \rceil = \gamma_{\times 2}(P_n)$. Hence, for each case, e is a critical edge of P_n as desired. □

3 $\gamma^-_{\times 2}$-stable Graphs and $\gamma^-_{\times 2}$-semi-stable Trees

In this section, we characterize the $\gamma^-_{\times 2}$-semi-stable trees. First we give a necessary and sufficient condition for $\gamma^-_{\times 2}$-stable graphs (respectively, $\gamma^-_{\times 2}$-semi-stable graphs).

Theorem 11 *A graph G with minimum degree $\delta(G) \geq 2$ (respectively, $\delta(G) = 1$) is $\gamma^-_{\times 2}$-stable (respectively, $\gamma^-_{\times 2}$-semi-stable) if and only if for every edge $uv \in E(G)$ (respectively, $uv \in F(G)$), there exists a $\gamma_{\times 2}(G)$-set D such that one of the following conditions holds:*

1) $u, v \in V(G) \setminus D$, or

2) $u, v \in D$ and neither u nor v is a leaf in $G[D]$, or

3) one of u and v is in D, say $u \in D$ and $v \notin D$, and $|N(v) \cap D| \geq 3$.

Proof. Let uv be any edge of $E(G)$ for which there is a $\gamma_{\times 2}(G)$-set D such that one of Conditions (1), (2) or (3) is verified. Then D is a DDS of $G - uv$ and so $\gamma_{\times 2}(G - uv) \leq |D| = \gamma_{\times 2}(G)$. Equality is obtained from Observation 2.

Now suppose that G is a $\gamma^-_{\times 2}$-stable graph. Let uv be any stable edge and D a $\gamma_{\times 2}(G - uv)$-set. Thus D is a $\gamma_{\times 2}(G)$-set. Clearly Condition (1) holds if $u, v \in V(G) - D$. If $u, v \in D$, then in $G - uv$, each of u and v has at least one neighbor in D. Hence each of u and v has in G at least two neighbors in $G[D]$, and so Condition (2) holds. Finally assume, without loss of generality, that $u \in D$ and $v \notin D$. Then v is dominated twice by D in $G - uv$ and so in G, v has at least three neigbhors in D. Hence Condition (3) follows. □

For the purpose of characterizing $\gamma^-_{\times 2}$-semi-stable trees, we define the family $\mathcal{H}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_p$ ($p \geq 1$) of trees, where $T_1 = P_2$, $T = T_p$, and, if $p \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following operations. (Note for $T_1 = P_2$, we consider one of the vertices as a support vertex.)

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_i.

- Operation $\mathcal{O}_2$: Attach a path $P_2 = xy$ by joining x to any support vertex of T_i.
- Operation $\mathcal{O}_3$: Attach a subdivided star SS_k ($k \geq 2$) of center vertex x by joining x to any vertex w of T_i, with the condition that if $k = 2$, then w belongs to a $\gamma_{\times 2}(T_i)$-set.

In the rest of the paper, we shall prove:

Theorem 12 *A nontrivial tree T is $\gamma^-_{\times 2}$-semi-stable if and only if $T \in \mathcal{H}$.*

We will use the following observations.

Observation 13 *Let T be a tree obtained from a nontrivial tree T' by adding a path $P_2 = uv$ attached by an edge vx to a support vertex x of T'. Then $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$.*

Proof. Let D be a $\gamma_{\times 2}(T)$-set. By Observation 1, D contains u, v, x and all leaves adjacent to x. Then $D - \{u, v\}$ is a DDS of T', and so $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 2$. If D' is any $\gamma_{\times 2}(T')$-set, then $D' \cup \{u, v\}$ is a DDS of T, and so, $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 2$. It follows that $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$. □

Observation 14 *Let T be a tree obtained from a nontrivial tree T' by adding a subdivided star SS_k ($k \geq 2$) of center vertex u, attached by an edge uw at any vertex w of T'. Then $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$.*

Proof. Let D be a $\gamma_{\times 2}(T)$-set. Then by Observation 1, D contains all vertices of SS_k except u (else replace u by w or a neighbor of w in T'). Hence $D \cap V(T')$ is a DDS of T' and $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 2k$. Also, if D' is any $\gamma_{\times 2}(T')$-set, then $D' \cup (V(SS_k) - \{u\})$ is a DDS of T' and so $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 2k$. It follows that $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$. □

Lemma 15 *If $T \in \mathcal{H}$, then T is a $\gamma^-_{\times 2}$-semi-stable tree.*

Proof. Let T be a tree of $\mathcal{H}$. Then T is obtained from a sequence T_1, T_2, ..., T_p ($p \geq 1$) of trees, where $T_1 = P_2$, $T = T_p$, and, if $p \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the three operations defined above. We use an induction on the number of operations performed to construct T. Clearly the property is vacuously true if $p = 1$. This establishes the basis case.

Assume now that $p \geq 2$ and that the result holds for all trees $T \in \mathcal{H}$ that can be constructed from a sequence of length at most $p - 1$, and let $T' = T_{p-1}$. By the inductive hypothesis, T' is a $\gamma^-_{\times 2}$-semi-stable tree, and hence every edge of $F(T')$ is stable in T'. Let T be a tree obtained from T' using one of the operations $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$. We consider the each of the three cases.

Case 1. T is obtained from T' by using Operation $\mathcal{O}_1$. Let y be the new added vertex. Clearly $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$ and $F(T) = F(T')$. Let $e = uv$ be any edge of $F(T)$. Since T' is $\gamma_{\times 2}^-$-semi-stable, by Theorem 11 there is a $\gamma_{\times 2}(T')$-set D' such that u, v satisfy Condition (1), (2) or (3). Since $D' \cup \{y\}$ is a $\gamma_{\times 2}(T)$-set for which u, v also satisfy Condition (1), (2) or (3), it follows that T is $\gamma_{\times 2}^-$-semi-stable.

Case 2. T is obtained from T' by using Operation $\mathcal{O}_2$, that is, the path xy is added to T' by the edge xz, where z is a support vertex of T'. Then by Observation 13, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$. Note that $F(T) = F(T') \cup \{xz\}$. Let $e = uv$ be any edge of $F(T)$. If $e \in F(T')$, then, as previously, since T' is $\gamma_{\times 2}^-$-semi-stable, there is a $\gamma_{\times 2}(T')$-set D' such that u, v satisfy Condition (1), (2) or (3). Then $D' \cup \{x, y\}$ is a $\gamma_{\times 2}(T)$-set for which u, v also satisfy Condition (1), (2) or (3). If $e = xz$, then y, x, z and all the leaf neighbors of z belong to every $\gamma_{\times 2}(T)$-set, and so x, z satisfy Condition (2) of Theorem 11. In each case, T is $\gamma_{\times 2}^-$-semi-stable.

Case 3. T is obtained from T' by using Operation $\mathcal{O}_3$, that is, the subdivided star SS_k ($k \geq 2$) of center vertex x is joined to T' by edge xw, where $w \in V(T')$. By Observation 14, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$. Let $\{x_1, x_2, ..., x_k\}$ be the set of support vertices of SS_k. Then $F(T) = F(T') \cup \{xw, xx_1, ..., xx_k\}$. Let $e = uv$ be any edge of $F(T)$. If $e \in F(T')$, then since T' is $\gamma_{\times 2}^-$-semi-stable, let D_{uv} denote a $\gamma_{\times 2}(T')$-set for which u, v satisfy Condition (1), (2) or (3) of Theorem 11. Then such a condition remains satisfied for u and v with respect to the $\gamma_{\times 2}(T)$-set $D_{uv} \cup (V(SS_k) - \{x\})$. Now assume that $e = xx_i$ for some i, where $1 \leq i \leq k$. If $k \geq 3$, then for every $\gamma_{\times 2}(T)$-set D, any edge xx_i satisfies either Condition (2) (if $x \in D$) or Condition (3) (if $x \notin D$) of Theorem 11. If $k = 2$, then by the construction, w belongs to a $\gamma_{\times 2}(T')$-set, say D'. Thus $D' \cup (V(SS_k) - \{x\})$ is a $\gamma_{\times 2}(T)$-set that contains at three neighbors of x, that is, xx_i satisfies Condition (3) of Theorem 11. Finally, if $e = wx$, then $\gamma_{\times 2}(T - wx) = \gamma_{\times 2}(T') + 2k = \gamma_{\times 2}(T)$. It follows that T is $\gamma_{\times 2}^-$-semi-stable. $\square$

Lemma 16 *If T is a $\gamma_{\times 2}^-$-semi-stable tree, then $T \in \mathcal{H}$.*

Proof. We use an induction on the order n of T. Clearly if $n = 2$, then T is the path P_2 that belongs to $\mathcal{H}$. Assume that every $\gamma_{\times 2}^-$-semi-stable tree T' of order $2 \leq n' < n$ is in $\mathcal{H}$. Let T be $\gamma_{\times 2}^-$-semi-stable tree of order n. If T is a $\gamma_{\times 2}^-$-semi-stable star $K_{1,k}$, then T can be obtained from $T_1 = P_2$ by $k - 1$ applications of Operation $\mathcal{O}_1$. If T is a double star, then T is obtained from $T_1 = P_2$ by using Operation $\mathcal{O}_2$ and zero or more applications of Operation $\mathcal{O}_1$. Hence both stars and doublestars are in $\mathcal{H}$. Thus we may assume that T has diameter at least four.

If any support vertex, say y, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to y. Clearly $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$, $F(T) = F(T')$ and T' is $\gamma_{\times 2}^-$-semi-stable. By induction on T', we have $T' \in \mathcal{H}$. It follows that $T \in \mathcal{H}$ because it is obtained from T' by using Operation $\mathcal{O}_1$. Thus we can assume henceforth that every support vertex is adjacent to exactly one leaf.

We now root T at leaf r of a longest path. Let v be a vertex at distance $\mathrm{diam}(T)-1$ from r on a longest path starting at r, and let u be the child of v on this path. Since $\mathrm{diam}(T) \geq 4$, let w, z be the parents of v and w, respectively. Clearly $d_T(v) = 2$. We consider the following cases.

Case 1. w is a support vertex. Let $T' = T - \{u, v\}$. Then $F(T') = F(T) - \{vw\}$. By Observation 13, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$. Suppose now that T' is not $\gamma^-_{\times 2}$-semi-stable. Then there is an edge $xy \in F(T')$ such that $\gamma_{\times 2}(T' - xy) > \gamma_{\times 2}(T')$. Note that the removing of xy from T' (respectively, T) provides two nontrivial subtrees $T'(x)$ and $T'(y)$ (respectively, $T(x)$ and $T(y)$) containing x and y, respectively. Without loss of generality, we can assume that $T'(y) = T(y)$, and so $T'(x)$ is a subtree of $T(x)$. Then $\gamma_{\times 2}(T' - xy) = \gamma_{\times 2}(T'(x)) + \gamma_{\times 2}(T'(y))$ and $\gamma_{\times 2}(T - xy) = \gamma_{\times 2}(T(x)) + \gamma_{\times 2}(T(y))$. It follows by Observation 13 that

$$\begin{aligned}
\gamma_{\times 2}(T - xy) &= \gamma_{\times 2}(T(x)) + \gamma_{\times 2}(T(y)) \\
&= \gamma_{\times 2}(T'(x)) + 2 + \gamma_{\times 2}(T'(y)) \\
&= 2 + \gamma_{\times 2}(T' - xy) \\
&> 2 + \gamma_{\times 2}(T') = \gamma_{\times 2}(T),
\end{aligned}$$

contradicting the fact that T is $\gamma^-_{\times 2}$-semi-stable. Therefore T' is $\gamma^-_{\times 2}$-semi-stable and so by induction on T', we have $T' \in \mathcal{H}$. Consequently, T is obtained from T' by using Operation $\mathcal{O}_2$, and hence $T \in \mathcal{H}$.

Case 2. w is not a support vertex but $d_T(w) \geq 3$. Our choice of v implies that T_w is a subdivided star SS_k, where $k = d_T(w) - 1 \geq 2$. Let $T' = T - T_w$. By Observation 14, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$. Now if T' has order two, then $T = SS_{k+1}$. Thus $T \in \mathcal{H}$, because it is obtained from $T_1 = P_2$ by using Operation $\mathcal{O}_3$. Thus we can assume that T' has order $n' \geq 3$. Using the same argument used in Case 1, it can be shown that T' is $\gamma^-_{\times 2}$-semi-stable, and so by induction on T', we have $T' \in \mathcal{H}$. If $k = 2$, then since $\gamma_{\times 2}(T - vw) = \gamma_{\times 2}(T)$, it follows, without loss of generality, that there is a $\gamma_{\times 2}(T - vw)$-set D such that $w \notin D$ and $z \in D$. Hence z belongs to a $\gamma_{\times 2}(T')$-set, for instance, $D \cap V(T')$ is such a set. Consequently, $T \in \mathcal{H}$ since it can be obtained from T' by using Operation $\mathcal{O}_3$.

Case 3. $d_T(w) = 2$. We shall show that this case cannot occur. Every edge of $F(T)$ satisfies one of the three conditions of Theorem 11, in particular, since $d_T(w) = 2$, edge wz satisfies either Condition (2) or (3). Assume that wz satisfies Condition (2). Then there is a $\gamma_{\times 2}(T)$-set S such that $w, z \in S$ and z has another neighbor besides w in S. But then $S \setminus \{w\}$ is a DDS of T with cardinality less than $\gamma_{\times 2}(T)$, a contradiction. Thus wz satisfies Condition (3), that is, there is a $\gamma_{\times 2}(T)$-set D such that $w \in D$, $z \notin D$ and z has at least three neighbors in D. Hence $d_T(z) \geq 3$. Now let $T' = T - \{u, v, w\}$. Then $D \cap V(T')$ is a DDS of T' implying that $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 3$. The equality follows from the fact that every $\gamma_{\times 2}(T')$-set can be extended to a DDS of T by adding the set $\{u, v, w\}$. If z is in a $\gamma_{\times 2}(T')$-set D', then $D' \cup \{u, v\}$ would be a DDS of T smaller than D. Hence z belongs to no $\gamma_{\times 2}(T')$-set. It follows that z is not a support vertex.

Since we have handled Cases 1 and 2 of this proof, any child, say w', of z that is not a support vertex must have the same properties w has. Hence $d_T(w') = 2$ and since z is not in any $\gamma_{\times 2}(T')$-set, w' is in a $\gamma_{\times 2}(T')$-set S' and $S = S' \cup \{w, v, u\}$ is a $\gamma_{\times 2}(T)$-set. But then $S \setminus \{w, w'\}) \cup \{z\}$ is a DDS of T with cardinality less than $\gamma_{\times 2}(T)$, a contradiction. Hence every child $b \neq w$ in $C(z)$ is a support vertex, that is, T_z is a tree obtained from a star $K_{1,t}$ ($t \geq 2$) by subdividing one edge twice and the remaining edges once.

Now let us consider the edge vw. Clearly, since T is $\gamma^-_{\times 2}$-semi-stable and $d_T(w) = d_T(v) = 2$, vw satisfies Condition (2) of Theorem 11. Thus there is a $\gamma_{\times 2}(T)$-set S such that $\{w, u, v, z\} \subseteq S$. But since every vertex in $C(z) \setminus \{w\}$ is a support vertex, $C(z) \subset S$, and hence $S \setminus \{w\}$ is a DDS of T smaller than S, a contradiction. This completes the proof. □

According to Lemmas 15 and 16, we have proven Theorem 12.

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