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A note on the connectivity of the Cartesian product of graphs

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Abstract

We present an example of two connected graphs for which the connectivity of the Cartesian product of the graphs is strictly greater than the sum of the connectivities of the factor graphs. This clarifies an issue from the literature. We then find necessary and sufficient conditions for the connectivity of the Cartesian product of the graphs to be equal to the sum of the connectivities of the individual graphs.

Throughout this paper we strive to use general terminology in graph theory from [3], and terminology concerning Cartesian products of graphs from [4].

Let $G = (V(G), E(G))$ be a graph, and let $\delta(G)$ denote the minimum degree among all vertices in G. The connectivity $\kappa(G)$ of G is the minimum size of $S \subseteq V(G)$ such that $G - S$ is disconnected or a single vertex.

The Cartesian product $G \Box H$ of two graphs G and H is the graph with vertex set $V(G \Box H) = V(G) \times V(H)$, and edge set $E(G \Box H)$ containing all pairs of the form $[(g_1, h_1), (g_2, h_2)]$ such that either $[g_1, g_2]$ is an edge in G and $h_1 = h_2$, or $[h_1, h_2]$ is an edge in H and $g_1 = g_2$. See [4] for a thorough discussion of Cartesian products of graphs.

How is the connectivity of $G \Box H$ related to the connectivity of G and the connectivity of H? This question was answered recently by the following result.

Theorem 1 (Špacapan [8]) *If G and H are nontrivial graphs, then*

$$\kappa(G \Box H) = \min\{\kappa(G)|V(H)|,\ \kappa(H)|V(G)|,\ \delta(G) + \delta(H)\}. \tag{1}$$

As Špacapan points out, it was claimed in [6] that, if G and H are connected and non-trivial, then $\kappa(G \Box H) = \kappa(G) + \kappa(H)$. We note that the same claim was made in [1, Theorem 3], [2, p. 37], and [5, p. 81]. Špacapan states that this claim is incorrect but he does not provide a counterexample.

We now present a counterexample to show that the claim is incorrect.

A friendship graph F_n may be formed by joining together n copies of the cycle C_3 at a common vertex; F_n has $2n+1$ vertices and $3n$ edges. In F_n, any two distinct vertices have exactly one neighbour in common. Apart from the common vertex, the degree of any vertex is 2; the common vertex has degree $2n$. We observe that $\kappa(F_n) = 1, n \geq 2$.

Let P_m be the path on m vertices, $m \geq 3$. The degree of each end-vertex is 1, the degree of every other vertex is 2, and $\kappa(P_m) = 1, m \geq 3$.

Hence, $\kappa(F_n) + \kappa(P_m) = 2$. However, according to Theorem 1, $\kappa(F_n \Box P_m) = 3, n \geq 2, m \geq 3$. Indeed, one can check manually that $\kappa(F_2 \Box P_3) = 3$. If $G = F_n$ and $H = P_m$ $(n \geq 2, m \geq 3)$ then $\kappa(G \Box H) > \kappa(G) + \kappa(H)$.

This is the desired counterexample.

The following inequality is due to G. Sabidussi. ([7, Lemma 2.3], [4, Cor. 5.2, p. 41]).

Lemma 1 (Sabidussi's inequality) *For any connected graphs G and H,*

$$\kappa(G \Box H) \geq \kappa(G) + \kappa(H). \tag{2}$$

Furthermore, there are examples of G and H for which $\kappa(G \Box H) = \kappa(G) + \kappa(H)$.

We now present necessary and sufficient conditions for $\kappa(G \Box H) = \kappa(G) + \kappa(H)$.

Theorem 2 *If G and H are two non-empty graphs then*

$$\kappa(G \Box H) = \kappa(G) + \kappa(H) \tag{3}$$

if and only if at least one of the following conditions is satisfied:

- *(C1)* G *or* H *is a singleton*
- *(C2)* $\kappa(G) = \delta(G)$ *and* $\kappa(H) = \delta(H)$
- *(C3)* $\kappa(G) = 1$ *and* H *is complete (or vice versa)*
- *(C4)* $\kappa(G) = 0$ *and* $\kappa(H) = 0$.

Proof: In the forward direction we compare the equality (3) with each of the terms in (1). In the reverse direction it is shown that each condition implies (3).

$(\Rightarrow)$ Suppose $\kappa(G \Box H) = \kappa(G) + \kappa(H)$. By (1), $\kappa(G) + \kappa(H)$ is bounded above by each of the terms in the minimum, and therefore, there are essentially two cases to be considered.

1. If $\kappa(G) + \kappa(H) = \delta(G) + \delta(H)$ then $\kappa(G) = \delta(G)$ and $\kappa(H) = \delta(H)$ since $\kappa(X) \leq \delta(X)$ in general. This leads to (C2).

2. We may suppose $\kappa(G) + \kappa(H) = \kappa(G)|H|$, from which $\kappa(H) = \kappa(G)(|H| - 1)$. There are now three subcases to consider.

   (a) If $\kappa(G) \geq 2$ then $\kappa(H) \geq 2(|H| - 1)$, but $|H| - 1 \geq \kappa(H)$, therefore $|H| - 1 = 0$ and H is a singleton. This leads to (C1).

   (b) If $\kappa(G) = 1$ then $\kappa(H) = |H| - 1$, therefore H is complete. This leads to (C3).

   (c) If $\kappa(G) = 0$ then $\kappa(H) = 0$. This leads to (C4).

($\Leftarrow$) Check each condition.

1. Assume (C1). If H is a singleton then $\kappa(H) = 0$, $G \Box H$ is isomorphic to G, and therefore (3) is true. Henceforth, we will assume that neither G nor H are singletons.

2. Assume (C2). If $\kappa(G) = \delta(G)$ and $\kappa(H) = \delta(H)$ then, using (2),

$$\kappa(G) + \kappa(H) = \delta(G) + \delta(H) = \delta(G \Box H) \geq \kappa(G \Box H) \geq \kappa(G) + \kappa(H),$$

   and therefore (3) is true.

3. Assume (C3). Suppose $\kappa(G) = 1$ and H is complete, say $H = K_m$ for some $m \geq 2$. Then G is connected and $|G| \geq 2$. By (1) we have

$$\begin{aligned} \kappa(G \Box K_m) &= \min\{\kappa(G)|K_m|,\ \kappa(K_m)|G|,\ \delta(G) + \delta(K_m)\} \\ &= \min\{m,\ (m-1)|G|,\ \delta(G) + m - 1\} \end{aligned} \tag{4}$$

   Under these conditions, $\kappa(G \Box K_m) = m$. Then $\kappa(G \Box K_m) = m = 1 + (m-1) = \kappa(G) + \kappa(K_m)$, and therefore (3) is true.

4. Assume (C4). If $\kappa(G) = 0$ and $\kappa(H) = 0$, then both G and H are disconnected, and therefore $G \Box H$ is disconnected, that is $\kappa(G \Box H) = 0$, and therefore (3) is true.

This completes the proof of Theorem 2.

Corollary 1 *If G and H are non-empty graphs such that $\kappa(G) = \delta(G)$ and $\kappa(H) = \delta(H)$ then $\kappa(G \Box H) = \kappa(G) + \kappa(H)$.*

This was first proved by G. Sabidussi [7, Lemma 2.4].

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References

[1] K. W. Cattermole, Bipartite communication networks with optimal connectivity, *Electronic Letters*, **8**(15) (1972), 385–388.

[2] K. W. Cattermole, Graph theory and communication networks, In: R.J. Wilson and L.W. Beineke, *Applications of Graph Theory*, London: Academic Press, 1979, 17–57.

[3] R. Diestel, *Graph Theory*, Third Ed., New York: Springer, 2005.

[4] W. Imrich, S. Klavžar and D. F. Rall, *Topics in Graph Theory: Graphs and their Cartesian Products*, Wellesley, MA: A. K. Peters Ltd., 2008.

[5] J. Lauri and R. Scapellato, *Topics in Graph Automorphisms and Reconstruction*, Cambridge: Cambridge University Press, 2003.

[6] Y. Niu and B. Zhu, Connectivities of Cartesian products of graphs, In Y. Alavi, D. R. Lick and J. Liu (eds) *Combinatorics, Graph Theory, Algorithms and Applications (Beijing, 1993)*, World Sci. Publ., River Edge, NJ, 1994, 301–305.

[7] G. Sabidussi, Graphs with given group and given graph-theoretical properties, *Canad. J. Math.*, **9**(1957), 515–525.

[8] S. Špacapan, Connectivity of Cartesian products of graphs, *Applied Mathematics Letters*, **21**(7)(2008), 682–685.

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