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On the number of minimal completely separating systems and antichains in a Boolean lattice

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Abstract

An (n)**completely separating system** $\mathcal{C}$ ((n)CSS) is a collection of blocks of $[n] = \{1, \ldots, n\}$ such that for all distinct $a, b \in [n]$ there are blocks $A, B \in \mathcal{C}$ with $a \in A \setminus B$ and $b \in B \setminus A$. An (n)CSS is minimal if it contains the minimum possible number of blocks for a CSS on $[n]$. The number of non-isomorphic minimal (n)CSSs is determined for $11 \leq n \leq 35$. This also provides an enumeration of a natural class of antichains.

1 Introduction

In this paper $\mathcal{C}$ will denote a completely separating system (CSS). CSSs were introduced by Dickson [4]. They were defined as an extension of a separating system as defined by Rényi [7]. An **(n)completely separating system ((n)CSS)** $\mathcal{C}$ on $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$, called blocks, such that for each $a, b \in [n]$ there are blocks $A, B \in \mathcal{C}$ with $a \in A \setminus B$ and $b \in B \setminus A$.

A common problem in Combinatorial Design/Extremal Set Theory is to determine the number of non-isomorphic instances of a given type of combinatorial design/extremal set system with certain fixed parameters. The main purpose of this paper is to determine the number of non-isomorphic minimal (n)CSSs for $11 \leq n \leq 35$, and coincidentally the number of antichains of a given type. The number of non-isomorphic instances for $1 \leq n \leq 10$ was determined in [8] using various structural properties of relevant CSSs. The use of structural properties allows the number of non-isomorphic minimum size CSSs to be determined by hand in reasonable time for some values of $n > 10$. This approach is illustrated in this paper. However, in most cases the number of such designs is too large for this approach. For example there are 5 643 146 designs for $n = 21$. Hence a search algorithm using backtracking is used for most of the cases considered here.

An introduction to completely separating systems (CSSs) is given in this section and some results on their structure and existence are presented in Section 2. The construction of all (n)CSSs for $n = 17$ to 20 is described in detail in Section 3. This begins to illustrate the variety of CSSs and the difficulty of their enumeration. A search algorithm is presented in Sections 4 and 5, with the results in Section 6.

The **volume** of a collection of sets $\mathcal{C}$ is $V(\mathcal{C}) = \sum_{A \in \mathcal{C}} |A|$. The integer $\mathbf{R(n)}$ is defined by $R(n) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is an } (n)\text{CSS}\}$. An (n)CSS for which $|\mathcal{C}| = R(n)$ is a **minimal** (n)CSS. An (n)CSS for which $|\mathcal{C}| = R(n) + s$ is an $(\mathbf{n}; \mathbf{s})$**CSS**. The **complementary CSS** of an (n)CSS $\mathcal{C}$ is $\overline{\mathcal{C}} = \{[n] \setminus A : A \in \mathcal{C}\}$.

Spencer [9] showed that

Lemma 1.1.

$$R(n) = \min\{t : \binom{t}{\lceil \frac{t}{2} \rceil} \geq n\}.$$

Explicit constructions of collections which achieve $R(n)$ were not supplied by Spencer.

Two (n)CSSs $\mathcal{C}$ and $\mathcal{D}$ are **isomorphic** if there exists a permutation π of $[n]$ such that $\mathcal{C}^\pi = \{A^\pi : A \in \mathcal{C}\} = \mathcal{D}$ where $A^\pi = \{\pi(x) : x \in A\}$. A p**-point** in a CSS is an element which occurs in exactly p blocks of the CSS. The **point profile** of a CSS $\mathcal{C}$ is a vector $(p_1^{s_1}, \ldots, p_t^{s_t})$ where s_i is the number of p_i-points in $\mathcal{C}$. The **block type** of a CSS $\mathcal{C}$ is a vector $(b_1^{s_1}, \ldots, b_t^{s_t})$ where s_i is the number of blocks of size b_i in $\mathcal{C}$.

CSSs have a dual formulation as antichains. An **antichain** on $[r]$ is a collection $\mathcal{A}$ of distinct subsets of $[r]$ such that for any distinct $A, B \in \mathcal{A}$, $A \not\subseteq B$. Let $\mathcal{A} = \{A_1, \ldots, A_r\}$ be a collection of subsets of $[n]$. An i**-set** in $\mathcal{A}$ is a set of size i. Cai [3] defined the **dual** $\mathcal{A}^*$ of $\mathcal{A}$ to be the collection $\mathcal{A}^* = \{X_1, \ldots, X_n\}$ of subsets of $[r]$ given by $X_i = \{k : i \in A_k\}$. Antichains are the duals of CSSs (Spencer [9]): If $\mathcal{A}$ is a CSS then its dual $\mathcal{A}^*$ is an antichain and vice versa. Define an antichain $\mathcal{A}$ in $2^{[r]}$ to be r**-native** if the size of the antichain exceeds the maximum size antichain on $[r-1]$.

The **squashed order** or **colex order** on a collection of sets $\mathcal{C}$ is defined by $A <_s B$ if the *largest* element of the symmetric difference of A and B is in B. The **shadow** of a collection of sets $\mathcal{A}$ with each set $A \in \mathcal{A}$ of cardinality k, is $\{B : |B| = k-1, B \subset A \text{ for some } A \in \mathcal{A}\}$.

A classic result on antichains is the Kruskal-Katona Theorem (see [2, Chapter 5]):

Theorem 1.2. *The cardinality of the shadow of m k-sets cannot be smaller than the cardinality of the shadow of the first m k-sets in squashed order. Further, the shadow of the first m k-sets in squashed order is a contiguous sequence of $(k-1)$-sets in squashed order beginning with the first set $\{1, 2, \ldots, k-1\}$.*

For $a_k > a_{k-1} > \ldots > a_t \geq t > 0$ the size of the shadow of the first $m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}$ k-sets in squashed order is given by $\binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_t}{t-1}$.

An important extension of the Kruskal-Katona Theorem involves the notion of a squashed antichain. An antichain $\mathcal{A}$ is **squashed** if for each i for which $\mathcal{A}$ contains i-sets, the collection A of proper i-subsets of sets in $\mathcal{A}$ precedes the collection B of i-sets in $\mathcal{A}$ in squashed order, and $A \cup B$ forms an initial segment of i-sets in squashed order. The number of i-sets in an antichain on $[r]$ can be denoted by p_i with $p_1, \ldots, p_r$ called the parameters of the antichain.

This allows the statement of the following theorem of Clements and Daykin et al. (see Theorem 10.2.1 in [1]).

Theorem 1.3. *There exists an antichain on $[r]$ with parameters $p_0, \ldots, p_r$ if and only if there is a squashed antichain on $[r]$ with the same parameters.*

2 Background results

Roberts and Rylands investigated the number of non-isomorphic minimum size (n)CSSs in [8]. The results in the next lemma and a complete catalogue of all minimum size (n)CSSs for $n \leq 10$ appear there.

Lemma 2.1. *Let $\mathcal{C}$ be a minimal CSS on $[n]$.*
i) Assume that $R(m) = R(n)$ for $m < n$. Then $n - m < |A| < m$ for each $A \in \mathcal{C}$.
ii) For $n \geq 5$, $\mathcal{C}$ contains at most one singleton block and $2n-1 \leq V(\mathcal{C}) \leq |\mathcal{C}|n - 2n + 1$. If there are no singleton blocks, then $2n \leq V(\mathcal{C}) \leq |\mathcal{C}|n - 2n$.

A very general statement about (n)CSSs is that as n increases, the structure of the CSSs becomes more varied and their number gets very large. However, when $n = \binom{r}{\lfloor r/2 \rfloor}$, and for values of n slightly smaller than this, there are few CSSs and they are easy to describe. As n decreases the number of (n)CSS increases until the value $\binom{r-1}{\lfloor (r-1)/2 \rfloor}$ is reached.

A well known and useful result is Sperner's Theorem [10] (see also [5]).

Theorem 2.2 (Sperner's Theorem). *The maximum size of an antichain on* $[r]$ *is* $\binom{r}{\lfloor r/2 \rfloor}$. *This is achievable in one way for* r *even, namely the collection of* $\frac{r}{2}$*-sets. When* r *is odd there are two ways to achieve this maximum: one consisting of* $\frac{r+1}{2}$*-sets and the other of* $\frac{r-1}{2}$*-sets.*

The CSS version of Sperner's Theorem can be stated as follows:
When $n = \binom{r}{\lfloor r/2 \rfloor}$ *there is one* (n)*CSS on* $[r]$ *when* r *is even, and it consists of* $\frac{r}{2}$*-points. When* r *is odd there are two* (n)*CSSs: one consisting of* $\frac{r+1}{2}$*-points and the other of* $\frac{r-1}{2}$*-points.*

Using Theorems 1.3 and 2.2, and with consideration of the dual antichains of CSSs on a given number of points, it is relatively easy to verify the following statements.

Removing one point from the (n)CSSs for $n = \binom{r}{\lfloor r/2 \rfloor}$ described in Theorem 2.2 shows that there is an $(n-1)$CSS when $R(n)$ is even and at least two $(n-1)$CSSs when $R(n)$ is odd. In these $(n-1)$CSSs all points occur in the same number of blocks, say p. These are the only $(n-1)$CSSs, and for $0 \le j < \lfloor \frac{r}{2} \rfloor$ all $(n-j)$CSSs contain p-points only. This is not the case for larger values of j.

To find all $(n-j)$CSSs for $0 \le j < \lfloor \frac{r}{2} \rfloor$ it is sufficient to find the number of non-isomorphic ways of removing j p-points from an (n)CSS. Note also that for a minimal (n)CSS with r blocks, a 1-point or an $(r-1)$-point can occur only when $n = \binom{r-1}{\lfloor (r-1)/2 \rfloor} + 1$, as it is only then that $R(n-1) < R(n)$.

3 Minimum size (n)CSSs for $11 \le n \le 20$

The number of non-isomorphic minimal (n)CSSs is derived here for $17 \le n \le 20$ within the more general context of $R(n) = 6$ for $11 \le n \le 20$. This is to illustrate some of the structural constraints that can aid the development of a catalogue of non-isomorphic minimal (n)CSSs. The computational approach is given in the next section.

The blocks in a CSS are shown here as rows in an array. All of the (n)CSSs will be minimal and so each will have 6 elements (the arrays have 6 rows). The sum of the volume of an (n)CSS and that of its complement is $6n$.

By Lemma 2.1 a minimal (n)CSS $\mathcal{C}$ with $n > 11$ has $n - 11 < |A| < 11$ for each $A \in \mathcal{C}$. By Section 2 if there is a 1-element or a 5-element then $n = 11$. Hence for $12 \le n \le 20$ the minimal (n)CSSs consist entirely of 2-points, 3-points or 4-points.

For b_4 4-sets and b_3 3-sets we find the maximum number of 2-sets, b_2, such that these sets form an antichain on $[6]$. The CSS is the dual of the antichain. We begin with zero 4-sets.

A maximal antichain on $[6]$ consisting only of 3-sets contains $\binom{6}{3} = 20$ sets. A maximal antichain on $[6]$ consisting only of 2-sets contains $\binom{6}{2} = 15$ sets. If an antichain is to have b_3 3-sets and the greatest possible number of 2-sets, then include the first b_3 3-sets in squashed order and the b_2 2-sets not covered by the 3-sets giving a total of $b_3 + b_2$ sets. This is a consequence of Theorem 1.3.

3-sets\4-sets	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	15	*9*	*6*	*5*	*5*	*5*	*2*	*1*	*1*	*1*	*0*	0	0	0	0	0
1	12	*7*	*5*	*5*	*3*	*3*	*1*	*1*	*0*	*0*	0	0	0			
2	10	*6*	*5*	*3*	*2*	*2*	*1*	*0*	*0*	0	0					
3	9	*6*	*5*	*2*	*2*	*2*	*1*	*0*	0	0						
4	9	*5*	*3*	*2*	*1*	*1*	*0*	0	0	0						
5	7	5	*2*	*1*	*1*	1	0	0								
6	6	5	*2*	*1*	1	1	0									
7	6	3	*1*	1	0	0	0									
8	5	2	1	0	0	0										
9	5	2	1	0	0	0										
10	5	1	0	0	0	0										
11	3	1	0	0												
12	2	1	0													
13	2	0	0													
14	1	0														
15	1	0														
16	1	0														
17	0															
18	0															
19	0															
20	0															

Table 1: The maximum number of 2-sets for the given number of 3-sets and 4-sets

Table 1 gives the value of b_2 in the first column of the body of the table. The values of b_3 are down the left hand side. The remainder of the table covers the cases in which the antichain contains at least one 4-set.

Consider the first b_4 4-sets in squashed order. They cover the first i 3-sets in squashed order where i can be determined by Theorem 1.2. Together with the next b_3 3-sets in squashed order these cover j 2-sets, which is the smallest number possible. Hence the maximum number of 2-sets in an antichain with b_4 4-sets and b_3 3-sets is $b_2 = \binom{6}{2} - j = 15 - j$. In Table 1 the values of b_4 are across the top, those of b_3 down the left side, and the values in the body of the table are those of b_2.

The second column of Table 1 can be obtained from the first by shifting it up 4 places as one 4-set covers four 3-sets. The third column can be obtained from the second by shifting it up 3 places as the second 4-set in squashed order covers a further three 3-sets, and so on.

A blank place in Table 1 indicates that the corresponding combination of 3-sets and 4-sets is not possible. A number in italics indicates that the combination can not give an (n)CSS for $n \geq 11$.

3.1 $R(n)$ for n from 17 to 20

The results presented are now used to find all (n)CSSs for $17 \leq n \leq 20$. The CSSs are represented as arrays, with blank spaces left to indicate entries removed.

3.1.1 $n = 19, 20$

By Theorem 2.2 there are unique antichains on [6] of sizes 19 and 20. This determines the unique (up to isomorphism) dual (20)CSS and the unique dual (19)CSS:

1	2	3	4	5	6	7	8	9	10
1	2	3	4	11	12	13	14	15	16
1	5	6	7	11	12	13	17	18	19
2	5	8	9	11	14	15	17	18	20
3	6	8	10	12	14	16	17	19	20
4	7	9	10	13	15	16	18	19	20

1	2	3	4	5	6	7	8	9	10
1	2	3	4	11	12	13	14	15	16
1	5	6	7	11	12	13	17	18	19
2	5	8	9	11	14	15	17	18	
3	6	8	10	12	14	16	17	19	
4	7	9	10	13	15	16	18	19	

.

The block types of these CSSs are 10^6 and $9^3 10^3$ respectively.

3.1.2 $n = 18$

By inspection of Table 1 every (18)CSS contains only 3-points. The (18)CSSs can be obtained from the (20)CSS by removing two points. These have block types 9^6, $8^1 9^4 10^1$ and $8^2 9^2 10^2$.

	2	3	4	5	6	7	8	9	10
	2	3	4	11	12	13	14	15	16
	5	6	7	11	12	13	17	18	19
2	5	8	9	11	14	15	17	18	
3	6	8	10	12	14	16	17	19	
4	7	9	10	13	15	16	18	19	

1	2	3	4	5	6	7	8	9	10
1	2	3	4		12	13	14	15	16
1	5	6	7		12	13	17	18	19
2	5	8	9		14	15	17	18	
3	6	8	10	12	14	16	17	19	
4	7	9	10	13	15	16	18	19	

1	2	3	4	5	6	7	8	9	10
1	2	3	4	11	12	13	14	15	16
1	5	6	7	11	12	13		18	19
2	5	8	9	11	14	15		18	
3	6	8	10	12	14	16		19	
4	7	9	10	13	15	16	18	19	

3.1.3 $n = 17$

For each $A \in \mathcal{C}$, $7 \leq |A| \leq 10$, and inspection of Table 1 shows that either all points are 3-points or there are 16 3-points with the 17th point a 2-point or a 4-point. Hence the possible volumes are 50, 51 and 52.

All (17)CSSs with only 3-points can be obtained from the (20)CSS of Section 3.1.1 by removing three points, or equivalently, by removing three 3-sets from the dual antichain. This yields the following seven non-isomorphic (17)CSSs:

1		3	4	5	6	7	8	9	10
1		3	4	11	12	13	14	15	16
1	5	6	7	11	12	13	17	18	
	5	8	9	11	14	15	17	18	
3	6	8	10	12	14	16	17		
4	7	9	10	13	15	16	18		

1	2	3	4	5	6		8	9	10
1	2	3	4		12	13	14	15	16
1	5	6			12	13	17	18	19
2	5	8	9		14	15	17	18	
3	6	8	10	12	14	16	17	19	
4		9	10	13	15	16	18	19	

1	2	3	4	5	6	7	8	9	10
1	2	3	4		12	13	14	15	16
1	5	6	7		12	13	17	18	
2	5	8	9		14	15	17	18	
3	6	8	10	12	14	16	17		
4	7	9	10	13	15	16		18	

1	2	3	4	5	6	7	8	9	
1	2	3	4	11	12		14	15	16
1	5	6	7	11	12		17	18	19
2	5	8	9	11	14	15	17	18	
3	6	8	12	14	16	17		19	
4	7	9	15	16	18			19	

1	2	3	4	5	6	7	8	9	10
1	2	3	4	11	12	13	14		16
1	5	6	7	11	12	13	17	18	
2	5	8	9	11	14		17	18	
3	6	8	10	12	14	16	17		
4	7	9	10	13		16	18		

1	2	3	4	5	6	7	8	9	10
1	2	3	4	11	12	13	14	15	16
1	5	6	7	11	12	13	17		
2	5	8	9	11	14	15	17		
3	6	8	10	12	14	16	17		
4	7	9	10	13	15	16			

1	2	3	4	5	6	7	8	9	10
1	2	3	4	11	12	13	14	15	
1	5	6	7	11	12	13	17	18	
2	5	8	9	11	14	15	17	18	.
3	6	8	10	12	14		17		
4	7	9	10	13	15		18		

The block types of these are $8^3 9^3$, $8^3 9^3$, $8^4 9^1 10^1$, $7^1 8^1 9^4$, $7^1 8^2 9^2 10^1$, $7^1 8^3 10^2$ and $7^2 9^3 10^1$. The first two CSSs can easily be seen to be non-isomorphic as the size of the union of their respective blocks of size 9 is different.

Now consider possible minimal (17)CSSs with 16 3-points and one point which is either a 2-point or a 4-point.

<u>Case 1:</u> Assume that there is a block of size 7 or 10. By considering the complementary CSS if necessary, it can be assumed that $1, \ldots, 10$ form a block. Then $1, \ldots, 10$ must be separated in the remaining five blocks, and at least nine of these points must appear as 2-points in these five blocks. The Roberts and Rylands catalogue [8] contains only one such (10)CSS, and it is this that is used in the first CSS below to separate $1, \ldots, 10$.

To completely separate $11, \ldots, 17$ in the last five rows all but one of these points must be a 3-point with the remaining being a 4-point, as the use of a 2-point will not allow complete separation. Only one of the (7)CSSs in the Roberts and Rylands catalogue [8] satisfies these conditions. The resulting (17)CSS, together with its complement are

1	2	3	4	5	6	7	8	9	10
1	2	3	4	11	12	13	14	15	16
1	5	6	7	11	12	13	17		
2	5	8	9	11	14	15	17		
3	6	8	10	12	14	16	17		
4	7	9	10	13	15	16	17		

1	2	3	5	6	7	11	13	15	
1	2	4	5	8	9	11	12	14	
1	3	4	6	8	10	12	13	16	
2	3	4	7	9	10	14	15	16	.
5	6	7	8	9	10	17			
11	12	13	14	15	16	17			

The block types of these CSSs are $8^4 10^2$ and $7^2 9^4$.

Case 2: Assume that all blocks are of size 8 or 9. The volume is either 50 or 52, so there are either 2 or 4 blocks of size 9. Assume that the first block contains $1, \ldots, 9$. These must be completely separated in the other 5 blocks. The catalogue in Roberts and Rylands [8] gives only one possibility, which forces all of the points to be 3-points. The eight elements $10, \ldots, 17$ must be separated with volume 23 or 25 as all but one must be a 3-point. Only one of the (8)CSS in the Roberts and Rylands [8] catalogue satisfies these conditions. There is only one way in which it can be included in the template to achieve complete separation, but the resulting CSS has blocks of size 7 and so does not give a new CSS.

Hence there are nine non-isomorphic minimal (17)CSSs. Volumes, block types and point profiles are shown here. Where a (17)CSS is isomorphic to its complement it is shown on a line by itself; where the complement is different the two appear on the same line.

Volume	Block Type	Point Profile	Complements: Volume	Block Type	Point Profile
51	$7^2 9^3 10^1$	3^{17}	51	$7^1 8^3 10^2$	3^{17}
51	$8^4 9^1 10^1$	3^{17}	51	$7^1 8^1 9^4$	3^{17}
51	$7^1 8^2 9^2 10$	3^{17}	51	$8^3 9^3$	3^{17}
51	$8^3 9^3$	3^{17}			
52	$8^4 10^2$	$3^{16} 4^1$	50	$7^2 9^4$	$2^1 3^{16}$

The enumeration of (n)CSSs for higher values of n cannot feasibly be tackled by hand. The CSSs for $11 \leq n \leq 35$ have been calculated by an exhaustive search. This was done independently by two of the authors, Montag and Grüttmüller, for $11 \leq n \leq 20$, and by Grüttmüller for $21 \leq n \leq 35$. Grüttmüller's method is the topic of the next two sections.

Note that the number of non-isomorphic minimum size CSSs for a fixed value of $r = R(n)$ corresponds exactly to the number of non-isomorphic r-native antichains.

4 Exhaustive search method

In this section, we describe the way in which an exhaustive search technique (backtracking) was applied to search for minimal (n)CSSs with n up to 20. We do this by building up feasible partial minimal (n)CSS in a systematic way. For more information on search techniques used in combinatorics see for example [6].

To avoid feasible (n)CSSs that are not minimal pruning methods will be used. Before proceeding, we need some more notation. Isomorphism of CSSs is an equivalence relation which partitions the set of all (n)CSSs into isomorphism classes. Our aim is to find from each isomorphism class a unique (canonical) representative (n)CSS. Let $\succeq$ be a total order relation on the set of all collections of r subsets of $[n]$. We call an (n)CSS $\mathcal{C}$ **canonical** if $\mathcal{C}$ is largest in its isomorphism class with respect to $\succeq$, that is $\mathcal{C} \succeq \mathcal{C}^\pi$ for all permutations π of $[n]$. Since it can be sometimes very time consuming to test all possible permutations, we introduce a further concept. Let $S \subseteq S_n$ be a set of permutations of the elements of $[n]$. An (n)CSS $\mathcal{C}$ is S**-canonical** if $\mathcal{C} \succeq \mathcal{C}^\pi$ for all $\pi \in S$. Clearly, if $\mathcal{C}$ is canonical then $\mathcal{C}$ is also S-canonical, but the opposite is in general not true. In particular, we used the set of all transpositions T for a fast, but incomplete isomorphism test.

Although any ordering $\succeq$ can be used it is beneficial to impose an ordering on the collections which is useful for pruning the backtrack search tree. For a subset $A \subseteq [n]$ define $\text{sep}(A) := \{(a,b) : a \in A, b \in [n] \setminus A\}$ the set of all pairs which are separated by A. Throughout this section we use the following order $\geq_{\text{sep}}$ on $2^{[n]}$. Let $A, B \in 2^{[n]}$. Then $A \geq_{\text{sep}} B$ if $|\text{sep}(A)| > |\text{sep}(B)|$ or $|\text{sep}(A)| = |\text{sep}(B)|$ and $A \leq_{\text{lex}} B$. Let $\mathcal{C}_1 = \{A_{1,1}, \ldots, A_{1,r}\}, \mathcal{C}_2 = \{A_{2,1}, \ldots, A_{2,r}\} \subseteq 2^{[n]}$ be two collections such that $A_{i,j} \geq_{\text{sep}} A_{i,j+1}$ for $i = 1, 2$ and $j = 1, \ldots, r-1$. Define $\succeq$ as follows: $\mathcal{C}_1 \succeq \mathcal{C}_2$ if and only if $\mathcal{C}_1 = \mathcal{C}_2$ or $A_{1,j^*} \geq_{\text{sep}} A_{2,j^*}$ for $j^* := \min\{j : A_{1,j} \neq A_{2,j}\}$.

Let the sets in $2^{[n]} \setminus \{\emptyset\} = \{A_1, \ldots, A_{2^n-1}\}$ be arranged in such a way that $A_i \geq_{\text{sep}} A_{i+1}$ for $i = 1, \ldots, 2^n - 2$. For a CSS $\mathcal{C}$ with r blocks let $\{c_j : j = 1, \ldots, r\}$ be the set of indices with $A_{c_j} \in \mathcal{C}$. The order of the sets in $\mathcal{C}$ is not fixed and thus $\mathcal{C}$ can be represented by the vector $\mathbf{c} = (c_1, \ldots, c_r)$ with $c_j < c_{j+1}$ for $j = 1, \ldots, r-1$. Note that we can not have $c_j = c_{j+1}$ for some $j \in [m-1]$ since there are no repeated sets in a minimal (n)CSS. For a permutation $\pi \in S_n$ define $\mathbf{c}^\pi = \text{sort}(\pi(c_1), \ldots, \pi(c_r))$ where $\pi(c_j) = j'$ if and only if $A^\pi_{c_j} = A_{j'}$ and the elements in $\mathbf{c}^\pi$ are sorted in increasing order.

A **partial CSS** $\mathcal{P}$ is a subcollection of a CSS $\mathcal{C}$. With the ordering above the following conditions are necessary for $\mathcal{P}$ to be a partial CSS of a canonical CSS $\mathcal{C}$ of minimum size r. Assume that $\mathcal{P}$ is represented by $(c_1, \ldots, c_m)$. Then

1. $\mathcal{P}$ is S-canonical for any $S \subseteq S_n$ or, equivalently, $(c_1, \ldots, c_m) \leq_{\text{lex}} (c_1, \ldots, c_m)^\pi$ for all $\pi \in S$; and
2. $|\bigcup_{j=1}^m \text{sep}(A_{c_j})| + |\text{sep}(A_{c_m+1})| \cdot (r-m) \geq |\{(a,b) : a, b \in [n], a \neq b\}| = n(n-1)$.

The second condition follows from the fact that the missing $r - m$ sets have to separate all pairs not already separated by $\mathcal{P}$ and that each set separates at most $|\text{sep}(A_{c_m+1})|$ pairs.

Now the backtracking algorithm below generates recursively all feasible partial CSSs exactly once and tries to extend as long as the necessary conditions are satisfied. There are a few pre-computations done in Step 0. and representations for sets are chosen which allow efficient implementations of operations like $\cup$ or $|\cdot|$. Clearly, this algorithm works only for n reasonably small.

Backtracking algorithm to find all canonical (n)CSSs with $r = R(n)$ sets

---

0. For each $A \in 2^{[n]}$ pre-compute $\text{sep}(A)$; for each $c \in [2^n - 1]$ and for each $\pi \in T$ pre-compute $\pi(c)$; define $\text{Sep}_0 := \emptyset$
1. **procedure** Search(m,$(c_1, c_2, \ldots, c_m)$)
2. **begin**
3. $\text{Sep}_m := \text{Sep}_{m-1} \cup \text{sep}(A_{c_m})$

4. **if** $m = r$, $|\mathrm{Sep}_m| = n(n-1)$ and $\mathcal{C} = \{A_{c_1}, \dots, A_{c_m}\}$ is canonical
5. **then** print solution $(c_1, \dots, c_m)$; return;
6. **else**
7. **for each** $c_{m+1} \in \{c_m + 1, \dots, 2^n - 2\}$ **do**
8. **if** $\{A_{c_1}, \dots, A_{c_{m+1}}\}$ is T-canonical and $|\mathrm{Sep}_m| + |\mathrm{sep}(A_{c_{m+1}})| \cdot (r-m) \geq n(n-1)$
9. **then** Search($m+1$,$(c_1, c_2, \dots, c_{m+1})$)
10. **end**

As a result of the algorithm, started with Search(1,(c_1)) for $c_1 = 1, \dots, 2^n - 2$, we were able to determine all minimal (n)CSSs for n up to 20. To extend the results to n up to 35 it requires a completely new approach which is explained in the next section. Note that this approach will use non-minimal $(n;s)$CSSs as ingredient substructures. These are constructed with a slight variation of the algorithm just explained (set $r = R(n) + s$ and allow $c_{m+1} = c_m$ in line 7.).

5 Union algorithm

It is a simple observation that if we take an arbitrary block X from an (n)CSS and consider the intersection $\mathcal{C}' = \{A \cap X : A \in \mathcal{C}, A \neq X\}$ and difference $\mathcal{C}'' = \{A \setminus X : A \in \mathcal{C}, A \neq X\}$ of the remaining blocks with X, then $\mathcal{C}'$ and $\mathcal{C}''$ are an $(|X|)$CSS and an $(n - |X|)$CSS, respectively. Note that $\mathcal{C}'$ and $\mathcal{C}''$ are not necessarily minimal and may contain empty sets. So an obvious idea to construct a minimal (n)CSS is to take an $(|X|)$CSS, an $(n-|X|)$CSS each on at most $R(n)-1$ blocks and to combine both in an appropriate way. What appropriate means is made more precise below.

Again, we define a total ordering $\geq_{\mathrm{card}}$ on $2^{[n]}$ as follows. Let $A, B \in 2^{[n]}$. Then $A \geq_{\mathrm{card}} B$ if and only if $|A| > |B|$, or $|A| = |B|$ and $A \leq_{\mathrm{lex}} B$. In turn, $\geq_{\mathrm{card}}$ induces an ordering on ordered collections of subsets from $[n]$ as follows. Let $\mathcal{A}_1 = \{A_{1,1}, \dots, A_{1,r}\}, \mathcal{A}_2 = \{A_{2,1}, \dots, A_{2,r}\} \subseteq 2^{[n]}$ be two collections such that $A_{i,j} \geq_{\mathrm{card}} A_{i,j+1}$ for $j = 1, \dots, r-1$. Define $\succeq$ as follows: $\mathcal{A}_1 \succeq \mathcal{A}_2$ if and only if $\mathcal{A}_1 = \mathcal{A}_2$ or $A_{1,j^*} \geq_{\mathrm{card}} A_{2,j^*}$ for $j^* := \min\{j : A_{1,j} \neq A_{2,j}\}$. The goal is to find a canonical representative in each isomorphism class. Note that the canonical predicate used here differs from the predicate defined in the previous section.

Now, let $\mathcal{C} = \{A_1, \dots, A_{R(n)}\}$ be a minimal (n)CSS, let $X = A_j \in \mathcal{C}$ be a block and $\varphi = \varphi(X) \in S_n$ an arbitrary permutation which maps X onto $[|X|]$. Define

$$\mathcal{C}'(X) = \mathcal{C}'_{\varphi,\pi,\sigma}(X) = \{A^{\pi\circ\varphi}_{\sigma(1)} \cap [|X|], \dots, A^{\pi\circ\varphi}_{\sigma(j-1)} \cap [|X|], A^{\pi\circ\varphi}_{\sigma(j+1)} \cap [|X|], \dots, A^{\pi\circ\varphi}_{\sigma(R(n))} \cap [|X|]\}$$

and

$$\mathcal{C}''(X) = \mathcal{C}''_{\varphi,\pi,\sigma}(X) = \{A^{\pi\circ\varphi}_{\sigma(1)} \setminus [|X|], \dots, A^{\pi\circ\varphi}_{\sigma(j-1)} \setminus [|X|], A^{\pi\circ\varphi}_{\sigma(j+1)} \setminus [|X|], \dots, A^{\pi\circ\varphi}_{\sigma(R(n))} \setminus [|X|]\}$$

where $\pi = \pi(X, \varphi) \in S_{|X|}$ and $\sigma = \sigma(X, \varphi, \pi) \in S_{R(n)}, \sigma(j) = j$ are chosen such that $\mathcal{C}'_{\varphi,\pi,\sigma}(X)$ is largest in its isomorphism class. Therefore, any minimal (n)CSS

$\mathcal{C}$ can be represented by a quadruple $(k_{\max}, \mathcal{C}', \mathcal{C}'', \sigma)$ where $k_{\max}$ is the size of a set X of maximum size in $\mathcal{C}$, $\mathcal{C}'$ is a $(k_{\max}; R(n) - r_1)$CSS isomorphic to $\mathcal{C}'(X)$, $\mathcal{C}''$ is a $(n - k_{\max}; R(n) - r_2)$CSS isomorphic to $\mathcal{C}''(X)$, and $\{[k_{\max}], A'_1 \cup (A''_{\sigma(1)} + k_{\max}), \ldots, A'_{R(n)-1} \cup (A''_{\sigma(R(n)-1)} + k_{\max})\}$ is isomorphic to $\mathcal{C}$. It is important to consider $\mathcal{C}'$ and $\mathcal{C}''$ as ordered collections since when putting both together we match the first set in $\mathcal{C}'$ with the first set in $\mathcal{C}''$, the second with the second and so forth. We may assume that the sets in $\mathcal{C}'$ are ordered with respect to $\geq_{\text{card}}$ and that the ordering of the sets in $\mathcal{C}''$ is described by σ.

Thus, when combining all possible choices for $k_{\max}, \mathcal{C}', \mathcal{C}''$ and σ at least one representative for each minimal (n)CSS is constructed. We only need to consider one of $\mathcal{C}$ or its complement $\overline{\mathcal{C}}$, namely the one with larger $k_{\max}$. Hence, we can assume that $(n+1)/2 \leq k_{\max}$ and $n - k_{\min} \leq k_{\max}$. The search spaces for C', C'' are the sets of all canonical $(k_{\max}; R(n) - r_1)$CSS respectively $(n - k_{\max}; R(n) - r_2)$CSS, where $R(k_{\max}) < r_1 \leq R(n)$ and $R(n - k_{\max}) < r_2 \leq R(n)$. These canonical non-minimal CSS are constructed with an algorithm very similar to the one in previous section. Finally, all possible permutations on the $R(n) - 1$ sets in the CSS are admissible for σ. Note that if $\mathcal{C}'$ or $\mathcal{C}''$ have less than $R(n) - 1$ sets, then we simply fill it up to size $R(n) - 1$ by adding some empty sets.

Unfortunately, the quadruple representation for a minimal (n)CSS is not unique. Thus, it is necessary to check if a CSS constructed is indeed new. This is achieved by building the canonical mate for each CSS and by storing it in a solution set for comparison. The details are given below.

Union algorithm to find all canonical (n)CSSs

---

procedure Generate()
begin
  Sol := $\emptyset$
  for $k_{\max} := \lfloor \frac{n+1}{2} \rfloor$ **to** $n - 1$ **do**
    for $r_1 := R(k_{\max}) + 1$ **to** $R(n)$ **do**
      for each $(k_{\max}; R(n) - r_1)$CSS $\mathcal{C}' = \{A'_1, \ldots, A'_{R(n)-1}\}$ **do**
        for $r_2 := R(n - k_{\max}) + 1$ **to** $R(n)$ **do**
          for each $(n - k_{\max}; R(n) - r_2)$CSS $\mathcal{C}'' = \{A''_1, \ldots, A''_{R(n)-1}\}$ **do**
            for each $\sigma \in S_{R(n)-1}$ **do**
              $\mathcal{C} := \{[k_{\max}], A'_1 \cup (A''_{\sigma(1)} + k_{\max}), \ldots, A'_{R(n)-1} \cup (A''_{\sigma(R(n)-1)} + k_{\max})\}$
              if ($\mathcal{C}$ is (n)-CSS **and**
                $n - k_{\max} \leq |A'_i \cup (A''_{\sigma(i)} + k_{\max})| \leq k_{\max}$ for $i = 1, \ldots, R(n) - 1$)
then
                $\hat{\mathcal{C}} :=$ MakeCanonical($\mathcal{C}$)
                if $\hat{\mathcal{C}} \notin$ Sol **then** print solution $\hat{\mathcal{C}}$; Sol := Sol $\cup \{\hat{\mathcal{C}}\}$ **end if**
              end if
            end for each // σ
          end for each // $\mathcal{C}''$
        end for // r_2

      end for each // $\mathcal{C}'$
    end for // r_1
  end for // $k_{\max}$
  for each $\mathcal{C} \in \mathrm{Sol}$ **do**
    if $k_{\max}(\mathcal{C}) > n - k_{\min}(\mathcal{C})$ **then** print solution $\bar{\mathcal{C}}$ **end if**
  end for // $\mathcal{C} \in \mathrm{Sol}$
end

An efficient implementation of the MakeCanonical() procedure is a key to success, so we describe it in some detail. For a given CSS $\mathcal{C} = \{A_1, \ldots, A_{R(n)}\}$ we need to find a permutation σ of the sets and a permutation π of the elements such that $\{A^\pi_{\sigma(1)}, \ldots, A^\pi_{\sigma(R(n))}\}$ is canonical. The first criterion of the canonical predicate is the size of the sets in $\mathcal{C}$ so we can arrange the sets in $\mathcal{C}$ with decreasing cardinality and may assume that σ permutes only sets within a cardinality class since applying π does not change the size of a set. Also it follows from the definition of canonical that it is possible to handle each cardinality class (called R_i in the following) separately starting with the class R_1 containing the sets of largest size. Now by induction it is easy to see that the set of all permutations Π which make $R_1, \ldots, R_{i-1}$ canonical can be represented as a collection $\mathcal{Z}(\Pi)$ of pairs $\{(Z^{\mathrm{dom}}{}_1, Z_1^{\mathrm{img}}), \ldots, (Z_m^{\mathrm{dom}}, Z_m^{\mathrm{img}})\}$ such that $\pi \in \Pi$ if and only if π maps the elements of Z_k^{dom} bijectively to the elements of Z_k^{img}. If the aim is to find the subset of Π that makes $R_1, \ldots, R_{i-1}, R_i$ canonical we have to find for each set $A \in R_i$ (in the order fixed by σ) a refinement of $\mathcal{Z}(\Pi)$, say $\mathcal{Z}(\hat{\Pi})$, such that for all $\pi \in \hat{\Pi}$ the set A^π is smallest possible. This can be achieved by shifting all elements of A to the left as far as possible with respect to $\mathcal{Z}(\Pi)$. More precisely, we map for each pair $(Z_k^{\mathrm{dom}}, Z_k^{\mathrm{img}})$ the elements which occur in both Z_k^{dom} and A to the smallest elements (first elements if we assume that sets are increasingly sorted) of Z_k^{img}. All necessary operations for this shifting are described in the sub-procedure $\mathrm{Shift}(\cdot,\cdot,\cdot)$. Note that distinct permutations σ of the sets in R_i lead to distinct $\mathcal{Z}(\Pi^\sigma)$ but possibly to the same canonical representation R_{shift} of R_i and it is not a priori clear which refinement of the $\mathcal{Z}(\Pi^\sigma)$s when shifting R_{i+1} will give the better canonical representation. So it is necessary to test in $\mathrm{MakeCanonical}(\cdot)$ for each R_i all set permutations σ (more precisely all permutations of the sets in $R_1, \ldots, R_{i-1}$ which produce the canonical $R_1^{\max}, \ldots, R_{i-1}^{\max}$ combined with all permutations of the sets in R_i), to save (in a stack) all σs and the corresponding $\mathcal{Z}(\Pi^\sigma)$s which give the overall best canonical representation $R_i^{\max}$ for later processing when handling R_{i+1}.

MakeCanonical procedure

---

procedure MakeCanonical($\mathcal{C} = \{A_1, \ldots, A_{R(n)}\}$)
begin
  Sort sets in $\mathcal{C}$ such that $|A_i| \geq |A_{i+1}|$ for $i = 1, \ldots, R(n) - 1$
  Form decomposition $\mathcal{R} = \{R_1, \ldots, R_m\}$ such that $\mathcal{C} = \bigcup_{i=1}^m R_i$, and
      $A, B \in R_i$ implies $|A| = |B|$ for $i = 1, \ldots, m$, and
      $A \in R_i, B \in R_{i+1}$ implies $|A| > |B|$ for $i = 1, \ldots, m-1$

stack^1 $\rightarrow$pushback($\{[n]\}, \{[n]\}$)
for $i := 1$ **to** m **do**
  $R_i^{\max} := R_i = \{A_{i,1}, \dots, A_{i,m_i}\}$
  while stack^i not empty **do**
    $(\mathcal{Z}^{\text{dom}}, \mathcal{Z}^{\text{img}}) :=$ stack^i $\rightarrow$popback()
    for each $\sigma \in S_{m_i}$ **do**
      $R_i^{\sigma} := \{A_{i,\sigma(1)}, \dots, A_{i,\sigma(m_i)}\}$
      $(R_{\text{shift}}, \mathcal{Z}^{\text{dom}}_{\text{shift}}, \mathcal{Z}^{\text{img}}_{\text{shift}}) := \text{Shift}(R_i^{\sigma}, \mathcal{Z}^{\text{dom}}, \mathcal{Z}^{\text{img}})$
      if $R_{\text{shift}} \succ R_i^{\max}$ **then**
        $R_i^{\max} := R_{\text{shift}}$
        stack^{i+1} $\rightarrow$clear()
      end if
      if $R_{\text{shift}} = R_i^{\max}$ **then**
        stack^{i+1} $\rightarrow$pushback($\mathcal{Z}^{\text{dom}}_{\text{shift}}, \mathcal{Z}^{\text{img}}_{\text{shift}}$)
      end if
    end for each // σ
  end while // stack^i not empty
end for // i
return $\mathcal{C} := \bigcup_{i=1}^{m} R_i^{\max}$
end

Shift procedure

---

procedure Shift($\{A_1, \dots, A_m\}$, $\mathcal{Z}^{\text{dom}}$, $\mathcal{Z}^{\text{img}}$)
begin
  $\mathcal{Z}^{\text{dom},0} := \mathcal{Z}^{\text{dom}}$; $\mathcal{Z}^{\text{img},0} := \mathcal{Z}^{\text{img}}$
  for $i := 1$ **to** m **do**
    $\mathcal{Z}^{\text{dom},i} := \emptyset$; $\mathcal{Z}^{\text{img},i} := \emptyset$
    $k_i := |\mathcal{Z}^{\text{dom},i-1}|$; let $\mathcal{Z}^{\text{dom},i-1} = \{Z_1^{\text{dom}}, \dots, Z_{k_i}^{\text{dom}}\}$ and
      let $\mathcal{Z}^{\text{img},i-1} = \{Z_1^{\text{img}}, \dots, Z_{k_i}^{\text{img}}\}$
    for $k = 1$ **to** k_i **do**
      $t := |Z_k^{\text{img}}|$; let $Z_k^{\text{img}} = \{z_1^{\text{img}}, \dots, z_t^{\text{img}}\}$ such that $z_\ell^{\text{img}} < z_{\ell+1}^{\text{img}}$
        for $\ell = 1, \dots, t-1$
      $\underline{Z}^{\text{dom}} := Z_k^{\text{dom}} \cap A_i$; $\overline{Z}^{\text{dom}} := Z_k^{\text{dom}} \setminus \underline{Z}^{\text{dom}}$;
      $d := |\underline{Z}^{\text{dom}}|$; $\underline{Z}^{\text{img}} := \{z_1^{\text{img}}, \dots, z_d^{\text{img}}\}$; $\overline{Z}^{\text{img}} := Z_k^{\text{img}} \setminus \underline{Z}^{\text{img}}$;
      if $\underline{Z}^{\text{dom}} \neq \emptyset$ **then**
        $\mathcal{Z}^{\text{dom},i} := \mathcal{Z}^{\text{dom},i} \cup \{\underline{Z}^{\text{dom}}\}$; $\mathcal{Z}^{\text{img},i} := \mathcal{Z}^{\text{img},i} \cup \{\underline{Z}^{\text{img}}\}$;
      end if
      if $\overline{Z}^{\text{dom}} \neq \emptyset$ **then**
        $\mathcal{Z}^{\text{dom},i} := \mathcal{Z}^{\text{dom},i} \cup \{\overline{Z}^{\text{dom}}\}$; $\mathcal{Z}^{\text{img},i} := \mathcal{Z}^{\text{img},i} \cup \{\overline{Z}^{\text{img}}\}$;
      end if
    end for // k
  end for // i

let π map for each $Z^{\text{dom}} \in \mathcal{Z}^{\text{dom},m}$ the ith element to the ith element in the corresponding $Z^{\text{img}} \in \mathcal{Z}^{\text{img},m}$
return $(\{A_1^\pi, \ldots, A_m^\pi\}, \mathcal{Z}^{\text{dom},m}, \mathcal{Z}^{\text{img},m})$
end

6 Results

Grüttmüller used the methods of the previous two sections to create a catalogue of minimal (n)CSSs for $11 \le n \le 35$. The number of minimal (n)CSSs for $7 \le n \le 10$ appears in Roberts and Rylands [8] and is included here for comparison. The values given here for $11 \le n \le 20$ were first calculated by Montag and Rylands in 2005 using a different algorithm to that described in Sections 4 and 5. The results are summarised in the following theorem.

Theorem 6.1. *For each $7 \le n \le 35$ the number of non-isomorphic CSSs which achieve $R(n)$ is shown in the row labelled d in Table 2 together with $R(n)$, and the minimum and maximum volumes for minimal (n)CSSs.*

n	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$R(n)$	5	5	5	5	6	6	6	6	6	6	6	6	6	6
d	18	7	2	2	1327	718	352	160	65	25	9	3	1	1
Min V	13	16	18	20	21	24	26	28	30	47	50	54	57	60
Max V	22	24	27	30	45	48	52	56	60	49	52	54	57	60

n	21	22	23	24	25	26
$R(n)$	7	7	7	7	7	7
d	5643146	2505350	1042262	406106	147540	49655
Min V	42	59	62	66	69	72
Max V	105	95	99	102	106	110

n	27	28	29	30	31	32	33	34	35
$R(n)$	7	7	7	7	7	7	7	7	7
d	15436	4455	1208	311	84	22	6	2	2
Min V	79	82	86	89	92	96	99	102	105
Max V	110	114	117	121	125	128	132	136	140

Table 2: The number of non-isomorphic minimal (n)CSSs, d, and maximum and minimum volumes.

The minimum and maximum volumes have been calculated and are shown as they are relevant to both combinatorial design theory and Sperner theory. The number of minimal (n)CSSs, d, correspond by duality to the number of non-isomorphic r-native antichains for $\binom{r-1}{\lceil (r-1)/2 \rceil} < n \le \binom{r}{\lceil r/2 \rceil}$. The minimum and maximum volumes apply equally to each class of CSSs and their corresponding dual class of antichains.

Figure 1 shows the number of (n)CSSs for each n. Note that the scale on the vertical axis is logarithmic. For the values shown here the number of (n)CSSs decreases monotonically for $\binom{t}{\lceil t/2\rceil} < n < \binom{t+1}{\lceil (t+1)/2\rceil}$.

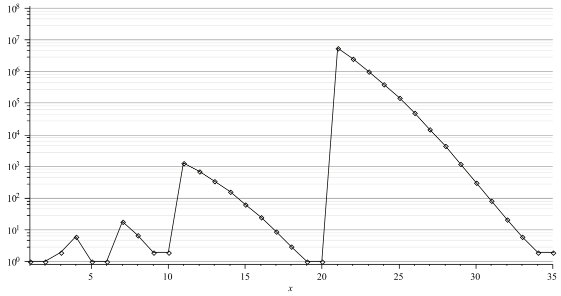

Figure 1: The number of non-isomorphic minimal (n)CSSs for $1 \leq n \leq 35$.

We conjecture that this monotonicity continues. The graph also suggests that the decrease is exponential. This conjecture about the monotonicity dualises to a conjecture on antichains: for each r, the number of non-isomorphic r-native antichains of size n is greater than the number of r-native antichains of size $n+1$ except when $n = \binom{r}{\lceil r/2\rceil} - 1$.

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