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An isomorphism theorem for digraphs

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Abstract

A seminal result by Lovász states that two digraphs A and B (possibly with loops) are isomorphic if and only if for every digraph X the number of homomorphisms $X \to A$ equals the number of homomorphisms $X \to B$. Lovász used this result to deduce certain cancellation properties for the direct product of digraphs.

We develop an analogous result for the class of digraphs without loops, and with weak homomorphisms replacing homomorphisms. We show that two digraphs A and B (without loops) are isomorphic if and only if the number of weak homomorphisms $X \to A$ equals the number of weak homomorphisms $X \to B$. This result is then applied to deduce a general cancellation property for the strong product of digraphs as well as graphs.

1 Introduction

A *digraph* A is a binary relation $E(A)$ on a finite vertex set $V(A)$; that is, $E(A) \subseteq V(A) \times V(A)$. For brevity, an ordered pair $(a, a') \in E(G)$ is denoted aa', and is visualized as an arrow pointing from a to a'. Elements of $E(A)$ are called *arcs*. A reflexive arc aa is called a *loop*, and is drawn as a closed curve beginning and ending at a. (In drawings, a loop is not embellished with an arrowhead.)

In this paper we denote by Γ_0 the class of digraphs which may have loops. The class $\Gamma \subset \Gamma_0$ consists of all digraphs without loops.

A *graph* is a symmetric digraph, that is a digraph A for which $aa' \in E(A)$ if and only if $a'a \in E(A)$. Given digraphs A and B, a *homomorphism* $f : A \to B$ is a map $f : V(A) \to V(B)$ for which $aa' \in E(A)$ implies $f(a)f(a') \in E(B)$. A *weak homomorphism* $f : A \to B$ is a map $f : V(A) \to V(B)$ for which $aa' \in E(A)$ implies that either $f(a)f(a') \in E(B)$ or $f(a) = f(a')$. Every homomorphism is a weak homomorphism, but not conversely.

As usual, an *isomorphism* is an bijective homomorphism whose inverse is also a homomorphism. We denote the condition of A and B being isomorphic as $A \cong B$.

There are several primary means of forming products of digraphs. Given digraphs A and B in Γ_0, the *direct product* $A \times B$ is the digraph whose vertex set is $V(A) \times V(B)$ and for which $(a,b)(a',b')$ is an arc precisely if $aa' \in E(A)$ and $bb' \in E(B)$. Figure 1(a) shows a direct product $A \times B$.

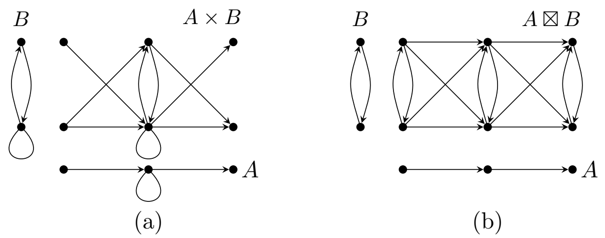

Figure 1: A direct product (a) and a strong product (b)

The *strong product*, denoted $A \boxtimes B$ is another primary product. It is the digraph whose vertex set is $V(A) \times V(B)$ and for which $(a,b)(a',b')$ is an arc precisely if one of the following conditions holds: $aa' \in E(A)$ and $bb' \in E(B)$; or $a = a'$ and $bb' \in E(B)$; or $aa' \in E(A)$ and $b = b'$. Figure 1(b) shows a strong product $A \boxtimes B$. See [2] for a survey of various products in the context of graphs.

In Γ_0, the direct product $A \times B$ is the maximal digraph on $V(A) \times V(B)$ for which both projections $V(A) \times V(B) \to V(A)$ and $V(A) \times V(B) \to V(B)$ are homomorphisms. Analogously, if we work only in the class Γ (as opposed to Γ_0), then $A \boxtimes B$ is the maximal digraph on $V(A) \times V(B)$ for which both projections $V(A) \times V(B) \to V(A)$ and $V(A) \times V(B) \to V(B)$ are weak homomorphisms. This seems to suggest that, whereas $\times$ is a natural product on Γ_0, it may be that $\boxtimes$ is a more natural product on Γ.

In this paper we draw an analogy between two categories. On one hand we have the category of digraphs Γ_0, whose product is $\times$ and whose morphisms are graph homomorphisms. On the other hand there is the category Γ with product $\boxtimes$ and where the morphisms are weak homomorphisms. We show that a major theorem in the first category (Theorem 1, below) has a corresponding version in the second category, and we give an application.

We denote by $\hom(A,B)$ the number of homomorphisms from A to B. Likewise, $\hom_w(A,B)$ denotes the number of weak homomorphisms from A to B. The following theorem by Lovász is proved in [4] and [1].

Theorem 1 *If A and B are digraphs in Γ_0, then $A \cong B$ if and only if* $\hom(X,A) = \hom(X,B)$ *for every digraph X.*

Lovász also noted the identity $\hom(X, A \times B) = \hom(X,A) \cdot \hom(X,B)$. From this he deduced various cancellation laws for the direct product. For instance, suppose $A \times C \cong B \times C$, where the digraph C has a loop. Then we conclude $A \cong B$, as follows: From $A \times C \cong B \times C$ we get $\hom(X, A \times C) = \hom(X, B \times C)$ for every X, and therefore $\hom(X, A) \cdot \hom(X, C) = \hom(X, B) \cdot \hom(X, C)$. But $\hom(X, C) \neq 0$ as the constant map sending $V(X)$ to a vertex with a loop in C is a homomorphism. So $\hom(X, A) = \hom(X, B)$ for all X, so $A \cong B$.

Our main intention is to prove the following variant of Lovász's theorem.

Theorem 2 *If A and B are digraphs in Γ, then $A \cong B$ if and only if* $\hom_w(X, A) = \hom_w(X, B)$ *for every digraph X.*

We will also show that $\hom_w(X, A \boxtimes B) = \hom_w(X, A) \cdot \hom_w(X, B)$ for any digraph X. Then, from reasoning parallel to Lovász's, we will prove that $A \boxtimes C \cong B \boxtimes C$ implies $A \cong B$ for digraphs in Γ.

In preparation for these results, we make several remarks concerning quotients.

2 Quotients

There are two notions of a digraph quotient, depending on whether we work in Γ_0 or Γ.

Suppose $A \in \Gamma_0$ and Ω is a partition of $V(A)$. The **quotient in Γ_0 of A by Ω** is another digraph in Γ_0, which we denote as A/Ω. Its vertex set is $V(A/\Omega) = \Omega$, and its edge set is

$$E(A/\Omega) = \{UV : U, V \in \Omega \text{ and } \exists\ uv \in E(A) \text{ with } u \in U,\ v \in V\}.$$

Similarly, we have a notion of a quotient in Γ. Its definition is the same as above, but loops are not allowed. Given $A \in \Gamma$ and a partition Ω of $V(A)$, the **quotient in Γ of A by Ω** is a digraph in Γ which we denote as A/Ω. Its vertex set is $V(A/\Omega) = \Omega$, and its edge set is

$$E(A/\Omega) = \{UV : U, V \in \Omega \text{ and } \exists\ uv \in E(A) \text{ with } u \in U,\ v \in V,\ U \neq V\}.$$

Figure 2 shows a digraph A, a partition Ω, and the quotients A/Ω in Γ_0 and Γ.

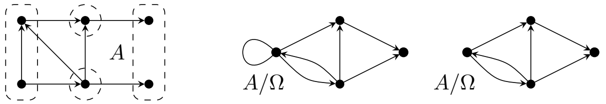

Figure 2: A digraph A and its quotients A/Ω in Γ_0 and Γ

We note a fundamental fact. For each statement, the one-sentence proof is straightforward.

Proposition 1 *Associated with any quotient A/Ω is a map $\lambda_\Omega : A \to A/\Omega$ defined as $\lambda_\Omega(u) = U$, where $u \in U \in \Omega$. If A/Ω is a quotient in Γ_0, then λ_Ω is a homomorphism. If A/Ω is a quotient in Γ, then λ_Ω is a weak homomorphism.*

The next lemma is stated for digraphs (and quotients) in Γ, but the same statement holds for digraphs (and quotients) in Γ_0, provided the term "weak homomorphism" is replaced with "homomorphism." The proof is nearly identical, but it is omitted since we only need the result for Γ.

Proposition 2 *Suppose $f : X \to A$ is a weak homomorphism between digraphs in Γ. Then there is a unique pair (Ω_f, f^*), where Ω_f is a partition of $V(X)$, and $f^* : X/\Omega_f \to A$ is an injective homomorphism for which $f = f^*\lambda_{\Omega_f}$.*

Proof. Suppose $f : X \to A$ is as stated. Let $\Omega_f = \{f^{-1}(a) : a \in V(A)\}$. Define the map $f^* : X/\Omega_f \to A$ as $f^*(U) = f(u)$, where $u \in U$. This is clearly well-defined, injective, and satisfies $f = f^*\lambda_{\Omega_f}$. Moreover, f^* is a homomorphism, as follows. Suppose $UV \in E(X/\Omega_f)$. This means $U \neq V$ and X has an edge uv with $u \in U$ and $v \in V$. Since f is a weak homomorphism we have either $f(u)f(v) \in E(A)$ or $f(u) = f(v)$. Now, $f^*(U)f^*(V) = f(u)f(v)$, and $f(u) \neq f(v)$ because f^* is injective. Thus $f^*(U)f^*(V) = f(u)f(v) \in E(A)$, so f^* is indeed a (injective) homomorphism.

Now we confirm uniqueness. Suppose there is a pair (Ω, g) for which $g : X/\Omega \to A$ is an injective homomorphism, and $f = g\lambda_\Omega$. Observe that $(\Omega, g) = (\Omega_f, f^*)$, as follows. Using the facts that $f = g\lambda_\Omega$ and g is injective, we see that two vertices u, v are in the same class of Ω_f if and only if $f(u) = f(v)$, if and only if $g\lambda_\Omega(u) = g\lambda_\Omega(v)$, if and only if $\lambda_\Omega(u) = \lambda_\Omega(v)$, if and only if u and v are in the same class in Ω. Thus $\Omega_f = \Omega$. To confirm $f^*(U) = g(U)$ for any U, take $u \in U$ and note $f^*(U) = f(u) = g\lambda_{\Omega_f}(u) = g\lambda_\Omega(u) = g(U)$. ■

3 Main Theorem

We now prove our main theorem, which hinges on the following lemma. The proof of our lemma and theorem follow along the lines of the proof of Theorem 1 that can be found in [1] (Theorem 2.11). The only substantial difference is that we work in Γ rather than Γ_0 and use weak homomorphism in place of homomorphisms. By *injective homomorphism* we mean a homomorphism that is injective as a map on vertex sets. (Any injective weak homomorphism is therefore a homomorphism.)

Lemma 1 *Suppose X and A are digraphs in Γ. Let $\mathscr{P}$ be the set of all partitions of $V(X)$ and let $\mathrm{inj}(X, A)$ be the number of injective homomorphisms $X \to A$. Then*

$$\hom_w(X, A) = \sum_{\Omega \in \mathscr{P}} \mathrm{inj}(X/\Omega, A),$$

where quotients are taken in Γ (as opposed to Γ_0).

Proof. Let $\mathrm{Hom}_w(X, A)$ be the set of all weak homomorphisms from X to A, so its cardinality is $\hom_w(X, A)$. Likewise, let $\mathrm{Inj}(X, A)$ be the set of all injective homomorphisms from X to A. Let

$$\Upsilon = \{(\Omega, f^*) : \Omega \in \mathscr{P},\ f^* \in \mathrm{Inj}(X/\Omega, A)\},$$

so $|\Upsilon| = \sum_{\Omega\in\mathscr{P}} \mathrm{inj}(X/\Omega, A)$. To finish the proof we produce a bijection $\beta : \mathrm{Hom}_w(X, A) \to \Upsilon$.

By Proposition 2, any $f \in \mathrm{Hom}_w(X, A)$ is associated with a unique pair $(\Omega_f, f^*) \in \Upsilon$, where $\Omega_f = \{f^{-1}(a) : a \in V(A)\}$ and $f^* : X/\Omega_f \to A$ is defined as $f^*(U) = f(u)$ for $u \in U$. Thus we have an map $\beta : \mathrm{Hom}_w(X, A) \to \Upsilon$ defined as $\beta(f) = (\Omega_f, f^*)$. This map is injective, for if $\beta(f) = \beta(g)$, then $(\Omega_f, f^*) = (\Omega_g, g^*)$ and Proposition 2 yields $f = f^*\lambda_{\Omega_f} = g^*\lambda_{\Omega_g} = g$.

To see that β is surjective, take any $(\Omega, f^*) \in \Upsilon$. Then the composition $f = f^*\lambda_\Omega$ of weak homomorphisms is a weak homomorphism. Following the definitions, we get $\beta(f) = (\Omega, f^*)$. ∎

Theorem 2 *If $A, B \in \Gamma$ and $\hom_w(X, A) = \hom_w(X, B)$ for every $X \in \Gamma$, then $A \cong B$.*

Proof. Suppose $\hom_w(X, A) = \hom_w(X, B)$ for every X. Our strategy is to show that this implies $\mathrm{inj}(X, A) = \mathrm{inj}(X, B)$ for every X. Then the proposition will follow because we get $\mathrm{inj}(B, A) = \mathrm{inj}(B, B) > 0$ and $\mathrm{inj}(A, B) = \mathrm{inj}(A, A) > 0$, so there are injective homomorphisms $A \to B$ and $B \to A$, whence $A \cong B$.

We use induction on $|V(X)|$ to show $\mathrm{inj}(X, A) = \mathrm{inj}(X, B)$ for all X. If $|V(X)| = 1$, then

$$\mathrm{inj}(X, A) = |V(A)| = \hom_w(X, A) = \hom_w(X, B) = |V(B)| = \mathrm{inj}(X, B).$$

If $|V(X)| > 1$, then Lemma 1 applied to the equation $\hom_w(X, A) = \hom_w(X, B)$ produces

$$\sum_{\Omega\in\mathscr{P}} \mathrm{inj}(X/\Omega, A) = \sum_{\Omega\in\mathscr{P}} \mathrm{inj}(X/\Omega, B).$$

Let T be the trivial partition of $V(X)$ consisting of $|V(X)|$ singleton sets. Then $X/T = X$ and the above equation becomes

$$\mathrm{inj}(X, A) + \sum_{\Omega\in\mathscr{P}-T} \mathrm{inj}(X/\Omega, A) \quad = \quad \mathrm{inj}(X, B) + \sum_{\Omega\in\mathscr{P}-T} \mathrm{inj}(X/\Omega, B).$$

The summations are equal by the inductive hypothesis, hence $\mathrm{inj}(X, A) = \mathrm{inj}(X, B)$. ∎

4 Application

We now prove that, for digraphs in Γ, the expression $A \boxtimes C \cong B \boxtimes C$ implies $A \cong B$. This cancellation result may not be entirely new—it was proved for graphs in [3]—but our approach appears to be novel, and it works for both graphs and digraphs. Our proof requires the following result.

Proposition 3 *If* $X, A, B \in \Gamma$, *then* $\hom_w(X, A \boxtimes B) = \hom_w(X, A) \cdot \hom_w(X, B)$.

Proof. Given that the projections $A \boxtimes B \to A$ and $A \boxtimes B \to B$ are weak homomorphisms, and compositions of weak homomorphisms are weak homomorphisms, it follows readily that any weak homomorphism $f : X \to A \boxtimes B$ has component form $f = (f_A, f_B)$ where f_A and f_B are weak homomorphisms from X to A and B, respectively. Conversely, any such map $f = (f_A, f_B)$ is easily seen to be a weak homomorphism. The result follows. ■

Theorem 3 *Suppose* $A, B, C \in \Gamma$. *If* $A \boxtimes C \cong B \boxtimes C$, *then* $A \cong B$.

Proof. If $A \boxtimes C \cong B \boxtimes C$, then certainly $\hom_w(X, A \boxtimes C) = \hom_w(X, B \boxtimes C)$ for every graph X. Thus $\hom_w(X, A) \cdot \hom_w(X, C) = \hom_w(X, B) \cdot \hom_w(X, C)$ by Proposition 3. But $\hom_w(X, C) \neq 0$ because any constant map $X \to C$ is a weak homomorphism. Therefore $\hom_w(X, A) = \hom_w(X, B)$ for every graph X, and Theorem 2 implies $A \cong B$. ■

As graphs are merely symmetric digraphs, Theorem 3 also holds also if A, B and C are graphs without loops. However, the requirement that there be no loops is absolutely essential, for cancellation can fail otherwise: Let C be the complete (symmetric) graph K_2 on two vertices, with loops added to each vertex. Let $A = C$ and let $B = K_2$. Note that $A \boxtimes C \cong B \boxtimes C$, but $A \ncong B$.

Remarks This paper is based on the first author's masters degree thesis, which was directed by the second author. We thank the referee for a prompt and thoughtful response.

The referee raises an interesting point. For $A \in \Gamma$, let $\mathscr{L}(A)$ denote A with loops added to all vertices. Then $\hom_w(X, A) = \hom(X, \mathscr{L}(A))$ for all $X \in \Gamma$, so

$$\hom_w(X, A) = \hom_w(X, B) \iff \hom(X, \mathscr{L}(A)) = \hom(X, \mathscr{L}(B)).$$

Now, if $\hom(X, \mathscr{L}(A)) = \hom(X, \mathscr{L}(B))$ holds for some $X \in \Gamma_0$, then this equality remains valid when loops are arbitrarily inserted to or deleted from X. Thus, if $\hom_w(X, A) = \hom_w(X, B)$ for all $X \in \Gamma$, then $\hom(X, \mathscr{L}(A)) = \hom(X, \mathscr{L}(B))$ for all $X \in \Gamma_0$, and $\mathscr{L}(A) \cong \mathscr{L}(B)$ by Theorem 1. Given that $\mathscr{L}(A) \cong \mathscr{L}(B)$ if and only if $A \cong B$ (for digraphs in Γ), it follows that Theorem 2 can be viewed as a consequence of Theorem 1.

The referee also notes that Theorem 3 can be proved as follows. First observe that $\mathscr{L}(A \boxtimes C) = \mathscr{L}(A) \times \mathscr{L}(C)$. Now if $A \boxtimes C \cong B \boxtimes C$, then $\mathscr{L}(A) \times \mathscr{L}(C) \cong \mathscr{L}(B) \times \mathscr{L}(C)$. Given that $\mathscr{L}(C)$ has a loop, we get $\mathscr{L}(A) \cong \mathscr{L}(B)$ by Lovász's cancellation result, hence $A \cong B$.

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