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Balanced independent sets in hypercubes

Mark Ramras

Department of Mathematics
Northeastern University
Boston, MA 02115
U.S.A.
ramras@neu.edu

Abstract

The n-dimensional hypercube Q_n has many maximal independent sets of vertices. We study the cardinality of those maximal independent sets which are *balanced*, i.e. exactly half of whose vertices have even weight, obtaining both upper and lower bounds for the maximum value. For $n \leq 7$ we obtain the exact value. For all odd n, we conjecture that the exact value is $2^{n-1} - \binom{n-1}{(n-1)/2}$. We also consider balanced independent subsets of the two middle levels of Q_n when n is odd and prove that a simply constructed balanced maximal independent set for that subgraph has the maximum cardinality of any balanced independent set. Its cardinality is $2 \cdot \binom{n-1}{(n-3)/2}$. Finally, one way in which balanced maximal independent sets in Q_n arise is from binary linear codes, and we use Hamming codes to find balanced maximal independent sets of *small* cardinality.

1 Preliminaries

Definition 1 *A set S of vertices in a graph G is* **independent** *if no two vertices in S are adjacent. An independent set of vertices S is* **maximal** *if S is not properly contained in any larger independent set in G.*

Definition 2 *By the* **neighborhood** *of a subset S of $V(G)$ we mean the set of vertices adjacent to at least one vertex in S. We denote this by $N(S)$. By the* **closed neighborhood** *of S we mean the subset $S \cup N(S)$, which is denoted by $\overline{N(S)}$.*

Maximal independent sets are precisely the independent dominating sets. These are the independent sets S whose closed neighborhood is the entire vertex set of G. They have been studied by many (see for example Haynes, Hedetniemi, and Slater [7], Zelinka [13], Arumugam and Kala [2], Jha [8]); in particular various people have been interested in determining the **independent domination number**, i.e., the **minimum** cardinality of an independent dominating set (see for example Allan and Laskar [1], Jha [8], Harant, Pruchnewski, Voigt [6], and Fischerman, Volkmann, Zverovich [4]). A subset of the vertices of a bipartite graph is called **balanced** if it has exactly half its elements in each of the partite sets. Our aim in this paper is to study balanced independent sets in hypercubes, especially to determine bounds on the maximum cardinality of balanced independent sets.

We first present a construction which produces maximal independent sets in bipartite graphs. We omit the proof, which is easy.

Proposition 1 *Let G be a bipartite graph with bipartition (X, Y). For any $X' \subset X$, let $Y' = Y \setminus N(X')$. Define $X'' = \{x \in X | N(x) \subseteq N(X')\}$, and let $S = X'' \cup Y'$.*

(1) *The set S is a maximal independent set in G and $X' \subset S$.*

(2) *If T is any maximal independent set in G, and $T = X' \cup Y'$, where $X' \subset X$ and $Y' \subset Y$, then $Y' = Y \setminus N(X')$ and $X' = X \setminus N(Y')$. Hence $X'' = X'$.*

Corollary 1 *For any subset C of X, C is the "X-part" of some maximal independent subset S of G (i.e. $C = X \cap S$) $\iff X \setminus C = N(D)$ for some subset D of Y.*

Definition 3 *By Q_n we mean the n-dimensional hypercube, i.e. the graph whose vertices are the binary strings of length n, in which two strings are adjacent if they differ in exactly one position. It will often be convenient to use an alternative description: the vertices are all the 2^n subsets of $\{1, 2, \dots, n\}$ and two subsets x and y are considered adjacent when their symmetric difference $x\Delta y$ has cardinality 1.*

Definition 4 *When a vertex v of Q_n is regarded as a subset of $\{1, 2, \dots, n\}$, its* **weight***, denoted by $wt(v)$, is the cardinality of v. When a vertex is viewed as a binary string, its weight is the sum of its coordinates.*

Definition 5 *A subset S of vertices of a bipartite graph G with bipartition (X, Y) is* **balanced** *if half the vertices of S belong to X and half belong to Y.*

Thus a subset of $V(Q_n)$ is balanced if exactly half of its elements have even weight.

Note: From now on we shall refer to a vertex of Q_n as **even** or **odd**, according to whether its weight is even or odd, and the **parity** of a vertex will mean the parity of its weight. Since adjacent vertices, viewed as binary strings, differ in precisely one position, their weights differ by one and therefore have opposite parity.

Example 1 *We give an example to show that when $G = Q_n$, the sets X' and X'' given in Proposition 1 need not be equal.*

In Q_5, thinking of the vertices as the subsets of $\{1, 2, 3, 4, 5\}$, let X be the set consisting of all subsets of even cardinality, and Y the set consisting of all subsets of odd cardinality. For $x \in X$ and $y \in Y$, x and y are adjacent precisely when the cardinality of their symmetric difference, $x\Delta y$, is 1. Let X' equal the set consisting of all 2-element subsets of $\{1, 2, 3, 4\}$, and the set $\{1, 5\}$. Then $N(X')$ equal the set of all 1-element and all 3-element subsets of $\{1, 2, 3, 4, 5\}$. It is easy to check that $X'' = X' \cup \{\{2,5\},\{3,5\},\{4,5\},\emptyset\}$. The maximal independent set S which this produces has 12 elements: $\emptyset$, $\{1,2,3,4,5\}$, and the ten 2-element subsets of $\{1,2,3,4,5\}$. S is clearly not balanced as it has eleven even vertices and one odd.

From now on we simplify notation for a subset of $\{1,2,\ldots,n\}$, writing $i_1 i_2 \cdots i_k$ instead of $\{i_1, i_2, \ldots, i_k\}$.

Another non-balanced maximal independent set T of Q_5 has nine vertices:

$$T = \{12, 13, 23\} \cup \{4, 1234\} \cup \{5, 1235\} \cup \{145, 2345\}.$$

Two other non-balanced maximal independent sets, both with eight vertices are:

$$T_1 = \{\emptyset, 12, 14, 25, 45, 1245\} \cup \{135, 234\} \text{ and}$$

$$T_2 = \{\emptyset, 12, 14, 25, 1245\} \cup \{135, 234, 345\}.$$

T_1 has 6 evens and 2 odds, while T_2 has 5 evens and 3 odds.

Q_5 also has balanced maximal independent sets S_1 and S_2 of sizes 8 and 10, respectively.

$$S_1 = \{\emptyset, 123, 14, 234, 15, 235, 45, 12345\}.$$

$$S_2 = \{\emptyset, 12345, 12, 345, 13, 245, 14, 235, 15, 234\}.$$

The next two examples illustrate different possibilities for maximal independent sets in Q_7.

Example 2 *This example is a non-balanced maximal independent set T in Q_7 with* $|\,T\,| = 42$.
$T_0 = \{\emptyset\}$, $T_1 = T_7 = \emptyset$
$T_2 = \{12, 13, 14, 15, 16, 17, 23, 24, 25, 26, 27\}$,
$T_3 = \{345, 346, 347, 356, 357, 367, 456, 457, 467, 567\}$,
$T_4 = \{1234, 1235, 1236, 1237, 1245, 1246, 1247, 1256, 1257, 1267\}$,
$T_5 = \{13456, 13457, 13467, 13567, 23456, 23457, 23467, 23567, 34567\}$,
$T_6 = \{124567\}$,
Hence $|\,T_0 \cup T_2 \cup T_4 \cup T_6\,| = 23$, *while* $|\,T_1 \cup T_3 \cup T_5 \cup T_7\,| = 19$. *Thus if we let* $T = \bigcup_{i=0}^{i=7} T_i$, *then* $|\,T\,| = 42$.

Example 3 *Our next example is a minor modification of the previous one. Define* T' *by* $T'_i = T_i$ *for* $0 \le i \le 4, T'_5 = T_5 \cup \{14567, 24567\}, T'_6 = \emptyset$, *and* $T'_7 = \{1234567\}$. *Then* $|\,T'_0 \cup T'_2 \cup T'_4 \cup T'_6\,| = 22$, *while*
$|\,T'_1 \cup T'_3 \cup T'_5 \cup T'_7\,| = 22$, *and so* T' *is a balanced maximal independent set of size* 44.

Remark 1 *One way in which balanced independent sets arise is from binary linear codes* (*see, for example,* [11]. A **binary linear code** is a subgroup $\mathcal{C}$ of Z_2^n, the group of binary strings of length n under the operation of coordinate mod 2 addition. Under the map $\phi : Z_2^n \longrightarrow Z_2$ given by $(x_1, \ldots, x_n) \mapsto \sum x_i \pmod 2$, the image of $\mathcal{C}$ under

ϕ is either $\{0\}$ or Z_2. In the first case, all elements of $\mathcal{C}$ are even. In the second, $\mathcal{C}/\mathcal{C} \cap (\operatorname{Ker}\phi) \simeq Z_2$ so $\mathcal{C} \cap (\operatorname{Ker}\phi) = \{x \in \mathcal{C} : x \text{ is even}\}$. Thus exactly half the elements of $\mathcal{C}$ are even and so $\mathcal{C}$ is balanced. $\mathcal{C}$ is an independent set if and only if every nonzero codeword has weight at least 2.

Definition 6 *A* **matching** *in a graph is a set of edges, no two of which are incident. A matching is a* **maximum** *matching if its cardinality is a maximum, among the cardinalities of all matchings of the graph. A matching is* **maximal** *if it is not a proper subset of any other matching.*

Lemma 1 *Let G be a bipartite graph with bipartition (X, Y). Assume that $|X| = |Y|$. Let M be a maximal matching of G and let $S = V(G) \setminus V(M)$. Then S is a balanced independent set.*

The proof is straightforward and we omit it.

2 Bounds on the maximum size of balanced independent sets

We are interested in maximal independent sets in the hypercube Q_n which are balanced. Note that the 2-coloring of Q_n is given by the parity of the vertices.

Proposition 2 *The following algorithm produces a balanced maximal independent set in Q_n.*

$\mathcal{S} = \{\emptyset, u\}$, *where u is odd and $wt(u) > 1$.*
While $\mathcal{S} \cup N(\mathcal{S}) \neq Q_n$
Do: Choose $w \in Q_n \setminus (\mathcal{S} \cup N(\mathcal{S}))$.
$\mathcal{S} := \mathcal{S} \cup \{w, u\Delta w\}$

Proof. We prove by induction on i that $\mathcal{S}_i$, the set produced after the i^{th} iteration, is balanced and independent. To this end, we first show that for any $v \in Q_n, v \in \mathcal{S}_i$ if and only if $u\Delta v \in \mathcal{S}_i$. Clearly this is true for $i = 0$. Assume it holds for $i = k$. Let $v \in \mathcal{S}_{k+1}$. If $v \in \mathcal{S}_k$ then by our induction hypothesis, $u\Delta v \in \mathcal{S}_k \subset \mathcal{S}_{k+1}$. So assume that $v \notin \mathcal{S}_k$. Then by the definition of $\mathcal{S}_{k+1}$, $u\Delta v \in \mathcal{S}_{k+1}$. Since $u\Delta u = \emptyset$, $u\Delta(u\Delta v) = v$. Thus the reverse implication holds as well.

Next we show that $w \in N(\mathcal{S}_i)$ if and only if $u\Delta w \in N(\mathcal{S}_i)$. Suppose that v is adjacent to w, $v \in \mathcal{S}_i$. Then $u\Delta v$ is adjacent to $u\Delta w$, and by what we have just proved, $u\Delta v \in \mathcal{S}_i$. Hence $u\Delta w \in N(\mathcal{S}_i)$. Once again, this also proves the reverse implication.

It follows that $w \in \mathcal{S}_i \cup N(\mathcal{S}_i)$ if and only if $u\Delta w \in \mathcal{S}_i \cup N(\mathcal{S}_i)$. Now since $wt(u)$ is odd, $\mathcal{S}_0 = \{\emptyset, u\}$ is balanced. Since $\mathcal{S}_{i+1} \setminus \mathcal{S}_i$ consists of two vertices of opposite parity, if $\mathcal{S}_i$ is balanced, so is $\mathcal{S}_{i+1}$.

Finally, we show that each $\mathcal{S}_i$ is an independent set. Since $wt(u) > 1$, $\mathcal{S}_0$ is independent. Assume this is true when $i = k$. Let $\mathcal{S}_{k+1} \setminus \mathcal{S}_k = \{w_k, u\Delta w_k\}$. By construction, $w_k \notin \mathcal{S}_k \cup N(\mathcal{S}_k)$. Hence neither is $u\Delta w_k$. Thus neither of them is adjacent to any vertex in $\mathcal{S}_k$. Furthermore, since $wt(u) > 1$, w_k and $u\Delta w_k$ are not adjacent. Hence $\mathcal{S}_{k+1}$ is an independent set.

Of course, eventually, $\mathcal{S}_k \cup N(\mathcal{S}_k) = Q_n$. Then $\mathcal{S}$ will be a *maximal* independent set. □

The cardinalities of the sets $\mathcal{S}$ produced by this algorithm can vary.

Example 4 *In Q_5, the following sets have their elements listed in the order in which they are adjoined in applying the algorithm.*

$$T_1 = \{\emptyset, \{1,2,3\}, \{1,4\}, \{2,3,4\}, \{1,5\}, \{2,3,5\}, \{4,5\}, \{1,2,3,4,5\}\}$$

$$T_2 = \{\emptyset, \{1,2,3,4,5\}, \{1,2\}, \{3,4,5\}, \{1,3\}, \{2,4,5\}, \{1,4\}, \{2,3,5\}, \{1,5\}, \{2,3,4\}\}$$

Remark 2 *Since $\{1\}$ has no neighbor in $V(Q_5) \setminus T_2, T_2$ provides a negative answer to the following question: If T is a balanced maximal independent set on Q_n does $Q_n \setminus T$ have a perfect matching?*

The construction in the previous proposition can be reinterpreted in game-theoretic terms. Two players, *Odd* and *Even*, take turns choosing vertices of Q_n, according to the following rules:
(1) *Odd* must always choose an odd vertex and *Even* must always choose an even one.
(2) A chosen vertex may not be adjacent to a vertex chosen earlier.
(3) The last player to make a valid selection wins.

Proposition 3 *If $n \geq 3$ the second player can always win.*

Proof. The second player decides on a vertex u whose weight is odd and at least 3. When the first player selects any vertex v, the second player responds with $u\Delta v$. The proof for the algorithm shows that whenever the first player's move is valid (i.e. the new vertex chosen by the first player is not adjacent to any of the vertices selected previously) then so is the second player's response. □

Remark 3 *If G is any graph which has a perfect matching and S is any independent set in G, then $|S| \leq \frac{1}{2} \cdot |G|$. For the perfect matching partitions the vertices of G into 2-sets, and at most one of each of these two members can belong to S. In particular, if S is an independent subset of Q_n, $|S| \leq 2^{n-1}$.*

Next we consider the question of how many odd and how many even vertices there can be in an independent subset of Q_n which is not monochromatic.

Proposition 4 *Let S be an independent set of vertices in Q_n, where $n \geq 2$. Suppose that the cardinality of S is at least $2^{n-1} - n + 2$. Then the elements of S are all of the same parity. Furthermore, for $n \geq 3$ this bound is sharp. In fact, there exists a maximal independent set S_n such that S_n has $2^{n-1} - n$ even vertices and 1 odd vertex.*

Proof. By induction on n. The statement is clearly true for $n = 2$, so assume $n \geq 3$ and that the result is true for $n-1$. For $i = 0, 1$ we denote by $Q^{(i)}$ the $n-1$ -dimensional subcube of Q_n consisting of all vertices whose n^{th} coordinate is i, and let $S_i = S \cap Q^{(i)}$. By our hypothesis, $\mid S_0 \mid + \mid S_1 \mid \geq 2^{n-1} - n + 2$. With no loss of generality we may assume that $\mid S_0 \mid \geq \mid S_1 \mid$. Since S_i is an independent subset of the $(n-1)$-cube $Q^{(i)}$, $2^{n-2} \geq \mid S_0 \mid$. Hence

$$\mid S_1 \mid \geq (2^{n-1} - n + 2) - 2^{n-2} \; = \; 2^{n-2} - n + 2$$

Suppose $\mid S_0 \mid = 2^{n-2}$. Then $\mid S_1 \mid \geq 2^{n-2} - (n-1) + 1$. Furthermore, since $n \geq 3$, $\mid S_0 \mid = 2^{n-2} \geq 2^{n-2} - n + 3$. Therefore by our induction hypothesis, the parity of the elements of S_0 is constant. Without loss of generality, since hypercubes are vertex-transitive, we may assume the parity is even. Then S_0 consists of all the even vertices of $Q^{(0)}$. Define

$$S_1^* = \{x \in Q^{(0)} \mid <x, y> \in E(Q_n), y \in S_1\}.$$

Since S is an independent set, $S_1^* \cap S_0 = \emptyset$. Hence all vertices of S_1^* are odd, and so all vertices of S_1 are even. Thus all vertices of S are even.

On the other hand, suppose $\mid S_0 \mid \leq 2^{n-2} - 1$. Then

$$\mid S_0 \mid \geq \mid S_1 \mid \geq 2^{n-2} - (n-1) + 2.$$

By induction, the parity is constant on each of S_0 and S_1. If the parity is the same on both subsets we are done. Otherwise, we may assume that the parity is even on S_0 and odd on S_1. Hence the parity is even on S_1^*, and therefore on $S_0 \cup S_1^*$. Thus

$$\mid S_0 \cup S_1^* \mid \leq \frac{1}{2} \mid Q^{(0)} \mid = 2^{n-2}.$$

But since $S_0 \cap S_1^* = \emptyset$,

$$\mid S_0 \cup S_1^* \mid = \mid S_0 \mid + \mid S_1^* \mid = \mid S_0 \mid + \mid S_1 \mid = \mid S \mid .$$

Hence $2^{n-1} - n + 2 \leq 2^{n-2}$. Therefore $2^{n-2} \leq n - 2$, which is a contradiction. Thus all elements of S have the same parity.

For the second statement we construct an independent subset S as in [12], Proposition 2. Let x be any odd vertex. Then x is adjacent to n vertices, all even. Let S consist of x and all *other* even vertices. Clearly S is independent and $\mid S \mid = 2^{n-1} - n + 1$.

Finally, it follows from the first statement that this S must be a maximal independent subset since for any larger independent subset all its vertices must have the same parity. □

Proposition 5 *Let $n \geq 3$ and let S be a balanced independent set in Q_n. Then $\mid S \mid \leq 2^{n-1} - 2n + 4$.*

Proof. By induction on n. If $n = 3$ and $\mid S \mid > 2$, then $\mid S \mid = 4$ and by the preceding proposition all elements of S are even, contradicting our hypothesis. So $\mid S \mid \leq 2$.

Now assume that $n > 3$ and the result is true for $n - 1$. Suppose that $\mid S \mid \geq 2^{n-1} - 2n + 6$. Let S_0 and S_1 be defined as in the proof of Proposition 4. Without loss of generality, we may assume that $\mid S_0 \mid \geq \mid S_1 \mid$. Then

$$\mid S_0 \mid \geq \frac{\mid S \mid}{2} \geq 2^{n-2} - n + 3 = 2^{(n-1)-1} - (n-1) + 2$$

By Proposition 4 since S_0 is independent in $Q^{(0)} \simeq Q_{n-1}$, all elements of S_0 have the same parity. Without loss of generality, say they are all even. Since by hypothesis, exactly half the elements of S are even,

$$|S_0| = (1/2) \mid S \mid = |S_1| .$$

Also, all elements of S_1 are odd. Thus all elements of S_1^* are even. As before, we have $|S_0 \cup S_1^*| = |S|$. Hence $Q^{(0)}$ has at least $|S|$ even elements. Therefore $|S| \leq 2^{n-2}$. So $2^{n-1} - 2n + 6 \leq 2^{n-2}$. Therefore $2^{n-2} \leq 2n - 6 = 2(n-3)$. So $2^{n-3} \leq n - 3$, which is impossible. Thus $|S| < 2^{n-1} - 2n + 6$. Since $|S|$ is even, we conclude that

$$|S| \leq 2^{n-1} - 2n + 4.$$

□

Corollary 2 *The maximum size of a balanced independent set:*
(1) *on Q_3 is* 2
(2) *on Q_4 is* 4 *and*
(3) *on Q_5 is* 10.
Furthermore, in each case there is a balanced independent dominating set of this size.

Proof. (1) Any vertex in Q_3 dominates exactly three of the four vertices of the opposite parity. Thus every balanced independent set consists of an antipodal pair, and is a dominating set. (Note: we say that two vertices of Q_n are **antipodal** if, regarded as subsets of $\{1, 2, \ldots, n\}$, one is the complement of the other, or, if regarded as binary strings, their coordinate-wise sum is the string of all ones.)

(2) By Proposition 5, if S is a balanced independent set in Q_4 then $|S| \leq 4$. It is easy to check that $S = \{\emptyset, 12, 134, 234\}$ is a balanced independent dominating set.

(3) By Proposition 5, if S is a balanced independent set in Q_5 then $|S| \leq 2^4 - 2 \cdot 5 + 4 = 10$. On the other hand, T_2 of Example 4 is a balanced independent dominating set of size 10. □

Lemma 2 *Let S be a balanced maximal independent set in Q_{n-1}. Let $S' = \{s\Delta\{1, n\} \mid s \in S\}$. Then $S \cup S'$ is a balanced maximal independent set in Q_n. Furthermore,*

$$\mid S \cup S' \mid / \mid Q_n \mid = 2 \cdot \mid S \mid / (2 \cdot \mid Q_{n-1} \mid) = \mid S \mid / \mid Q_{n-1} \mid .$$

Proof. Clearly S and $S' \cong S$ are both independent in Q_n. First we show that $S \cup S'$ is an independent set. Suppose $t \in S$, $s' \in S'$, and t and s' are adjacent. Since the elements of S are subsets of $\{1, 2, \ldots, n-1\}$, for all $s \in S$ we have that $n \notin s$. Hence $n \in s\Delta\{1, n\}$. In particular, $n \in t\Delta\{1, n\}$. Since $s' \in S$, $s' = s\Delta\{1, n\}$ for some $s \in S$. Therefore $n \in s'$, so since $|\, t\Delta s' \,| = 1, t\Delta s' = \{n\}$. Since $s' = s\Delta\{1, n\}$, $\{n\} = t\Delta s\Delta\{1, n\}$. Thus $\emptyset = \{n\}\Delta\{n\} = t\Delta s\Delta\{1, n\}\Delta\{n\} = t\Delta s\Delta\{1\}$. Hence $t\Delta s = \{1\}$. But this means that t and s are adjacent, contradicting the independence of S. So $S \cup S'$ is independent.

Note that for any $s \in S$,the weight of s and the weight of $s\Delta\{1, n\}$ have the same parity. Thus since S is balanced, so is S', and hence so is $S \cup S'$.

Next we show the maximality of $S \cup S'$. By hypothesis, S dominates Q_{n-1}. The isomorphism of S' with S means that S' dominates $Q_{n-1}\Delta\{n\}$. Thus $S \cup S'$ dominates Q_n.

The last two assertions follow immediately from the fact that

$$|\, S \,| = |\, S' \,| \,.$$

□

By a_n we denote the maximum cardinality of a balanced maximal independent set in Q_n. For $n \geq 3$, Lemma 2 says that $a_{n+1} \geq 2 \cdot a_n$. Now by Corollary 2, $a_3 = 2$, $a_4 = 4$, and $a_5 = 10$, and so by Lemma 2, we have $a_6 \geq 20$.

Remark 4 *For $n \geq 3$, view Q_n as $Q_{n-2} \times Q_2$. Thus if $X = V(Q_{n-2})$, $V(Q_n) = X_{00} \cup X_{10} \cup X_{01} \cup X_{11}$, where $X_{00} = X$, $X_{10} = X\Delta\{n-1\}$, $X_{01} = X\Delta\{n\}$ and $X_{11} = X\Delta\{n-1, n\}$. For any subset S of $V(Q_n)$, S is partitioned into the four subsets $S_{ij} = S \cap X_{ij}$, where $i, j \in \{0, 1\}$.*

Proposition 6 *The maximum size of a balanced independent set on Q_6 is 20, and there is a balanced independent dominating set of this size.*

Proof. It follows from Corollary 2 and Lemma 2 that Q_6 has a balanced independent dominating set of size 20. Suppose S is a balanced independent set in Q_6 of size 22. By Remark 4, S is partitioned into the four subsets S_{00}, S_{10}, S_{01} and S_{11}. Since $|\, S \,| = 22$, one of these four subsets must have at least $\lceil 22/4 \rceil = 6$ elements. Without loss of generality, suppose $|\, S_{00} \,| \geq 6$. Then by Proposition 4 all the vertices of S_{00} have the same parity. With no loss of generality we may assume these vertices are all even. They are adjacent to at least 6 of the 8 odd vertices in each of X_{10} and X_{01}, so that each of S_{10} and S_{01} has at most 2 odd vertices. Since S has 11 odd vertices, S_{11} must have at least 7 of them, and therefore S_{10} and S_{01} each have at most one even vertex. Furthermore, by Proposition 4, S_{11} has no even vertices. But since S_{00} has at most 8 even vertices, S can have at most 10 even vertices, which is a contradiction. Thus we obtain the desired conclusion. □

Lemma 3 *If T contains 2 elements of one parity, it contains at most 8 of the other.*

Proof. With no loss of generality we may assume that $x, y \in T$ are both even. Then $|\,N(\{x,y\})\,| = |\,N(x)\,| + |\,N(y)\,| - |\,N(x) \cap N(y)\,| = 2 \cdot 5 - |\,N(x) \cap N(y)\,|$. Since the only possible members of $N(x) \cap N(y)$ are $x \cup y$ and $x \cap y$, $|\,N(x) \cup N(y)\,| \leq 2$. Hence $|\,N(\{x,y\})\,| \geq 10 - 2 = 8$. Thus at most $16 - 8 = 8$ odd vertices can belong to the independent subset T of Q_5. □

Definition 7 *For $0 \leq i \leq n$ we denote by L_i the set of vertices of Q_n of weight i.*

We need to make a slight digression about "upper shadows" and "lower shadows" of subsets of $L_i(Q_n)$. Our notation and terminology follow [3], § 5. For $S \subset L_i(Q_n)$, the **upper shadow** of S, denoted by $\partial_u(S)$, is $N(S) \cap L_{i+1}(Q_n)$. The **lower shadow of** S, denoted by $\partial_\ell(S)$, is $N(S) \cap L_{i-1}(Q_n)$. For the next lemma we will need a lower bound on the size of the upper shadow of S, in a very special case. The result we need here is the analog, for upper shadows, of the Kruskal-Katona Theorem [9],[10] for lower shadows. **Colexicographical order** ("colex" for short) on $L_i(Q_n)$ means reverse lexicographical order on the i-element subsets of $\{1, 2, \ldots, n\}$. That is, writing a as $a_1 a_2 \cdots a_i$, where $a_1 < a_2 < \cdots a_i$, colex order defines a to be less than b if $a_k < b_k$ where $k = \max\{j\,|a_j \neq b_j\}$. Paraphrasing [3], the two forms of the Kruskal-Katona Theorem are:

Theorem 1 [3, p. 34] *For a subset S of $L_i(Q_n)$, with $1 \leq i \leq n-1$, let T_f be the set of the first $|\,S\,|$ elements of $L_i(Q_n)$ in colex order and let T_ℓ be the set of the last $|\,S\,|$ elements of $L_i(Q_n)$ in colex order. Then*

(1) $|\,\partial_u(S)\,| \geq |\,\partial_u(T_\ell)\,|$;

(2) $|\,\partial_\ell(S)\,| \geq |\,\partial_\ell(T_f)\,|$.

Lemma 4 *If $S \subset V(Q_5)$, $|\,S\,| \geq 3$ and all elements of S have the same parity, then $|\,N(S)\,| \geq 10$.*

Proof. Let $x \in S$. The map $w \mapsto x\Delta w$ is an automorphism of Q_n. So, letting $S' = x\Delta S = \{x\Delta s\,|\,s \in S\}, |\,S'\,| = |\,S\,| \geq 3$, and $\emptyset \in S'$. Furthermore, the automorphism takes $N(S)$ onto $N(S')$, so $|\,N(S)\,| = |\,N(S')\,|$. So we may assume that $\emptyset \in S$.

Case 1. Suppose $y \in S \cap L_4$. Then $N(S) \supset N(\emptyset) \cup N(\{y\})$, and since the distance between $\emptyset$ and y is 4, these two neighbor sets are disjoint. Hence $|\,N(S)\,| \geq \deg(\emptyset) + \deg(y) = 10$.

Case 2. $|\,S \cap L_2\,| \geq 2$. Then $S \supset \{\emptyset, y, z\}$, with $y, z \in L_2$. $N(S) \supset L_1 \cup \partial_u(\{y,z\})$, so $|\,N(S)\,| \geq 5 + |\,\partial_u(\{y,z\})\,|$. Since the last two elements of $L_2(Q_5)$ in colex order are 35 and 45, $|\,\partial_u(\{y,z\})\,| \geq |\,\partial_u(\{35, 45\})\,| = |\,\{135, 235, 145, 245, 345\}\,| = 5$. Thus $|\,N(S)\,| \geq 10$. □

Corollary 3 *If the independent subset T of Q_5 contains at least 3 elements of one parity, it has at most 6 of the other.*

Proof. By Lemma 4, these 3 elements of T dominate at least 10 of the 16 elements of Q_5 of the opposite parity. Hence at most 6 elements of the opposite parity can belong to T. □

Lemma 5 *If $T \subset V(Q_5)$ is independent and $|\,T\,| = 12$, then T has at least 11 vertices of the same parity.*

Proof. T cannot be balanced since, by Corollary 2, the maximum size of a balanced independent subset in Q_5 is 10. Without loss of generality, we may assume that T has at least 7 odd vertices, and therefore at most 5 evens. If it has exactly 2 evens, then they dominate at least 8 odds and T has at most $16 - 8 = 8$ odds. But then $|\,T\,| \leq 2 + 8 = 10 < 12 = |\,T\,|$; contradiction. Let T_0 be the set of even vertices of T. Then $3 \leq |\,T_0\,| \leq 5$. By Lemma 4, $|\,N(T_0\,| \geq 10$. So T can have at most $16 - 10 = 6$ odd vertices. Therefore $|\,T\,| \leq 5 + 6 = 11 < 12$, a contradiction. Thus T has at most one even vertex. □

Proposition 7 *The maximum size of a balanced independent set on Q_7 is 44, and there is a balanced independent dominating set of this size.*

Proof. It is easy to see that $L_0 \cup L_2 \cup L_5 \cup L_7$ is a balanced independent dominating set of size $2[\binom{7}{0} + \binom{7}{2}] = 2[1 + 21] = 44$. So suppose S is a balanced independent set of size 46. Using Remark 4, $V(Q_7)$ is partitioned into the four 5-cubes $X_{00} = X$, X_{10}, X_{01}, and X_{11}. Thus, as before, S is partitioned into the 4 subsets S_{ij}. One of these 4 must have at least $\lceil 46/4 \rceil = 12$ vertices. Suppose first that one of the 4 has at least 13 vertices. Without loss of generality we may assume $|\,S_{00}\,| \geq 13$. Then by Proposition 4, all the vertices of S_{00} have the same parity, which, again without loss of generality we may assume is even. Thus S_{10} and S_{01} must each have at most 3 odd vertices. Since Q_5 has exactly 16 odd vertices, S_{11} has at most 16 odds. Thus S has at most $0 + 3 + 3 + 16 = 22 < 23$ odd vertices, contradicting the assumption that S is balanced and $|\,S\,| = 46$. So $\sum_{i,j \in \{0,1\}} |\,S_{ij}\,| = 46$, and each summand is ≤ 12. The only two possibilities are: (a) two summands are 12 and the other two are 11, or (b) three summands are 12 and the other is 10.

S is our supposed balanced subset of Q_7 of size 46. We may assume that $|\,S_{00}\,| = 12$. So by the last lemma, S_{00} has at least 11 vertices of the same parity, say odd. These dominate at least 11 evens in each of X_{10} and X_{01}, leaving at most 5 evens in each of X_{10} and X_{01} that can belong to S. Since S_{00} has at most 1 even vertex, for S to have 23 even vertices, S_{11} must have at least $23 - (5 + 5 + 1) = 12$ even vertices. But $|\,S_{11}\,| \leq 12$, so S_{11} has no odd vertices. Furthermore, the 12 evens in S_{11} dominate 12 odds in each of X_{10} and X_{01}. So each of S_{10} and S_{01} has at most $16 - 12 = 4$ odds. Therefore S has at most $12 + 4 + 4 + 0 = 20 < 23$ odds. Hence S is not balanced. □

We now present a lower bound for the maximum size of a balanced independent set when n is odd. It agrees with the maximum found for $n = 3, 5$, and 7.

Proposition 8 *For n odd there is a balanced independent dominating set S with $|\,S\,| = 2^{n-1} - \binom{n-1}{(n-1)/2}$.*

Proof. If $n \equiv 1 \pmod 4$ let

$$S = \bigcup_{\substack{i \text{ odd} \\ 1 \le i \le (n-3)/2}} L_i \cup \bigcup_{\substack{i \text{ even} \\ (n+3)/2 \le i \le n-1}} L_i.$$

Clearly S is independent and dominates Q_n. Since by Pascal's Identity $\binom{n-1}{j-1} + \binom{n-1}{j} = \binom{n}{j}$, we have

$$\begin{aligned} |\, S\,| &= \sum_{\substack{i \text{ odd} \\ 1 \le i \le (n-3)/2}} \binom{n}{i} + \sum_{\substack{i \text{ even} \\ (n+3)/2 \le i \le n-1}} \binom{n}{i} \\ &= \sum_{\substack{i \text{ odd} \\ 1 \le i \le (n-3)/2}} \left[\binom{n-1}{i-1} + \binom{n-1}{i}\right] + \sum_{\substack{i \text{ even} \\ (n+3)/2 \le i \le n-1}} \left[\binom{n-1}{i-1} + \binom{n-1}{i}\right] \\ &= \left[\sum_{0 \le j \le n-1} \binom{n-1}{j}\right] - \binom{n-1}{(n-1)/2}. \end{aligned}$$

By the Binomial Theorem, the first term is 2^{n-1}, yielding the desired result.

If $n \equiv 3 \pmod 4$ let

$$S = \bigcup_{\substack{i \text{ even} \\ 0 \le i \le (n-3)/2}} L_i \cup \bigcup_{\substack{i \text{ odd} \\ (n+3)/2 \le i \le n}} L_i.$$

Then once again it is clear that S is an independent dominating set. By the proof in the preceding case, together with the fact that $\binom{n}{0} = 1 = \binom{n-1}{0}$ and $\binom{n}{n} = 1 = \binom{n-1}{n-1}$, we again obtain the desired cardinality for S. □

Conjecture 1 *For n odd, the maximum cardinality of a balanced independent set in Q_n is $2^{n-1} - \binom{n-1}{(n-1)/2}$.*

Definition 8 *Let n be odd, say $n = 2k+1$. By the* **middle level subgraph of** Q_n *we mean the subgraph induced by the vertices of weights k and $k+1$. We denote it by $Q_n(k, k+1)$.*

Remark 5 *The construction of S in Proposition 8 completely avoided the vertices in the two middle levels, i.e. the vertices of $L_k \cup L_{k+1}$. Since the middle binomial coefficients are the largest, we might try to increase $|S|$ by including as many vertices as possible from the two middle levels. In fact, we shall construct a balanced maximal independent set for subgraph $Q_n(k, k+1)$ and extend it to a balanced maximal independent set for Q_n by adjoining to it all the vertices in levels i where $i \le k-2$ and $i \equiv k \pmod 2$, and in levels j where $j \ge k+3$ and $j \equiv k+1 \pmod 2$.*

Proposition 9 *Let $n = 2k+1$ and let S be a balanced maximal independent subset of $Q_n(k, k+1)$. Then*

$$S \cup \bigcup_{\substack{i \le k-2 \\ i \equiv k \pmod 2}} L_i \cup \bigcup_{\substack{j \ge k+3 \\ j \equiv k+1 \pmod 2}} L_j$$

is a balanced maximal independent subset of Q_n.

Next we construct a balanced maximal independent set for this "middle-level" subgraph.

Proposition 10 *Assume that* $n = 2k+1$. *Let* $\mathcal{B}$ *be the family of* $(k+1)$*-element subsets of* $\{1, 2, \ldots, 2k\}$, *and let* $\mathcal{A} = \overline{\mathcal{B}} = \{\overline{b} \mid b \in \mathcal{B}\}$, *where* $\overline{b}$ *denotes the complement of* b *in the set* $\{1, 2, \ldots, 2k+1\}$. *Finally, set* $\mathcal{S} = \mathcal{A} \cup \mathcal{B}$. *Then* $\mathcal{S}$ *is a balanced maximal independent set in* $Q_n(k, k+1)$ *and* $|\,S\,| = 2 \cdot \binom{n-1}{k-1}$. *Furthermore,*

$$\frac{|\mathcal{S}|}{|Q_n(k, k+1)|} = \frac{k}{2k+1}.$$

Proof. Clearly $|\mathcal{A}| = |\mathcal{B}|$. Since for all $a \in \mathcal{A}$, $|a| = k$, while for all $b \in \mathcal{B}$, $|b| = k+1$, $\mathcal{S}$ is balanced. (In fact, it is symmetric with respect to the antipodal map.) An alternate description of $\mathcal{A}$ is as the set of k-element subsets of $\{1, 2, \ldots, 2k+1\}$ which *do* contain the element $2k+1$. To show that $\mathcal{S}$ is independent, we must show that for all b, $b' \in \mathcal{B}$, $\overline{b}$ is not a subset of b'. But $2k+1 \notin b$ and $2k+1 \notin b'$. Thus $2k+1 \in \overline{b}$, and hence $\overline{b}$ is not a subset of b'.

To see that $\mathcal{S}$ is maximal, suppose $|x| = k$ and $x \notin \mathcal{S}$. Then $x \notin \mathcal{A}$. So $2k+1 \notin x$. Let $y = x \cup \{j\}$ where $j \neq 2k+1$ and $j \notin x$. Then $y \in \mathcal{B}$ and y is adjacent to x, so that $\mathcal{S} \cup \{x\}$ is not independent.

On the other hand, suppose $|z| = k+1$ and $z \notin \mathcal{S}$. Then $z \notin \mathcal{B}$. So by the definition of $\mathcal{B}$, $2k+1 \in z$. Let x be any k-element subset of z which contains $2k+1$ as an element. Then $x \in \mathcal{A}$ and x is adjacent to z. Thus $\mathcal{S} \cup \{z\}$ is not independent. This establishes the maximality of $\mathcal{S}$.

Now $|\mathcal{S}| = 2|\mathcal{B}| = 2\binom{2k}{k+1} = 2\binom{2k}{k-1} = 2\binom{n-1}{k-1}$. Finally, the ratio of $|\mathcal{S}|$ to $|Q_{2k+1}(k, k+1)|$ is

$$\frac{2\binom{2k}{k+1}}{2\binom{2k+1}{k+1}} = \frac{(2k)!}{(k-1)!(k+1)!}\frac{(k+1)!k!}{(2k+1)!} = \frac{k}{2k+1}. \qquad \square$$

Definition 9 *A graph* G *is* k**-regular** *if the degree of each vertex is* k.

Lemma 6 *If* G *is a* k*-regular bipartite graph, with* $k > 0$, *and* S *is an independent subset of* $V(G)$ *then* $|\,S\,| \leq \frac{1}{2}|\,V(G)\,|$.

Proof. Let (X, Y) be the bipartition of G. Since every edge of G has one end in X and the other in Y, counting the edges of G by their X-endpoints and then by their Y-endpoints, we have $k|\,X\,| = |\,E(G)\,| = k|\,Y\,|$. Hence $|\,X\,| = |\,Y\,|$. Next, we will show that $|\,N(S \cap X\,| \geq |\,S \cap X\,|$. The number of edges with one end in $S \cap X$ is $k|\,S \cap X\,|$. By definition, each of these edges has its other end in $N(S \cap X)$. The number of edges with one end in $N(S \cap X)$ is $k|\,N(S \cap X)\,|$. This is at least the number of edges between $N(S \cap X)$ and X. So $|\,N(S \cap X)\,| \geq |\,|S \cap X\,|$. Now suppose that $|\,S\,| > \frac{1}{2}|\,V(G)\,|$. Since there is no edge between $S \cap X$ and $S \cap Y$, $S \cap Y \subset Y \setminus N(S \cap X)$. Therefore $|\,S \cap Y\,| \leq |\,Y\,| - |\,N(S \cap X)\,| \leq |\,Y\,| - |\,S \cap X\,|$ Thus $|\,S \cap X\,| + |\,S \cap Y\,| \leq |\,Y\,|$. Hence $|\,S\,| \leq |\,Y\,| = \frac{1}{2}|\,V(G)\,|$. $\square$

Corollary 4 *The asymptotic upper bound of $\frac{1}{2}$ given by Proposition 10 is the best possible.*

Proof. $\lim_{k\to\infty} k/(2k+1) = 1/2$. □

Remark 6 *Using Pascal's Identity and the Binomial Theorem, it is not hard to see that the conjectured bound of $2^{n-1} - \binom{n-1}{(n-1)/2}$ from Proposition 8 and the bound one gets from combining the results of Propositions 9 and 10 are the same.*

Theorem 2 *If $n = 2k+1$ and S is a balanced independent set in $Q_n(k, k+1)$ then $|\,S\,| \leq 2 \cdot \binom{n-1}{k-1} = 2 \cdot \binom{n-1}{(n-3)/2}$.*

Proof. Let $S_{k+1} = S \cap L_{k+1}$ and $S_k = S \cap L_k$. By Theorem 1(1), $|\,N(S_{k+1})\,| \geq |\,N(T_{k+1})\,|$, where T_{k+1} consists of the first $|\,S_{k+1}\,|$ elements of L_{k+1} in colex order. Let $B_{k+1} = \{B \in L_{k+1} \,|\, n \notin B\}$. Then B_{k+1} is precisely the set of the first $\binom{n-1}{k+1} = \binom{n-1}{k-1}$ elements of L_{k+1} in colex order. Hence by the preceding proposition,

$$|\,L_k \setminus N(B_{k+1})\,| = |\,B_{k+1}\,| = \binom{n-1}{k-1}.$$

Now suppose that

$$|\,S_{k+1}\,| \geq |\,B_{k+1}| + 1 = |\,B_{k+1} \cup \{1, 2, \ldots, k, n\}\,|.$$

Note that $\{1, 2, \ldots, k, n\}$ is the $(|\,B_{k+1}\,| + 1)^{st}$ element of L_{k+1} in colex order. Let $C_{k+1} = B_{k+1} \cup \{1, 2, \ldots, k, n\}$. Then

$$|\,S_{k+1}\,| \geq |\,C_{k+1}\,| = |\,B_{k+1}\,| + 1 = \binom{n-1}{k-1} + 1 = |\,C_{k+1}\,|.$$

By Theorem 1(1), since C_{k+1} is precisely the set of the first $\binom{n-1}{k-1} + 1$ elements of L_{k+1} in colex order,

$$|\,N(S_{k+1})\,| \geq |\,N(C_{k+1})\,| = |\,N(B_{k+1}) \cup \{k\text{–subsets of } \{1, 2, \ldots, k, n\} \text{ that contain } n\}\,|$$

$$= |\,N(T_{k+1})\,| + k.$$

Therefore

$$|\,L_k \setminus N(S_{k+1})\,| \leq |\,L_k\,| - |\,N(B_{k+1})\,| - k = |\,S_{k+1}\,| - k < |\,S_{k+1}\,|.$$

But for S to be an independent set, $S_k \subset L_k \setminus N(S_{k+1})$. Hence $|S_k| \leq |L_k \setminus N(S_{k+1})| < |S_{k+1}|$, and so S is not balanced. □

As an immediate corollary of Proposition 10 and Theorem 1 we have:

Corollary 5 *For $n = 2k+1$, the maximum size of a balanced independent set in $Q_n(k, k+1)$ is $2 \cdot \binom{n-1}{k-1}$.*

3 Upper bounds for the *minimum* size of a balanced independent dominating set

Corollary 6 *For $n \geq 3$ there is a balanced maximal independent set S in Q_n with $|S| = \frac{1}{4}|Q_n| = 2^{n-2}$.*

Proof. By induction on n. For $n = 3$ we have $S = \{\emptyset, \{1, 2, 3\}\}$. Lemma 2 provides the inductive step. □

It is easy to construct a balanced independent set in Q_n of size 2^{n-2}. Let A be any subset of Q_{n-1} consisting of exactly half of all the even vertices (so $|A| = \frac{1}{4}|Q_{n-1}|$). (For example, we can take A to be the subgroup of Z_2^{n-1} of index 2 consisting of the vertices of even weight.) Let F_0 be the image of Q_{n-1} in Q_n under the map $\phi_0 : x \mapsto x0$, and let F_1 be the image of Q_{n-1} under the map $\phi_1 : x \mapsto x1$. Then precisely half the odd vertices of F_1 are dominated by $\phi_0(A)$. Denote by B the remaining half of the odd vertices of F_1, and let $S = \phi_0(A) \cup B$. Clearly S is a balanced independent set, of size $(\frac{1}{4})|Q_n| = 2^{n-2}$. However, S need not be maximal, i.e. S need not dominate Q_n. For example, in Q_4 let $A = N(\{1\}) = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 4\}\}$. Then $\{2, 3, 4\} \notin N(A)$. Now let $B = \{\{1, 2, 3, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}$. Then $S = A \cup B$ is a balanced independent set which does not dominate Q_5. However, if we let $A' = A \cup \{1, 5\}$ and $B' = B \cup \{2, 3, 4\}$, then $S' = A' \cup B'$ is a balanced maximal independent set in Q_5 of size 10.

However, a much lower upper bound for the minimum size of a balanced independent dominating set can be given, using Hamming's Theorem [5] on perfect single-error correcting binary linear codes. A binary linear code of length n is **single-error correcting** if, as vertices of Q_n, the distance between any two codewords is at least three. It is **perfect** if every non-codeword is adjacent to exactly one codeword. Thus a binary linear code of length n is perfect single-error correcting if and only if the closed neighborhoods of the codewords partition $V(Q_n)$. It is easy to see that if Q_n has a perfect single-error correcting code then $n = 2^k - 1$ for some k. Hamming's Theorem [5] is the converse.

Theorem 3 *If $n = 2^k - 1$ then there exists a perfect single-error correcting binary linear code of length n.*

Proposition 11 *Let $n \geq 3$ and let k be an integer such that $2^k - 1 \leq n$. Then there is a balanced maximal independent set S with $\mid S \mid / \mid Q_n \mid = 1/2^k$.*

Proof. Let $m = 2^k - 1$. Then by Theorem 3 Q_m has a perfect single-error correcting code, C. Thus C is a maximal independent set. Since C is a subgroup of Q_m under $+$ and contains vertices whose weights have opposite parity, its vertices of even weight form a subgroup of index 2 in C. Hence exactly half the vertices of C have even weight, i.e. C is balanced. Furthermore, $\mid C \mid \cdot (1 + m) = \mid Q_m \mid$, so

$$\mid C \mid / \mid Q_m \mid = 1/(1 + m) = 1/2^k.$$

So Q_m has a maximal independent set S which is balanced, and

$$| S | / | Q_m |= 1/2^k.$$

Now if $n > m$, repeated application of Lemma 2 produces a balanced independent set T in Q_n such that

$$| T | / | Q_n |=| S | / | Q_m |= 1/2^k$$

□

To conclude, there are two related open questions. The first was implied in Conjecture 1: whether for n odd, the maximum cardinality of a balanced independent set in Q_n is $2^{n-1} - \binom{n-1}{(n-1)/2}$. The second is whether, for all n (or at least for all odd n), given any two balanced independent sets of maximum cardinality in Q_n, there is an automorphism of Q_n that takes one set to the other.

References

[1] R. B. Allan and R. Laskar, On domination and independent domination numbers of a graph, *Discrete Math.* **23** no. 2 (1978), 73–76.

[2] S. Arumugam and R. Kala, Domination parameters of hypercubes, *J. Indian Math. Soc.* **65** nos. 1-4 (1998), 31–38.

[3] B. Bollobás, *Combinatorics: set systems, hypergraphs, families of vectors, and combinatorial probability*, Cambridge Univ. Press, 1986.

[4] M. Fischerman, L. Volkmann and I. Zverovich, Unique irredundance, domination and independent domination in graphs, *Discrete Math.* **305** nos. 3–6 (2005), 190–200.

[5] R.W. Hamming, Error detecting and error correcting codes, *Bell System Tech. J.* **29** (1950), 147–160.

[6] J. Harant, A. Pruchnewski and M. Voigt, On dominating sets and independent sets of graphs, *Combin. Probab. Comput.* **8** no. 6 (1999), 547–553.

[7] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, Inc., 1998.

[8] P. K. Jha, Smallest independent dominating sets in Kronecker products of cycles, *Discrete Appl. Math.* **113** nos. 2–3 (2001), 303–306.

[9] G. O. H. Katona, A theorem on finite sets, in *Theory of Graphs* (P. Erdős and G. O. H. Katona, eds.) Akadémiai Kradó, Budapest, 1968.

[10] J. B. Kruskal, The number of simplices in a complex, in *Mathematical Optimization Techniques*, Univ. California Press, Berkeley, 1963, 251–278.

[11] V. Pless, *Introduction to the theory of error-correcting codes*, Wiley Interscience, 1998.

[12] M. Ramras, Bipartite dominating sets in hypercubes, *Ars Combinatoria* **77** (2005), 169–180.

[13] B. Zelinka, Some remarks on domination in cubic graphs, *Discrete Math.* **158** nos. 1–3 (1996), 249–255.

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