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A note on full transversals and mixed orthogonal arrays*

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Abstract

We investigate a packing problem in M-dimensional grids, where bounds are given for the number of allowed entries in different axis-parallel directions. The concept is motivated from error correcting codes and from more-part Sperner theory. It is also closely related to orthogonal arrays. We prove that some packing always reaches the natural upper bound for its size, and even more, one can partition the grid into such packings, if a necessary divisibility condition holds. We pose some extremal problems on maximum size of packings, such that packings of that size always can be extended to meet the natural upper bound.

1 The concept of full transversals

Let us be given positive integers $n_1, n_2, \ldots, n_M$ and $L_1, L_2, \ldots, L_M$, such that

$$\frac{L_1}{n_1+1} \leq \frac{L_2}{n_2+1} \leq \cdots \leq \frac{L_M}{n_M+1} \leq 1. \tag{1}$$

Let $\Pi = [0, n_1] \times \cdots \times [0, n_M]$, and let I denote a subset of Π. We call an $I \subseteq \Pi$ an $(L_1, L_2, \ldots, L_M)$-*transversal*, if there are no $L_i + 1$ elements of I, any two of them differing from each other only in the i^{th} coordinate, for any $i = 1, 2, \ldots, M$. Identifying I with a 0-1 valued function defined on Π, namely the indicator function of I, an alternative definition of the $(L_1, L_2, \ldots, L_M)$-transversal is that for any k, fixing $i_1, \ldots, \widehat{i_k}, \ldots, i_M$ in an arbitrary fashion (where $\widehat{\ }$ denotes a missing entry),

$$\sum_{i_k=0}^{n_k} I(i_1, \ldots, i_k, \ldots, i_M) \leq L_k \tag{2}$$

holds. We talk about L-*transversals*, if $L_1 = L_2 = \cdots = L_M = L$.

Lemma 1.1. *If I is an $(L_1, \ldots, L_M)$-transversal, then for every $i = 1, \ldots, M$*

$$|I| \leq \frac{L_i}{n_i+1} \prod_{j=1}^{M} (n_j+1). \tag{3}$$

Proof. The statement is almost trivial: deleting the i^{th} coordinate from the elements of I leaves at most $(n_1+1)\cdots(n_M+1)/(n_i+1)$ distinct elements. No one of these

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elements may have more than L_i different pre-images, since otherwise I would fail being an $(L_1, \ldots, L_M)$-transversal. $\square$

We call an $(L_1, \ldots, L_M)$-transversal I *full*, if I shows equality in (3) for $i = 1$.

One motivation for this research is that $(L_1, L_2, \ldots, L_M)$-transversals define homogeneous M-part $(n_1, n_2, \ldots, n_M; L_1, L_2, \ldots, L_M)$-Sperner families, see [1], [2], [6] or [7]. Although full transversals were important for the study of 2-part Sperner families [1], [6], [7], they turned less useful for M-part Sperner families [2]. However some positive results for the existence problem of full transversals might lead to new results in that theory. In the last section we will discuss the connection between full transversals and (generalized) orthogonal arrays.

For $L = 1$, a more general notion of a *d-fold transversal* is a subset $I \subset \Pi$ with the property that for every two distinct elements of the d-fold transversal, there exist at least $d + 1$ coordinates in which they differ. In other words the minimum Hamming distance of I is $d+1$, which means that I is a $\left\lfloor \frac{d}{2} \right\rfloor$- error correcting code. The d-fold transversal is related to d-fold M-part Sperner families in [2] like transversals are related to M-part Sperner families.

The main concern of this note is the existence of full transversals. If we relax the definition of an $(L_1, L_2, \ldots, L_M)$-transversal to a non-negative real function defined on Π, and modify the definition of $|I|$ to

$$\sum_{i_1} \sum_{i_2} \cdots \sum_{i_M} I(i_1, i_2, \ldots, i_M),$$

then the analogue of (3) still holds. Defining the relaxation of full transversals by equality in the analogue of (3) for $i = 1$, and setting $I(i_1, i_2, \ldots, i_M) = \frac{L_1}{n_1+1}$, one finds that the relaxed transversals always exist. Therefore our 0-1 full transversal problem is about the feasibility of a particular integer program, which is a packing problem. We give a very explicit solution to the 0-1 full transversal problem, unlike [8], which invoked matching theory to settle the special case $M = 2$, $L_1 = L_2$. Furthermore, if L_1 divides $n_1 + 1$, which is a necessary condition for partitioning Π into full transversals, we construct such a partition.

We raise an extremal problem: what is the smallest size of a transversal in Π that cannot be extended into a full transversal of Π?

2 Existence of full transversals

For a real number α, let $\langle \alpha \rangle$ be its fractional part, i.e.

$$\langle \alpha \rangle = \alpha - \lfloor \alpha \rfloor.$$

We will prove the following lemma:

Lemma 2.1. *Let* $\alpha, \beta, \mu \in \mathbb{R}$ *and* $0 \le \mu < 1$, $0 \le \beta \le 1 - \mu$. *Then*

$$\left| \left\{ i \in \mathbb{Z} : 0 \le i \le n, \left\langle \alpha + \frac{i}{n+1} \right\rangle \in [\beta, \beta + \mu) \right\} \right| \le \lceil (n+1)\mu \rceil \qquad (4)$$

and if $(n+1)\mu$ is integral, then equality holds.

Proof. $\alpha, \alpha + \frac{1}{n+1}, \alpha + \frac{2}{n+1}, \ldots, \alpha + \frac{n}{n+1}$ are $n+1$ evenly spaced points at distance $\frac{1}{n+1}$ in the interval $[\alpha, \alpha+1)$.

Divide the interval $[\beta, \beta+\mu)$ into $k+1 := \lceil (n+1)\mu \rceil$ subintervals, k of which are of length $\frac{1}{n+1}$, as follows:

$$\left[\beta, \beta + \frac{1}{n+1}\right), \ldots, \left[\beta + \frac{k-1}{n+1}, \beta + \frac{k}{n+1}\right), \left[\beta + \frac{k}{n+1}, \beta + \mu\right).$$

Since any half-open interval that is contained in $[0,1)$ and has length at most $\frac{1}{n+1}$ contains at most one of the points of the form $\left\langle \alpha + \frac{j}{n+1} \right\rangle$, and intervals of length $\frac{1}{n+1}$ do contain one such point, the lemma follows. ☐

Theorem 2.2. *Let $n_1, \ldots, n_M$ and $L_1, \ldots, L_M$ be positive integers satisfying inequality (1) and set $\mu = \frac{L_1}{n_1+1}$. Then, for any β with $0 \le \beta \le 1 - \mu$,*

$$I = \left\{ (i_1, \ldots, i_M) \in \Pi : \left\langle \sum_{j=1}^{M} \frac{i_j}{n_j+1} \right\rangle \in [\beta, \beta+\mu) \right\}$$

is a full transversal.

Proof. Let $j \in \{1, 2, \ldots, M\}$ and fix $i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_M$ such that $i_k \in \{0, 1, \ldots, n_k\}$ for all $k \in \{1, 2, \ldots, M\} \setminus \{j\}$. Set

$$\alpha_j = \sum_{k \in \{1,2,\ldots,M\} \setminus \{j\}} \frac{i_k}{n_k+1}.$$

Then $(n_j+1)\mu \le (n_j+1) \cdot \frac{L_j}{n_j+1} = L_j$, so by Lemma 2.1, we have that

$$\left| \left\{ i_j \in \mathbb{Z} : 0 \le i_j \le n_j, \left\langle \alpha_j + \frac{i_j}{n_j+1} \right\rangle \in [\beta, \beta+\mu) \right\} \right| \le L_j,$$

with other words I is a transversal.

Now assume that $j = 1$. Then $(n_1+1)\mu = L_1$, which is integral, and by Lemma 2.1 we have that

$$\left| \left\{ i_1 \in \mathbb{Z} : 0 \le i_1 \le n_1, \left\langle \alpha_1 + \frac{i_1}{n+1} \right\rangle \in [\beta, \beta+\mu) \right\} \right| = L_1,$$

with other words I is a full transversal. ☐

Corollary 2.3. *Let $n_1, \ldots, n_M$ and $L_1, \ldots, L_M$ be positive integers satisfying inequality (1) and set $A = \left\lfloor \frac{n_1+1}{L_1} \right\rfloor$. Then Π can be decomposed into A full transversals and one additional transversal, where this additional transversal is empty if $\frac{n_1+1}{L_1}$ is integral.*

Proof. Let with $\mu = \frac{L_1}{n_1+1}$, for $k = 0, 1, \ldots, A-1$,

$$I_k = \left\{ (i_1, \ldots, i_M) \in \Pi : \left\langle \sum_{j=1}^{M} \frac{i_j}{n_j+1} \right\rangle \in [k\mu, (k+1)\mu) \right\}$$

and

$$I_{k+1} = \left\{ (i_1, \ldots, i_M) \in \Pi : \left\langle \sum_{j=1}^{M} \frac{i_j}{n_j+1} \right\rangle \in [A\mu, 1) \right\}.$$

Clearly, $\Pi = \bigcup_{j=1}^{k+1} I_j$. By Theorem 2.2, $I_1, \ldots, I_k$ are full transversals and, since $1 - A\mu \leq \mu$, I_{k+1} is a transversal. □

3 Extendability of transversals into full transversals

There are many transversals that cannot be extended into full transversals. Set $M = 2, n_1 = n_2 = n \geq 2$ and let L be an integer with $2 \leq L \leq n$. Set $m = \max(n-L+1, L-1)$, so $1 \leq n-m \leq L-1$. Consider a full L-transversal on $[0,m]^2$ (since $L \leq m+1$, such a transversal exists). This is an L-transversal of $[0,n] \times [0,n]$, which cannot be extended into a full L-transversal of $[0,n]^2$: as we can only add elements from $[m+1,n]^2$, no extension can have more elements than

$$L(m+1) + (n-m)^2 = L(n+1) + (n-m)(n-m-L) < L(n+1).$$

We give below a sufficient condition for the case $M = 2$, which guarantees that every transversal can be extended into a full transversal. Note that for $M = 2$, (1) boils down to $L_1(n_2+1) \leq L_2(n_1+1)$. We will use a stronger condition than this, namely

$$L_1(n_2+1) + (L_1-1)(L_2-1) \leq L_2(n_1+1). \tag{5}$$

Theorem 3.1. *If Condition* (5) *holds, then any* (L_1, L_2)*-transversal can be extended into a full transversal.*

Proof. Assume that we have an (L_1, L_2)-transversal I, which is not full, so $|I| < L_1(n_2+1)$. There is a column, say column j, such that there are only $x \leq L_1 - 1$ elements in this column from I. We claim that there is a row i with at most $L_2 - 1$ elements in I and with $(i,j) \notin I$ — now transversal I can be extended by adding (i,j). Were the claim false, I had x entries in column j and n_1+1-x rows containing L_2 additional entries each, so $|I| \geq x + (n_1+1-x)L_2 = (n_1+1)L_2 - x(L_2-1) \geq (n_1+1)L_2 - (L_1-1)(L_2-1) \geq L_1(n_2+1)$, where the last inequality follows from (5). We have the contradiction. □

Observe that the opening example of this section fails inequality (5) just by 1 when $L = 2$. We are left with an intriguing open problem: if not all transversals of Π can be extended into a full transversal of Π, then what is the smallest size of a transversal of Π that cannot be extended into a full transversal of Π? More precisely, we define $\mathcal{P} := \mathcal{P}(n_1, \dots, n_M; L_1, \dots, L_M)$ as the set of $(L_1, \dots, L_M)$-transversals in Π that can not be extended into a full transversal, and

$$t(n_1, \dots, n_M; L_1, \dots, L_M) = \begin{cases} \min_{I \in \mathcal{P}} |I|, & \text{if } \mathcal{P} \neq \emptyset \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Obviously, $t(n_1, \dots, n_M; n_1+1, \dots, n_M+1)$ is undefined. The beginning paragraph of this section shows that $t(n, n; L, L) \leq L \max(n - L + 2, L)$ for $2 \leq L \leq n$. Theorem 3.1 implies that $t(n, n; 1, 1)$ is undefined, which does not extend to $M = 3$:

Proposition 3.2. *For $1 \leq L \leq n$, $2 \leq n$ we have that*

$$t(n, n, n; L, L, L) \leq \begin{cases} L\left(\left\lceil \frac{n}{2} \right\rceil + 1\right)^2, & \text{if } L \leq \left\lceil \frac{n}{2} \right\rceil + 1 \\ L^3, & \text{otherwise.} \end{cases}$$

Proof. Let

$$m = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & \text{if } L \leq \left\lceil \frac{n}{2} \right\rceil + 1 \\ L - 1, & \text{otherwise.} \end{cases}$$

Since $L \leq m+1$, by Theorem 2.2 there is a full L-transversal of $[0, m]^3$. Let I be such a full L-transversal, then $|I| = L(m+1)^2$. Also, I is an L-transversal in $[0, n]^3$. To prove our theorem, it is enough to show that I can not be extended into a full L-transversal of $[0, n]^3$.

Assume, contrary to our statement, that I' is a full L-transversal in $[0, n]^3$ with $I \subseteq I'$. Then $|I'| = L(n+1)^2$. For $k = 1, 2, 3$, let

$$\begin{aligned} \Pi_k^{(1)} &= A_1 \times A_2 \times A_3, \text{ where } A_k = [0, n] \text{ and for } j \neq k, A_j = [m+1, n], \\ J_k &= I' \cap \Pi_k^{(1)} = \{(i_1, i_2, i_3) \in I' : \text{if } j \neq k \text{ then } i_j > m\}, \\ I_k &= \{(i_1, i_2, i_3) \in I' : |\{j : i_j > m\}| = k\}, \\ \Pi_k^{(2)} &= B_1 \times B_2 \times B_3, \text{ where } B_k = [0, n] \text{ and for } j \neq k, B_j = [0, m]. \end{aligned}$$

Clearly, $|I'| = |I| + |I_1| + |I_2| + |I_3|$, and, since J_k is an L-transversal on $\Pi_k^{(1)}$, $|J_k| \leq L(n-m)^2$. Since $I' \cap \Pi_k^{(2)}$ is an L-transversal on $\Pi_k^{(2)}$, it has at most $L(m+1)^2 = |I|$ elements, and, since $I \subseteq \Pi_k^{(2)}$, $I' \cap \Pi_k^{(2)} = I$. But then $I = I' \cap \bigcup_k \Pi_k^{(2)} = I \cup I_1$, from which $|I_1| = 0$. Also, $\bigcup_k J_k = I_2 \cup I_3$ and for $k \neq k'$, $J_k \cap J_{k'} = I_3$, implying

$$\begin{aligned} 3L(n-m)^2 &\geq \sum_k |J_k| = \left|\bigcup_k J_k\right| + \sum_{k \neq k'} |J_k \cap J_{k'}| - |J_1 \cap J_2 \cap J_3| \\ &= |I_2 \cup I_3| + 3|I_3| - |I_3| = |I_2| + 3|I_3| \geq |I_2| + |I_3| + |I_1| \\ &= |I'| - |I| = L(n+1)^2 - L(m+1)^2. \end{aligned}$$

Thus, we get

$$(n+1)^2 \leq (m+1)^2 + 3(n-m)^2. \tag{6}$$

Assume first that $L \leq \lceil \frac{n}{2} \rceil + 1$. Equation (6) gives that

$$(n+1)^2 \leq \left(\left\lceil \frac{n}{2} \right\rceil + 1\right)^2 + 3\left\lfloor \frac{n}{2} \right\rfloor^2.$$

For even n, this means $(n+1)^2 < n^2 + n + 1$, a contradiction. For odd n, we get

$$\begin{aligned} (n+1)^2 &\leq \left(\frac{n+3}{2}\right)^2 + \frac{3(n-1)^2}{4} = \frac{4n^2+12}{4} = n^2 + 2 + 1 \\ &\leq n^2 + n + 1, \end{aligned}$$

which is also a contradiction.

Now, if $\lceil \frac{n}{2} \rceil + 1 < L \leq n$, using $k = n - m$ equation (6) gives

$$(n+1)^2 \leq (n+1-k)^2 + 3k^2 = (n+1)^2 + 2k(2k - (n+1)),$$

which is a contradiction, since $2k < n$.

Therefore I cannot be extended to a full transversal. □

4 Connections with orthogonal arrays

In this section we give evidence of the close connection between full transversals and orthogonal arrays. Consider r sets S_i $(i = 1, ..., r)$ of s_i symbols and take a matrix T of size $N \times r$ where the i^{th} column draws its elements from the set S_i. This matrix is a *mixed* (or *asymmetrical*) *orthogonal array* (the notion of *orthogonal array with variable numbers of symbols* is also used), of strength d, *constraints* r and *index set* $\mathbb{L}$ if for any choice of d different columns $j_1, \ldots, j_d$ each possible sequence $(a_{j_1}, \ldots, a_{j_d}) \in S_{j_1} \times \cdots \times S_{j_d}$ appears exactly $\lambda(j_1, \ldots, j_d) \in \mathbb{L}$ times. In case of equal symbol set sizes and fixed λ we have the classical definition of orthogonal arrays, introduced by C.R. Rao (see [10, 11]). The name was coined by K.A. Bush [3, 4]. C.-S. Cheng [5] seems to be the first to consider variable sizes. The standard reference work for orthogonal arrays is the book of A.S. Hedayat, N.J.A. Sloane and J. Stufken [9]. These arrays are widely used in planning experiments or fractional factorial designs.

It turns out that certain full transversals are suitable to design mixed orthogonal arrays:

Proposition 4.1. *Assume that parameters $M; L_1, ...L_M; n_1, ..., n_M$, in Theorem 2.2 satisfy $\frac{n_1+1}{L_1} = ... = \frac{n_M+1}{L_M}$. Then the construction in Theorem 2.2 is a mixed orthogonal array with the symbol sets $S_i = \{0, ..., n_i\}$ of strength $M-1$, constraints M and index set $\mathbb{L} = \{L_1, ..., L_M\}$.*

If L_1 divides $n_1 + 1$, using Corollary 2.3, we can partition the set of row vectors representing the elements of Π into $(n_1+1)/L_1$ disjoint orthogonal arrays with the symbol sets $S_i = \{0, \ldots, n_i\}$ of strength $M-1$, constraints M and index set $\mathbb{L} = \{L_1, \ldots, L_M\}$.

Proof. In view of Theorem 2.2 we can construct full transversals with these parameters. Such a full transversal can be described as an $N \times M$ matrix $X = (x_{ij})$, where

$$N = L_1(n_2+1)\cdots(n_M+1) = L_i(n_1+1)\cdots\widehat{(n_i+1)}\cdots(n_M+1),$$

and the i^{th} row vector $(x_{i1}, \dots, x_{iM})$ is the i^{th} element of our full transversal. Clearly, the j^{th} column of this matrix draws its elements from S_i. Taking $M-1$ columns of the matrix is the same as leaving out one column. If we leave out the j^{th} column from the matrix, then — since we started with a full transversal — each $(a_1, \dots, \widehat{a_j}, \dots, a_M) \in S_1 \times \cdots \times \widehat{S_j} \times \cdots \times S_M$ appears exactly L_j times in the remaining matrix. Thus, this matrix actually is a mixed orthogonal array with the symbol sets $S_i = \{0, \dots, n_i\}$ of strength $M-1$, constraints M and index set $\mathbb{L} = \{L_1, \dots, L_M\}$. The rest follows. ☐

Even those transversals that are not full have a connection to the theory of arrays, namely to *packing arrays* (see B. Stevens and E. Mendelsohn [12]).

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