AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **47** (2010), Pages 189–195

Extraconnectivity of hypercubes (II)*

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Abstract

The g-extraconnectivity $\kappa_g(G)$ of a simple connected graph G is the minimum cardinality of a subset of $V(G)$, if any, whose deletion disconnects G in such a way that every remaining component has at least g vertices. In this paper, we determine $\kappa_g(Q_n)$ for $n+2 \leq g \leq 2n$, $n \geq 4$, where Q_n denotes the n-dimensional hypercube.

1 Introduction

In a network, traditional connectivity is an important measure since it can correctly reflect the fault tolerance of network systems with few processors. However, it always underestimates the resilience of large networks. There is a discrepancy because the occurrence of events which would disrupt a large network after a few processor or link failures is highly unlikely. Thus the disruption envisaged occurs in a worst case scenario. To overcome this shortcoming, Harary introduced the concept of conditional connectivity [5]. One special case of the conditional connectivity is the extraconnectivity [3, 4], which has received a lot of attention in recent years (see the nice survey [6] and the references therein), and is the focus of this paper.

* The research is supported by NSFC (No. 10671165) and SRFDP (No. 20050755001).
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For a connected graph G, a vertex set F is a *vertex cut* of G if $G - F$ is disconnected. A *g-extra cut* of G is a vertex cut F of G such that every component of $G - F$ has at least g vertices. The cardinality of the minimum g-extra cut of G is the *g-extraconnectivity* of G, denoted by $\kappa_g(G)$. A minimum g-extra cut of G is abbreviated as a *κ_g-cut* of G. The traditional connectivity of G is exactly $\kappa_1(G)$. It should be noted that the above definition of $\kappa_g(G)$ is different from the one given in [4], which requires every component to have *more than* g vertices. However, the essence is the same. We choose the current definition in order to facilitate the statements and the usage of the results in [12]. To the present time, there is no known polynomial-time algorithm for finding $\kappa_g(G)$ even for $g = 2$. We suspect that the problem of determining $\kappa_g(G)$ $(g \geq 2)$ is NP-hard.

The hypercube is a well known topology for computer networks which has attracted a lot of attention over the past four decades; see [2, 3] and [7]–[13] for the details. In an n-dimensional hypercube Q_n, each vertex can be represented by a binary string of length n, known as its coordinate. Two vertices are adjacent in Q_n if and only if their coordinates differ at exactly one position. The hypercube Q_n has 2^n vertices and $n2^{n-1}$ edges. It is known that $\kappa_1(Q_n) = n$, $\kappa_2(Q_n) = 2n - 2$ for $n \geq 3$, and $\kappa_3(G) = 3n-5$ for $n \geq 6$ (see [3, 9, 13]). In [11], the authors proved that if $n \geq 4$, then $\kappa_g(Q_n) = gn - 2(g-1) - \binom{g-1}{2} = -\frac{1}{2}g^2 + (n - \frac{1}{2})g + 1$ for $1 \leq g \leq n-3$, and $\kappa_g(Q_n) = \frac{n(n-1)}{2}$ for $n-2 \leq g \leq n+1$. In this paper we further determine $\kappa_g(Q_n)$ for $n+2 \leq g \leq 2n$. To be more precise, if $n \geq 5$, then $\kappa_g(Q_n) = -\frac{1}{2}g^2+(2n-\frac{3}{2})g-(n^2-2)$ for $n+2 \leq g \leq 2n-4$, and $\kappa_g(Q_n) = (n-1)(n-2)$ for $2n-3 \leq g \leq 2n$.

Following Latifi [8], for a fixed index $i \in \{1, 2, \ldots, n\}$, we can express Q_n as $D_0 \bigodot D_1$, where D_0 (respectively D_1) is the $(n-1)$-dimensional subcube of Q_n (simply called $(n-1)$-subcube) induced by those vertices whose i^{th} coordinates are 0 (respectively 1). Sometimes we use $X^{i-1}0X^{n-i}$ and $X^{i-1}1X^{n-i}$ to denote D_0 and D_1, respectively, where $X \in Z_2$. Notice that there is a perfect matching M between D_0 and D_1. For each vertex v, we call the vertex v' which is matched to v by M as the *out neighbor* of v. For a subgraph A of Q_n which is completely contained in D_0 (or D_1), denote by A' the subgraph of Q_n induced by $\{v' \mid v \in V(A)\}$. For vertex $u \in V(Q_n)$, let $u_{i_1 i_2 \ldots i_s}$ denote the vertex whose coordinate differs from that of u at the $i_1^{th}, i_2^{th}, \ldots, i_s^{th}$ positions. For $u \in V(G)$, denote by $N_G(v)$ the set of neighbors of v in G; for a subgraph A of G, denote by $N_G(A) = (\bigcup_{v \in V(A)} N_G(v)) \setminus V(A)$ the set of neighbors of A in G; let $C_G(A) = N_G(A) \cup V(A)$. For a vertex set $S \subseteq V(G)$, denote by $G[S]$ the subgraph of G induced by S. We follow Bondy and Murty [1] for any other terminology not given here.

2 Minimum neighborhood problem

Given a graph G and an integer g with $1 \leq g \leq |V(G)|$, let $\theta_G(g)$ denote the minimum number of vertices adjacent to a set of g vertices of G. The minimum neighborhood problem (MNP) is to determine $\theta_G(g)$. The MNP not only is of interest in its own right, but the result obtained is useful in system-level fault diagnosis and in the analysis of fault tolerance of interconnection networks; see [7, 12]. Somani et al. [10] and Yang and Meng [11] presented the following result for an n-dimensional hypercube Q_n:

Theorem 2.1. $\theta_{Q_n}(g) = -\frac{1}{2}g^2 + (n - \frac{1}{2})g + 1$ *for* $1 \le g \le n + 1$.

Recently, X. Yang et al. [12] showed the following result:

Theorem 2.2. $\theta_{Q_n}(g) = -\frac{1}{2}g^2 + (2n - \frac{3}{2})g - (n^2 - 2)$ *for* $n + 2 \le g \le 2n$.

We shall show in the next section that $\kappa_g(Q_n) = -\frac{1}{2}g^2 + (2n - \frac{3}{2})g - (n^2 - 2)$ for $n + 2 \le g \le 2n$.

3 Extraconnectivity of hypercube

X. Yang et al. in [12] also reported some properties of the following two families of quadratic functions which were defined as

$$\begin{aligned} p_n(x) &= -\frac{1}{2}x^2 + (n - \frac{1}{2})x + 1; \\ q_n(x) &= -\frac{1}{2}x^2 + (2n - \frac{3}{2})x - (n^2 - 2). \end{aligned}$$

Property 1. $(x - 2) + q_n(x - 1) = q_{n+1}(x)$.

Property 2. $x + q_n(x) \ge q_{n+1}(x)$ for $x \le 2n + 1$.

Property 3. $2n + q_n(2n) = q_{n+1}(2n + 1) + 1 = q_{n+1}(2n + 2) + 2$.

Property 4. $p_n(x) + p_n(a - x) \ge q_{n+1}(a)$ for $0 < a \le 2n + 2, \frac{a}{2} \le x \le n + 1$.

Property 5. $p_n(x) + q_n(a - x) \ge q_{n+1}(a)$ for $a \ge n + 4, 2 \le x \le a - n - 2$.

Remark 3.1. *$q_n(x)$ is increasing when $x \le 2n - 3$ and $q_n(2n - 3) = q_n(2n) = (n - 1)(n - 2) < q_n(2n - 2) = q_n(2n - 1)$.*

Combining Theorem 2.2 with the definition of $q_n(x)$, we have the following corollary.

Corollary 3.2. *Assume that A is a subgraph of Q_n, $n \ge 5$. If $|V(A)| = g$, then $|N_{Q_n}(A)| \ge -\frac{1}{2}g^2 + (2n - \frac{3}{2})g - (n^2 - 2) = q_n(g)$ for $n + 2 \le g \le 2n$.*

Lemma 3.3 ([12])**.** *Let G be a graph with n vertices. Then $\theta_G(g + 1) \ge \theta_G(g) - 1$ for $1 \le g \le n - 1$.*

Lemma 3.4 ([11])**.** *Assume that $n \ge 4$, $B \subseteq Q_n$ and $|V(B)| \ge n$. If $|V(Q_n) \setminus C_{Q_n}(B)| \ge n$, then $|N_{Q_n}(B)| \ge n(n - 1)/2$.*

Lemma 3.5. *Assume that* $n \geq 5$ *and* $A \subseteq Q_n$. *If* $|V(A)| \geq 2n$ *and* $|V(Q_n) - C_{Q_n}(A)| \geq |V(A)|$, *then* $|N_{Q_n}(A)| \geq q_n(2n)$.

Proof. By induction on n. If $n = 5$, then $q_n(2n) = 12$. By Lemma 3.4, we have $|N_{Q_n}(A)| \geq n(n-1)/2 = 10$. Since $|V(Q_n) - C_{Q_n}(A)| \geq |V(A)|$, we have $10 \leq |V(A)| \leq 11$ (note that $|V(Q_n)| = 2^5 = 32$). If $|V(A)| = 10$, then $|N_{Q_n}(A)| \geq q_n(2n)$ by Theorem 2.2. If $|V(A)| = 11$, then since $|N_{Q_n}(A)| \geq q_n(2n)$, we have $|N_{Q_n}(A)| \geq q_n(2n) - 1 = 11$ by Lemma 3.3. But $32 = |V(Q_n)| = |V(A)| + |N_{Q_n}(A)| + |V(Q_n) - C_{Q_n}(A)| \geq 11 + 11 + 11$, a contradiction. Assume that the result holds for all $n < M$. We next show that the result is true for $n = M$.

Suppose $|N_{Q_n}(A)| \leq q_n(2n) - 1 = (n-1)(n-2) - 1$. We shall derive a contradiction. Let $F = N_{Q_n}(A)$, $F_0 = F \cap V(D_0)$ and $F_1 = F \cap V(D_1)$ (assume $Q_n = D_0 \bigodot D_1$). Then either $|F_0| < \frac{(n-1)(n-2)-1}{2}$ or $|F_1| < \frac{(n-1)(n-2)-1}{2}$ since $(n-1)(n-2)-1$ is odd. Without loss of generality, we assume $|F_0| < \frac{(n-1)(n-2)-1}{2}$.

Since $|F_0| < \frac{(n-1)(n-2)-1}{2}$, $D_0 - F_0$ contains at most one component with order at least $n-1$, by Lemma 3.4. Assume that $G_1, G_2, \ldots, G_s$ are all components of $D_0 - F_0$ such that $|V(G_i)| < n-1$ and let G^* denote the subgraph $D_0 - (F \cup V(G_1 \cup \cdots \cup G_s))$.

Claim 1. $\sum_{i=1}^{s} |V(G_i)| < n - 1$.

Note that $2^{n-1} - |F_0| > 4(n-1)$ for $n \geq 6$. If $G^* = \emptyset$, then we can take some integers $i_1, i_2, \ldots, i_t$ such that $|V(\cup_{j=1}^{t} G_{i_j})| \geq n-1$ and $|V(\cup_{i=1}^{s} G_i) \setminus V(\cup_{j=1}^{t} G_{i_j})| \geq n-1$; then we have $|F_0| \geq \frac{(n-1)(n-2)}{2}$ by Lemma 3.4, a contradiction. That is, $G^* \neq \emptyset$ and $|V(G^*)| \geq n-1$. It can be seen that $\sum_{i=1}^{s} |V(G_i)| < n-1$ and G^* is connected by Lemma 3.4 and the assumption $|F_0| < \frac{(n-1)(n-2)-1}{2}$.

Suppose $\sum_{i=1}^{s} |V(G_i)| = N$. If $N = 0$, then $A \subset D_1 - F_1$ since $|V(Q_n) - C_{Q_n}(A)| \geq |V(A)|$. If $|V(D_1 - F_1 \cup V(A))| \geq 2(n-1)$, then $|F_1| \geq q_{n-1}(2(n-1))$ by induction. Note that $V(A') \subset F_0$ and $|V(A')| \geq 2(n-1)$, that is, $|F_1| + |F_0| \geq q_{n-1}(2(n-1)) + 2(n-1) = q_n(2n) + 2$ by Property 3, a contradiction. If not, $|V(D_1 - F_1 \cup V(A))| < 2(n-1)$, then we have $|V(A')| = |V(A)| = 2^{n-1} - |F_1| - |V(D_1 - F_1 \cup V(A))| \leq |F_0|$, that is, $|F_1| + |F_0| \geq 2^{n-1} - |V(D_1 - F_1 \cup V(A))| > 2^{n-1} - 2(n-1) > q_n(2n) = n^2 - 3n + 2$ for $n \geq 6$, a contradiction. Therefore, we always assume $N \neq 0$ in the following discussion.

Assume that $C_1, C_2, \ldots, C_k$ are the components of $D_1 - F_1$ with order less than $2(n-1)$, and $C_{k+1}, \ldots, C_l$ the components of $D_1 - F_1$ with order at least $2(n-1)$, respectively.

Claim 2. Each of the components $C_{k+1}, \ldots, C_l$ is connected to G^* and $|V((\cup_{i=1}^{k} C_i))| < 2(n-1) - N$.

If $D_1 - F_1$ has more than one component with order at least $2(n-1)$, assume that one of them is A_1 such that it is disconnected to G^*; then $|N_{Q_{n-1}}(A_1)| \geq q_{n-1}(2(n-1)) = n^2 - 5n + 6 \leq |F_1|$ by induction. If $N \geq 2$, then $|F_0| \geq 2n - 4$, that is, $|F_1| + |F_0| \geq q_{n-1}(2(n-1)) + 2n - 4 \geq q_n(2n)$, a contradiction. Thus A_1 is connected to G^*. If $N = 1$ and $|F_0| < 2n-4$, but $|V(A_1')| \geq 2(n-1) > |F_0|+1$, that is, A_1 is connected to G^*. Note that $|F_0| \geq p_{n-1}(N)$ and $|F_1| \geq |N_{D_1}(\cup_{i=1}^k C_i)|$. Then $|V(\cup_{i=1}^k C_i)| < 2(n-1)-N$ since $p_{n-1}(N)+q_{n-1}(2(n-1)-N) \geq q_n(2(n-1)) > q_n(2n)$ and Remark 3.1.

Suppose that $D_1 - F_1$ has exactly one component with order at least $2(n-1)$, say A_1. If $|D_1 - F_1 \cup V(A_1)| \geq 2(n-1)$, then by an argument similar to the above paragraph, we have that A_1 is connected to G^*. Thus $|D_1 - F_1 \cup V(A_1)| < 2(n-1) - N$ since $p_{n-1}(N) + q_{n-1}(2(n-1) - N) \geq q_n(2(n-1)) > q_n(2n)$. Note that A_1 is disconnected to G^*; then $A_1' \subset F_0 \cup (\cup_{i=1}^s G_i)$, that is, $|F_0| + N \geq |V(A_1)| > 2^{n-1} - |F_1| - (2(n-1) - N)$. It is easy to see that $|F_1| + |F_0| > 2^{n-1} - 2(n-1) > q_n(2n) = n^2 - 3n + 2$ for $n \geq 6$, a contradiction.

Claim 2 implies that $A \subset (\cup_{i=1}^s G_i) \cup (\cup_{i=1}^k C_i)$, that is, $|A| < N + (2(n-1) - N) < 2n$, a contradiction.

Assume that $D_1 - F_1$ has no component with order at least $2(n-1)$. Without loss of generality, we can assume that $C_1, \ldots, C_{s_1}$ are the components which are disconnected to G^* of $\{C_1, \ldots, C_s\}$. Clearly, $2n - N \leq \sum_{i=1}^{s_1} |V(C_i)|$ since $|V(A)| \geq 2n$ and $|V(Q_n) - C_{Q_n}(A)| \geq |V(A)| \geq 2n$; and $\sum_{i=1}^{s_1} |V(C_i)| < F_0 + N$ since $C_1, \ldots, C_{s_1}$ are disconnected to G^*. If $\sum_{i=s_1+1}^{s} |V(C_i)| < 2(n-1)$, then $2^{n-1} - |F_1| - |F_0| - N < 2(n-1)$, that is $|F_0| + |F_1| > 2^{n-1} - 2(n-1) - N > (n-1)(n-2) = q_n(2n)$ for $n \geq 6$, a contradiction. Thus $2n - N \leq \sum_{i=1}^{s_1} |V(C_i)|$ and $\sum_{i=s_1+1}^{s} |V(C_i)| \geq 2(n-1)$. Since $|V(Q_n) - C_{Q_n}(A)| \geq |V(A)|$, we have $A \subset (C_1 \cup \cdots \cup C_{s_1}) \cup (V(D_0 - V(G^*) \cup F_0))$. Noting that $p_{n-1}(N) + q_{n-1}(2(n-1) - N) > q_n(2n)$ and

$$q_n(2(n-1) - N) \leq q_n(2(n-1) - N + 1) \leq \cdots \leq q_n(2(n-1) - 1),$$

we have $\sum_{i=1}^{s_1} |V(C_i)| \geq 2(n-1)$, that is, $|F_1| \geq q_{n-1}(2(n-1)) = (n-2)(n-3)$ by induction. If $|F_0| \geq 2n - 4$, then we have $|F_0| + |F_1| > q_n(2n)$, a contradiction. If $|F_0| < 2n - 4$, that is, $N \leq 1$, then since $2(n-1) \leq \sum_{i=1}^{s_1} |V(C_i)|$ and $V({C_1}' \cup \cdots \cup {C_{s_1}}') \subset V(D_0 - V(G^*) \cup F_0$ since C_i is disconnected to G^*, we have $2(n-1) \leq 2n-4+1$, a contradiction.

Combining the above arguments, we complete the proof. □

Corollary 3.6. *For any connected subgraph A of Q_n, if $|V(A)| \geq 2n$ and $|V(Q_n) - C_{Q_n}(A)| \geq 2n$, then $|N_{Q_n}(A)| \geq q_n(2n)$.*

Lemma 3.7 ([13])**.** *Any two vertices in $V(Q_n)$ have exactly two common neighbors for $n \geq 3$ if they have any.*

Lemma 3.8 ([11])**.** *Let $0 \leq g \leq n$, $A \subseteq Q_n$ and $A \cong K_{1,g}$. Then $|N_{Q_n}(A)| = (g+1)n - 2g - \binom{g}{2}$.*

Lemma 3.9 ([12])**.** *Let $A = Q_n[\{u = 0^n, u_1, \ldots, u_n, u_{12}, u_{13}, \ldots, u_{1g-n}\}], n + 2 \leq g \leq 2n$. Then $|N_{Q_n}(A)| = q_n(g)$.*

In the following, we say that a graph G has *property $\mathcal{P}_g$* if each component of G has size at least g.

Lemma 3.10. *Let* $A = Q_n[\{u = 0^n, u_1, \ldots, u_n, u_{12}, u_{13}, \ldots, u_{1g-n}\}], n+2 \leq g \leq 2n$. *Then* $N_{Q_n}(A)$ *is a* κ_g*-cut of* Q_n $|N_{Q_n}(A)| = q_n(g)$.

Proof. By Lemma 3.9 and $|V(A)| = g$, it is sufficient to show that $Q_n - C_{Q_n}(A)$ has property $\mathcal{P}_g$.

Note that $|Q_n - C_{Q_n}(A)| = 2^n - q_n(g) - |A| > g$ for $n+2 \leq g \leq 2n$ and $n \geq 5$. If $Q_n - C_{Q_n}(A)$ is connected, then we can see that $Q_n - C_{Q_n}(A)$ has property $\mathcal{P}_g$. Let $a = (x_1x_2 \ldots x_n), b = (y_1y_2 \ldots y_n)$ be two vertices of $Q_n - C_{Q_n}(A)$. Without loss of generality, we can assume $x_{i_1} = 0, x_{i_2} = 0, \ldots, x_{i_l} = 0$ and other coordinates of a are equal to 1. Similarly, assume $y_{j_1} = 0, y_{j_2} = 0, \ldots, y_{j_k} = 0$ and other coordinates of b are equal to 1. It is not difficult to see that $P = aa_{i_1}a_{i_1i_2}\ldots a_{i_1i_2\ldots i_l}$ (clearly, $a_{i_1i_2\ldots i_l} = 1^n$) and $Q = bb_{j_1}b_{j_1j_2}\ldots b_{j_1j_2\ldots j_k}$ (clearly, $b_{j_1j_2\ldots j_k} = 1^n$) form a path of $Q_n - C_{Q_n}(A)$ between a and b (see Lemma 2.6 of [11] for the detail.). Thus $Q_n - C_{Q_n}(A)$ is connected. □

Theorem 3.11. *If* $n \geq 5$*, then* $K_g(Q_n) = q_n(g)$ *for* $n+2 \leq g \leq 2n-4$*, and* $K_g(Q_n) = q_n(2n)$ *for* $2n-3 \leq g \leq 2n$.

Proof. We first show that the theorem is true for $n+2 \leq g \leq 2n-4$. By Lemma 3.10, we have that $K_g(Q_n) \leq q_n(g)$. Let F be the κ_g-cut of Q_n. Assume that A ($|A| \geq g$) is the smallest component of $Q_n - F$. Clearly, $|N_{Q_n}(A)| \geq q_n(g)$ if $n+2 \leq |A| \leq 2n-4$ by Corollary 3.2, and $|N_{Q_n}(A)| \geq q_n(g)$ if $|A| \geq 2n-3$ by Corollary 3.6 and Remark 3.1. Therefore, $K_g(Q_n) = q_n(g)$ for $n+2 \leq g \leq 2n-4$.

From Remark 3.1, Corollary 3.6 and Lemma 3.9, we have that $K_g(Q_n) = q_n(2n)$ for $2n-3 \leq g \leq 2n$. □

Acknowledgements

The authors wish to thank the anonymous referees for their very useful suggestions.

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(Received 24 Sep 2009; revised 8 Mar 2010)