AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **49** (2011), Pages 217–226

# On the rainbow $k$-connectivity of complete graphs*

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**Abstract**

A path in an edge-colored graph $G$, where adjacent edges may be colored the same, is called a rainbow path if no two edges of the path are colored the same. For a $\kappa$-connected graph $G$ and an integer $k$ with $1 \leq k \leq \kappa$, the rainbow $k$-connectivity $rc_k(G)$ of $G$ is defined as the minimum integer $j$ for which there exists a $j$-edge-coloring of $G$ such that every two distinct vertices of $G$ are connected by $k$ internally disjoint rainbow paths. This paper is to investigate the rainbow $k$-connectivity of complete graphs. We improve the upper bound of $f(k)$ from $(k+1)^2$ by Chartrand et al. to $ck^{\frac{3}{2}} + C(k)$, where $c$ is a constant, $C(k) = o(k^{\frac{3}{2}})$, and $f(k)$ is the integer such that if $n \geq f(k)$ then $rc_k(K_n) = 2$. Recently, we saw that Dellamonica et al. got the best possible upper bound $2k$, which is linear in $k$. However, our proof is more structural or constructive.

## 1 Introduction

All graphs considered in this paper are simple, finite and undirected. Let $G$ be a nontrivial connected graph with an edge coloring $c : E(G) \rightarrow \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path of $G$ is called *rainbow* if no two edges of it are colored the same. A well-known result shows that in every $\kappa$-connected graph $G$ with $\kappa \geq 1$, there are $k$ internally disjoint $u-v$ paths connecting any two distinct vertices $u$ and $v$ for every integer $k$ with $1 \leq k \leq \kappa$. Chartrand et al. [2] defined the *rainbow $k$-connectivity* $rc_k(G)$ of $G$ to be the minimum integer $j$ for which there exists a $j$-edge-coloring of $G$ such that for every two distinct vertices

* Supported by NSFC, PCSIRT and the “973” program.

$u$ and $v$ of $G$, there exist at least $k$ internally disjoint $u-v$ rainbow paths. It is clearly well-defined.

The concept of rainbow $k$-connectivity has application in transferring information of high security in communication networks. For details we refer to [2] and [4].

By the definition of rainbow $k$-connectivity $rc_k(G)$, we know that it is almost impossible to derive the exact value or a nice bound of the rainbow $k$-connectivity for a general graph $G$. So one investigates the rainbow $k$-connectivity of some classes of special graphs, such as complete graphs, and complete multipartite graphs (see [2]). In [2], Chartrand et al. studied the rainbow $k$-connectivity of the complete graph $K_n$ for various pairs $k, n$ of integers, and they derived the following result:

**Theorem 1.1** ([2]) *For every integer $k \geq 2$, there exists an integer $f(k)$ such that if $n \geq f(k)$, then $rc_k(K_n) = 2$.* ∎

They obtained an upper bound $(k+1)^2$ for $f(k)$, namely $f(k) \leq (k+1)^2$. This paper is to continue their investigation, and the following result is derived:

**Theorem 1.2** *For every integer $k \geq 2$, there exists an integer $f(k) = ck^{\frac{3}{2}} + C(k)$ where $c$ is a constant and $C(k) = o(k^{\frac{3}{2}})$ such that if $n \geq f(k)$, then $rc_k(K_n) = 2$.* ∎

From Theorem 1.2, we can obtain an upper bound $ck^{\frac{3}{2}} + C(k)$ for $f(k)$, where $c$ is a constant and $C(k) = o(k^{\frac{3}{2}})$, that is, we improve the upper bound of $f(k)$ from $O(k^2)$ to $O(k^{\frac{3}{2}})$, a considerable improvement. Recently, Dellamonica et al. [3] got the best possible upper bound $2k$, which is linear in $k$. However, our proof is more structural or constructive, and informative.

For notation and terminology not defined here, we refer to [1].

## 2 Proof of Theorem 1.2

In [2], the authors derived the following two propositions:

**Proposition 2.1** ([2]) *For $n \geq 4$, $rc_2(K_n) = 2$ and for $n \geq 5$, $rc_3(K_n) = 2$.* ∎

**Proposition 2.2** ([2]) *Let $n \geq 6$ be an integer. Then*

$$rc_4(K_n) = \begin{cases} 3 & \text{if } n = 6, 7 \\ 2 & \text{if } n \geq 8. \end{cases}$$

∎

It is easy to show that Theorem 1.2 holds for the case $2 \leq k \leq 4$ by the above two propositions, so we assume $k \geq 5$.

Let $K_{\ell[r]}(r \geq 2)$ denote the complete $\ell$-partite graph each part of which contains $r$ elements and $\ell_0 = \lceil \max\{\sqrt{\frac{k}{2}+1}, \frac{k-1}{r}+2\} \rceil$ $(\geq 3)$. Let $V = \bigcup_{s=1}^{\ell} V_s$ where $V_s = \{u_{s,1}, u_{s,2}, \ldots, u_{s,r}\}$ $(1 \leq s \leq \ell)$ is the vertex set of each part, and $U_j = \{u_{1,j}, u_{2,j}, \ldots, u_{l,j}\}$ $(1 \leq j \leq r)$. We will give our result by means of the following four steps:

**Step 1.** We will show that for every integer $k \geq 5$, if $\xi \geq \ell_0$, then $rc_k(K_{\xi^2[r]}) = 2$ (Proposition 2.3), which gives a result on the rainbow $k$-connectivity of the complete $\ell$-partite graph $K_{\ell[r]}$ where the number of parts $\ell = \xi^2$ is a square number.

**Step 2.** We will obtain a similar result for the general complete $\ell$-partite graph $K_{\ell[r]}$ using Proposition 2.3: for every integer $k \geq 5$, if $\ell \geq {\ell_0}^2$, then $rc_k(K_{\ell[r]}) = 2$ (see Lemma 2.4). Here the number of parts $\ell$ is not always a square number.

**Step 3.** Let $G'$ be a complete $(\ell+1)$-partite graph with $\ell$ parts of order $r$ and one part of order $p$ where $0 \leq p \leq r-1$, that is, $G'$ is obtained from $K_{\ell[r]}$ by adding a new part with $p$ vertices. We will obtain a similar result: for every integer $k \geq 5$, if $\ell \geq {\ell_0}^2$, then $rc_k(G') = 2$ (Proposition 2.5). We can see that Lemma 2.4 is a special case (when $p = 0$) of Proposition 2.5.

**Step 4.** We derive Theorem 1.2 from Proposition 2.5.

We first consider the graph $G = K_{\xi^2[r]}$ where $\xi \geq \ell_0 \geq 3$ is an integer. Now $V = \bigcup_{s=1}^{\xi^2} V_s$ where $V_s = \{u_{s,1}, u_{s,2}, \ldots, u_{s,r}\}$ $(1 \leq s \leq \xi^2)$. Let $U_{i,j} = \{u_{(i-1)\xi+1,j}, \ldots, u_{i\xi,j}\}$, and $\overline{V_i} = \bigcup_{t=1}^{\xi} V_{(i-1)\xi+t}$, where $1 \leq i \leq \xi$, $1 \leq j \leq r$. Then the subgraph of $G$ induced by $U_{i,j}$, say $G_{i,j}$, is the complete graph $K_\xi$; the subgraph of $G$ induced by $U_j$, say $G_j$, is the complete graph $K_{\xi^2}$, and $G_{i,j}$ is a subgraph of $G_j$. See Figure 2.1 for the case $k = 5$, $\xi = 3$, $r = 4$; the vertex subsets $U_1$, $U_{1,1}$, $V_1$ and $\overline{V_2}$ are shown in the figure and the subgraph $G_{1,1}$ is $K_3$, $G_1$ is $K_9$.

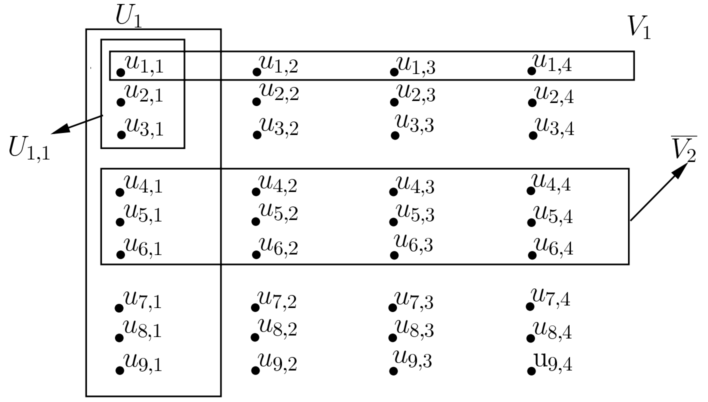


Figure 2.1 The figure for the case $k = 5$, $\xi = 3$, $r = 4$ (we omit the edges).

**Proposition 2.3** *For every integer $k \geq 5$, if $\xi \geq \ell_0$, then $rc_k(K_{\xi^2[r]}) = 2$, where $r \geq 2$.*

*Proof.* At first, we give $G$ a 2-edge-coloring. Similar to the edge coloring in the proof of Theorem 1.1 in [2], we first give a 2-edge-coloring of each $G_j$ $(1 \leq j \leq r)$ as follows: We assign the edge $uv$ of $G$ the color 1 if either $uv \in E(G_{i,j})$ for some $i$ $(1 \leq i \leq \xi)$ or if $uv = u_{(i_1-1)\xi+t,j}\ u_{(i_2-1)\xi+t,j}$ for some $i_1, i_2, t$ with $1 \leq i_1, i_2, t \leq \xi$ and $i_1 \neq i_2$. All other edges of $G_j$ are assigned the color 2. For example, in the graph $G$ of Figure 2.1, we choose $i_1 = 1$, $i_2 = 2$, $t = 2$, $j = 1$, $\xi = 3$, and then edge $u_{(i_1-1)\xi+t,j}\ u_{(i_2-1)\xi+t,j} = u_{2,1}u_{5,1}$ receives the color 1; all edges with color 1 in $G_1$ are shown in Figure 2.2. With a similar argument to Theorem 1.1 in [2], we can obtain:

**There are $\xi - 1$ disjoint rainbow $u - v$ paths for any two vertices $u, v$ in the subgraph $G_j$ including one path of length 1 and $\xi - 2$ paths of length 2. (*)**

For other edges, that is, the edges between distinct $G_j$s, we use the above two colors as follows. Let $uv = u_{i_1,j_1}u_{i_2,j_2}$ where $u_{i_1,j_1} \in G_{j_1}$, $u_{i_2,j_2} \in G_{j_2}$; we give it the color different from the color of edge $u_{i_1,j_1}u_{i_2,j_1}$ of graph $G_{j_1}$ or $u_{i_1,j_2}u_{i_2,j_2}$ of graph $G_{j_2}$ (since edges $u_{i_1,j_1}u_{i_2,j_1}$ and $u_{i_1,j_2}u_{i_2,j_2}$ have the same color by the above coloring). For example, we choose $i_1 = 1$, $i_2 = 5$, $j_1 = 1$, $j_2 = 2$, edges $u_{i_1,j_1}u_{i_2,j_1} = u_{1,1}u_{5,1}$ and $u_{i_1,j_2}u_{i_2,j_2} = u_{1,2}u_{5,2}$ get the same color 2 as shown above, so edges $u_{i_1,j_1}u_{i_2,j_2} = u_{1,1}u_{5,2}$ and $u_{i_1,j_2}u_{i_2,j_1} = u_{1,2}u_{5,1}$ get color 1.

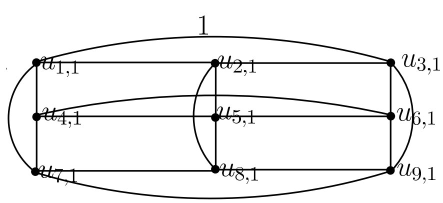


Figure 2.2 The figure for the edges with color 1 in $G_1$.

To complete our proof, it suffices to show that there are at least $k$ disjoint rainbow paths connecting any two vertices $u$ and $v$ of $G$. So we now count the number of disjoint rainbow $u-v$ paths for any two vertices $u, v$ of $G$. Without loss of generality, let $u = u_{1,1}$. We consider four cases:

**Case 1**. $v = u_{1,j}$ $(2 \leq j \leq r)$, that is, $u$ and $v$ are in the same part. An example of $v = u_{1,2}$ is shown in Figure 2.1.

By the coloring, edges $u_{1,1}u_{s,j}$ and $u_{s,j}u_{1,j}$ have distinct colors where $2 \leq s \leq \xi^2$, and so path $u_{1,1}, u_{s,j}, u_{1,j}$ is a rainbow $u - v$ path. Similarly, path $u_{1,1}, u_{s,1}, u_{1,j}$ is a rainbow $u - v$ path for each $2 \leq s \leq \xi^2$. Clearly any two such rainbow paths are disjoint, so there are at least $2(\xi^2 - 1) \geq k$ disjoint rainbow $u - v$ paths in $G$.

**Case 2**. $v = u_{s,1}$ $(2 \le s \le \xi^2)$, that is, $u$ and $v$ are in the same $G_j$ (here $j = 1$). An example of $v = u_{5,1}$ is shown in Figure 2.1.

By **(*)**, in subgraph $G_1$ there are $\xi - 1$ disjoint rainbow $u - v$ paths; in each subgraph $G_j$ where $2 \le j \le r$, since there are $\xi - 2$ disjoint rainbow $u_{1,j} - u_{s,j}$ paths $u_{1,j}, y, u_{s,j}$ of length two for some $y$'s where $y \in G_j$, and the color of $u_{1,j}y(yu_{s,j})$ is different from that of $u_{1,1}y(yu_{s,1})$, we obtain $\xi - 2$ disjoint rainbow $u_{1,1} - u_{s,1}$ paths $u_{1,1}, y, u_{s,1}$. Then there are at least $r(\xi - 2) + 1 \ge k$ disjoint rainbow $u - v$ paths in $G$.

**Case 3**. $v = u_{s,j_0}$ where $2 \le s \le \xi, 2 \le j_0 \le r$, that is, $u$ and $v$ belong to the same $\overline{V_i}$ (here $i = 1$) but in distinct parts and distinct $G_j$'s. Without loss of generality, let $j_0 = 2$; an example of $v = u_{2,2}$ is shown in Figure 2.1.

At first, the edge $u_{1,1}u_{s,2}$ is a rainbow $u - v$ path. Next we consider the monochromatic $u_{1,1} - u_{s,1}$ path $u_{1,1}, y, u_{s,1}$ in $G_1$, then each path $u_{1,1}, y, u_{s,2}$ is a rainbow $u_{1,1} - u_{s,2}$ path of length two. In $U_1$, we choose $y = u_{(i-1)\xi+t,1}$ where $1 \le i \le \xi, 2 \le t \le \xi$ and $t \ne s$; there are $\xi(\xi - 2)$ such paths. Similarly, there are $\xi(\xi - 2)$ disjoint monochromatic $u_{1,2} - u_{s,2}$ paths $u_{1,2}, y, u_{s,2}$ where $y \in U_2$ and we can obtain another $\xi(\xi - 2)$ disjoint rainbow $u - v$ paths. In each subgraph $G_j$ where $3 \le j \le r$, there are $\xi - 2$ disjoint rainbow $u_{1,j} - u_{s,j}$ paths $u_{1,j}, y, u_{s,j}$ of length two, then paths $u_{1,1}, y, u_{s,2}$ are disjoint rainbow $u - v$ paths since the color of $u_{1,j}y$ $(yu_{s,j})$ is different from that of $u_{1,1}y$ $(yu_{s,2})$.

So the number of disjoint rainbow $u - v$ paths in $G$ is at least

$$\begin{aligned} 1 + 2\xi(\xi - 2) + (r - 2)(\xi - 2) &= 1 + (2\xi + r - 2)(\xi - 2) \\ &\ge 1 + r(\xi - 2) \\ &\ge k. \end{aligned}$$

**Case 4**. $v = u_{(i_0-1)\xi+t_0,j_0}$ where $2 \le i_0 \le \xi$, $1 \le t_0 \le \xi$, $2 \le j_0 \le r$; that is, $v$ and $u$ are in distinct $\overline{V_i}$'s and $G_j$'s (here $v$ is not in $\overline{V_1}$ and $G_1$).

**Subcase 4.1**. $t_0 = 1$; an example of $v = u_{4,2}$ is shown in Figure 2.1.

The edge $uv$ is a rainbow $u - v$ path. In subgraph $G_1$, we find the monochromatic $u - u_{(i_0-1)\xi+1,1}$ path $u, y, u_{(i_0-1)\xi+1,1}$. We choose any vertex of $U_1 \setminus \{U_{1,1} \cup U_{i,1}\}$ to be $y$. Since the color of edge $yu_{(i_0-1)\xi+1,1}$ is different from that of $yv$, each path $u, y, v$ is rainbow, so we get $\xi(\xi - 2)$ disjoint rainbow $u - v$ paths. Similarly, in subgraph $G_{j_0}$, we can find monochromatic paths $u_{1,j_0}, y, u_{(i_0-1)\xi+1,j_0}$, and get another $\xi(\xi - 2)$ disjoint rainbow $u - v$ paths. In each subgraph $G_j$ where $j \ne 1, j_0$, we find the disjoint rainbow $u_{1,j} - u_{(i_0-1)\xi+1,j}$ paths $u_{1,j}, y, u_{(i_0-1)\xi+1,j}$ of length two. Since the color of $u_{1,j}y$ $(yu_{(i_0-1)\xi+1,j})$ is different from that of $u_{1,1}y$ $(yu_{(i_0-1)\xi+1,1})$, respectively, there are $\xi - 2$ disjoint rainbow $u - v$ paths $u_{1,1}, y, u_{(i_0-1)\xi+1,1}$.

So the number of disjoint rainbow $u - v$ paths in $G$ is at least

$$1 + 2\xi(\xi - 2) + (r - 2)(\xi - 2) \;=\; 1 + (2\xi + r - 2)(\xi - 2) \;>\; 1 + r(\xi - 2) \;\ge\; k.$$

**Subcase 4.2**. $t_0 \neq 1$; an example of $v = u_{5,2}$ is shown in Figure 2.1.

The edge $uv$ is a rainbow $u-v$ path. In subgraph $G_1$, we find the monochromatic $u - u_{(i_0-1)\xi+t_0,1}$ path $u, y, u_{(i_0-1)\xi+t_0,1}$. Let $y = u_{(i-1)\xi+t,1}$ where $i \neq 1, i_0, t \neq 1, t_0$ or $y = u_{t_0,1}$ or $u_{(i_0-1)\xi+1,1}$. Since the color of edge $yu_{(i_0-1)\xi+t_0,1}$ is different from that of $yv$, each path $u, y, v$ is rainbow, and so there are $(\xi-2)(\xi-2)+2$ disjoint rainbow $u-v$ paths. Similarly, in subgraph $G_{j_0}$, we can find monochromatic paths $u_{1,j_0}, y, u_{(i_0-1)\xi+1,j_0}$, and get other $(\xi-2)(\xi-2)+2$ disjoint rainbow $u-v$ paths. In each subgraph $G_j$ where $j \neq 1, j_0$, we find the disjoint rainbow $u_{1,j} - u_{(i_0-1)\xi+t_0,j}$ paths $u_{1,j}, y, u_{(i_0-1)\xi+t_0,j}$ of length two. Since the color of $u_{1,j}y$ and $yu_{(i_0-1)\xi+t_0,j}$ is different from that of $u_{1,1}y$ and $yu_{(i_0-1)\xi+t_0,j_0}$, respectively, there are $\xi - 2$ disjoint rainbow $u-v$ paths $u_{1,1}, y, u_{(i_0-1)\xi+t_0,j_0}$ (here $y \in G_j$ where $j \neq 1, j_0$). So the number of disjoint rainbow $u-v$ paths in $G$ is

$$
\begin{aligned}
5+2(\xi-2)(\xi-2)+(r-2)(\xi-2) &= 5+(2\xi+r-6)(\xi-2) \\
&\geq 5+r(\xi-2) \\
&> k.
\end{aligned}
$$

∎

Next we will consider the case for the general complete multipartite graph $G \cong K_{\ell[r]}$ with equal parts, where $\ell \geq \ell_0^2$. Let $\xi$ be the integer satisfying $\xi^2 \leq \ell \leq (\xi+1)^2$ where $\xi \geq \ell_0$. Then by Proposition 2.3, we only need to consider the case $1 \leq q = \ell - \xi^2 \leq 2\xi$. Let $G$ be obtained from $K_{\xi^2[r]}$ by adding $q$ new parts $P_i$ $(1 \leq i \leq q)$. Let the vertex set of each corresponding new part be $V(P_i) = \{v_{i,1}, v_{i,2}, \ldots, v_{i,r}\}$ $(1 \leq i \leq q)$. We now **update** $\overline{V_i}$ as follows: if $1 \leq q \leq \xi$, let the new $\overline{V_i}$ be the union of the old $\overline{V_i}$ and $V(P_i)$; if $\xi < q \leq 2\xi$, let the new $\overline{V_i}$ be the union of the old $\overline{V_i}$ and $V(P_i) \cup V(P_{\xi+i})$ (if this exists). Similarly, we **update** $U_{i,j}, U_j$ $(1 \leq i \leq \xi, 1 \leq j \leq r)$ (and also $G_j$, $G_{i,j}$). See Figure 2.3 for the case $k=5$, $\xi=3$, $r=4$ and $\ell=13$; the vertex subsets $V_1$, $V(P_1)$ and new vertex sets $U_1$, $U_{1,1}$, $\overline{V_2}$ are shown in the figure and the new subgraph $G_{1,1}$ is $K_5$, and $G_1$ is $K_{13}$.

By a similar argument (but a little more complicated) to the proof of Proposition 2.3, we derive the following lemma which will be used in the sequel:

**Lemma 2.4** *For every integer $k \geq 5$, if $\ell \geq \ell_0^2$, then $rc_k(K_{\ell[r]}) = 2$, where $r \geq 2$.*

*Proof.* At first, we give a 2-edge-coloring to graph $G$ as follows: For each $G_j$ $(1 \leq j \leq r)$ (similar to the proof of Theorem 1.1 in [2]), we assign the edge $uv$ of $G$ the color 1 if $uv \in E(G_{i,j})$ for some $i$ $(1 \leq i \leq \xi)$ or $uv = u_{(i_1-1)\xi+t,j}\ u_{(i_2-1)\xi+t,j}$ or $uv = v_{i_1,j}\ v_{i_2,j}$ or $uv = v_{\xi+i_1,j}\ v_{\xi+i_2,j}$ for some $i_1, i_2, t$ with $1 \leq i_1, i_2, t \leq \xi$ and $i_1 \neq i_2$. For example, all edges with color 1 in the subgraph $G_1$ of graph $K_{13[4]}$ are shown in Figure 2.4. All other edges of $G_j$ are assigned the color 2. For other edges of $G$, that is, edges between distinct $G_j$'s, in a similar way to the proof of Proposition 2.3, we use the above two colors as follows. If $uv = u_{i_1,j_1}\ u_{i_2,j_2}$ where $u_{i_1,j_1} \in G_{j_1}$, $u_{i_2,j_2} \in G_{j_2}$, we give it the color different from that of edge $u_{i_1,j_1}\ u_{i_2,j_1}$ of graph $G_{j_1}$ or $u_{i_1,j_2}\ u_{i_2,j_2}$ of graph $G_{j_2}$ (since edges $u_{i_1,j_1}\ u_{i_2,j_1}$ and $u_{i_1,j_2}\ u_{i_2,j_2}$ have

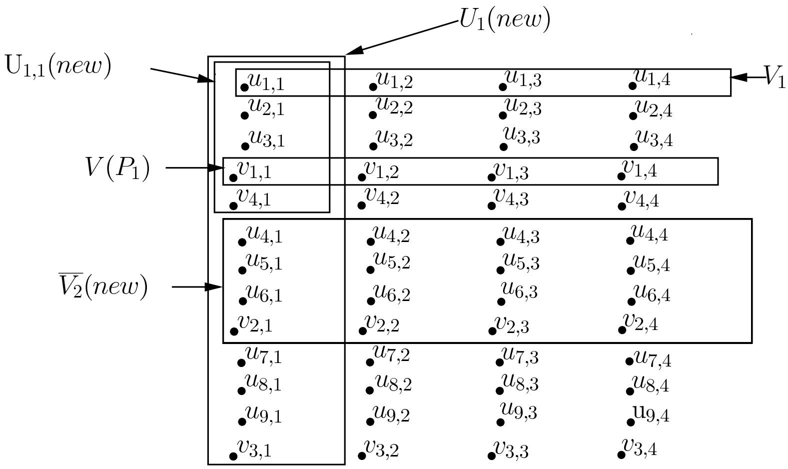


Figure 2.3 The figure for the case $k = 5$, $\xi = 3$, $r = 4$ and $\ell = 13$ (we omit the edges).

the same color by the coloring). Similarly, we color the edges for the cases in which $uv = u_{i_1,j_1}v_{i_2,j_2}$ and $uv = v_{i_1,j_1}v_{i_2,j_2}$.

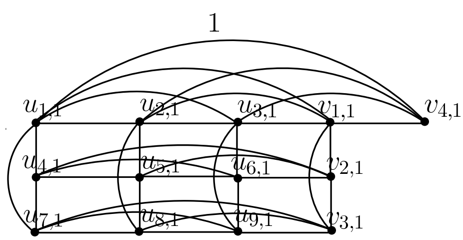


Figure 2.4 The figure for the edges with color 1 in subgraph $G_1$.

By the proof of our Proposition 2.3, there are at least $k$ disjoint rainbow paths connecting any two vertices $u, v \in V(G) \setminus \bigcup_{i=1}^{q} V(P_i)$. We need to count the number of disjoint rainbow paths between $u$ and $v$ where $u \in \bigcup_{i=1}^{q} V(P_i)$ and $v \in V(G)$. Without loss of generality, let $u = v_{1,1}$, and we consider four cases, which are similar to the four cases in the proof of our Proposition 2.3.

**Case 1**. $v \in V(P_1)$, that is, $u$ and $v = v_{1,j_0}$ $(2 \leq j_0 \leq r)$ are in the same part (here the part is $P_1$). An example of $v = v_{1,2}$ is shown in Figure 2.3.

$G$ contains the disjoint rainbow $u - v$ paths $v_{1,1}, y, v_{1,j_0}$ where $y \in U_1 \cup U_{j_0} \setminus \{v_{1,1}, v_{1,j_0}\}$. So the number of these paths is at least $2\xi^2 \geq 2\ell_0^2 > 2(\ell_0^2 - 1) \geq k$.

**Case 2**. $v \in U_1$, that is, $v$ and $u$ are in the same subgraph $G_1$. An example of $v = v_{2,1}$ is shown in Figure 2.3.

In $G_1$, there are at least $\xi - 1$ disjoint rainbow $u - v$ paths. In each $G_j$ $(2 \leq j \leq r)$ there are at least $\xi - 2$ disjoint rainbow $u' - v'$ paths $u', y, v'$ of length two where $u'$, $v'$ are in the same part with $u$, $v$, respectively. So we get at least $\xi - 2$ disjoint rainbow $u - v$ paths $u, y, v$, where $y \in G_j$ $(2 \leq j \leq r)$. Then the number of disjoint rainbow $u - v$ paths is at least $1 + r(\xi - 2) \geq 1 + r(\ell_0 - 2) \geq k$.

**Case 3**. $v \in \overline{V_1}$ but not in $U_1 \cup V(P_1)$. Without loss of generality, let $v \in U_{j_0}$.

Initially, edge $uv$ is a $u - v$ path.

**Subcase 3.1**. $v = v_{\xi+1,j_0}$. An example of $v = v_{4,2}$ is shown in Figure 2.3.

Similar to Case 3 in the proof of Proposition 2.3, we find the monochromatic $u - v_{\xi+1,1}$ path of length two in the subgraph $G_1$ and get at least $\xi^2$ disjoint rainbow $u - v$ paths. Similarly, there are another $\xi^2$ disjoint rainbow $u - v$ paths by finding the monochromatic $v_{1,j_0} - v_{\xi+1,j_0}$ paths in $G_{j_0}$. In each $G_j$ $(j \neq 1, j_0)$, similar to Case 3 in the proof of Proposition 2.3, we get at least $\xi - 2$ disjoint rainbow $u - v$ paths of length two by finding rainbow $v_{1,j} - v_{\xi+1,j}$ paths in $G_j$. So the number of disjoint rainbow $u - v$ paths is at least

$$\begin{aligned} 1 + 2\xi^2 + (r-2)(\xi - 2) &> 1 + 2\xi(\xi - 2) + (r-2)(\xi - 2) \\ &= 1 + (2\xi + r - 2)(\xi - 2) \\ &\geq 1 + (2\ell_0 + r - 2)(\ell_0 - 2) \\ &> 1 + r(\ell_0 - 2) \ > \ k. \end{aligned}$$

**Subcase 3.2**. $v = u_{s_0,j_0}$ where $1 \leq s_0 \leq \xi, 2 \leq j_0 \leq r$. An example of $v = u_{3,2}$ is shown in Figure 2.3.

By a similar procedure to Subcase 3.1, we can get at least $1 + 2\xi(\xi - 1) + (r - 2)(\xi - 2) > 1 + (2\xi + r - 2)(\xi - 2) \geq 1 + (2\ell_0 + r - 2)(\ell_0 - 2) > 1 + r(\ell_0 - 2) \geq k$ disjoint rainbow $u - v$ paths in graph $G$.

**Case 4**. $v$ is not in $\overline{V_1}$ and $U_1$, without loss of generality, let $v \in \overline{V_2} \setminus U_1$. We consider two subcases similar to Case 4 in the proof of Proposition 2.3.

**Case 4.1**. $v = v_{2,j_0}$ where $2 \leq j_0 \leq r$. An example of $v = v_{2,2}$ is shown in Figure 2.3.

Similar to Subcase 4.1 in the proof of Proposition 2.3, we can get at least $1+2\xi(\xi - 2)+(r-2)(\xi-2) = 1+(2\xi+r-2)(\xi-2) \geq 1+(2\ell_0+r-2)(\ell_0-2) > 1+r(\ell_0-2) \geq k$ disjoint rainbow $u - v$ paths in $G$.

**Case 4.2**. $v \neq v_{2,j_0}$. An example of $v = u_{4,2}$ is shown in Figure 2.3.

Similar to Subcase 4.2 in the proof of Proposition 2.3, we can get at least $1+2(\xi - 1)(\xi - 2) + (r-2)(\xi - 2) = 1 + (2\xi + r - 4)(\xi - 2) \geq 1 + r(\xi - 2) \geq 1 + r(\ell_0 - 2) \geq k$ (since $\xi \geq 2$) disjoint rainbow $u - v$ paths in $G$. ∎

From Lemma 2.4, we derive the following result.

**Proposition 2.5** *Let $G'$ be a complete $(\ell+1)$-partite graph with $\ell$ parts of order $r$ and a part of order $p$ where $0 \le p < r$. Then for every integer $k \ge 5$, if $\ell \ge \ell_0^2$, then $rc_k(G') = 2$.*

*Proof.* For the case $p = 0$, the conclusion clearly holds by Lemma 2.4, so we assume $p \ge 1$. We know that $G'$ can be obtained from $G \cong K_{\ell[r]}$ (graph of Lemma 2.5) by adding $p$ new vertices: $w_1, w_2, \ldots, w_p$ $(1 \le p \le r-1)$ and edge $vw_j$ where $1 \le j \le p$ and $v \in V(G)$. For example, let the graph $G'$ be a complete 14-partite graph with 13 parts of order $r = 4$ and a part of order $p = 2$. Then $G'$ can be obtained from the graph $G$ in Figure 2.3 by adding a new part with $p = 2$ vertices.

We color the new edges as follows: Give color 2 to edges between $U_j$ and $w_j$, and color 1 to the other edges.

Now we count the number of disjoint rainbow $u - v$ paths between any two vertices of $G$. By Lemma 2.4, we need to consider the case that $u = w_j, v \in V(G')$ $(1 \le j \le p)$. Without loss of generality, let $u = w_1$.

**Case 1**. $v \in V(G)$.

**Subcase 1.1**. $v \in U_1$; without loss of generality, let $v = u_{1,1}$.

Then $G$ contains the $u-v$ path $u, v$ as well as the $u-v$ rainbow paths $w_1, y, u_{1,1}$ of length two where $y \in \overline{V_1} \setminus \{u_{1,1}, \ldots, u_{1,r}\}$. So the number of disjoint rainbow $u-v$ paths is at least $1 + r(\xi - 1) > 1 + r(\ell_0 - 2) \ge k$.

**Subcase 1.2**. $v$ is not in $U_1$; without loss of generality, let $v = u_{1,2}$.

Then $G$ contains the $u-v$ path $u, v$ as well as the $u-v$ rainbow paths $w_1, y, u_{1,2}$ where $y \in (U_1 \cup U_2) \setminus (U_{1,1} \cup U_{1,2} \cup \{u_{(i-1)\xi+1,j_0}\}_{i=1}^{\xi})$ where $j_0 = 1, 2$, or $y \in \{u_{(i-1)i_1+1,j_1}\}_{i=2}^{\xi}$ $(j_1 \ne 1, 2)$, or $y = u_{s,j}$ where $2 \le s \le \xi, j \ne 1, 2$. So the number of disjoint rainbow $u - v$ paths is at least

$$\begin{aligned} 1 + 2(\xi-1)(\xi-1) + 2(\xi-1)(r-2) &= 1 + (\xi-1)(2\xi+2r-6) \\ &> 1 + (\xi-2)(2\xi+2r-6) \\ &\ge 1 + (\xi-2)r > k \end{aligned}$$

(here we let $r \ge 2$). The proof for the case that $v = v_{1,2}$ or $v_{\xi+1,2}$ is similar.

**Case 2**. $v = w_j$ where $j \ne 1$.

Then $G$ contains the $u - v$ rainbow paths $w_1, y, w_j$ where $y \in U_1 \cup U_j$. So the number of disjoint rainbow $u - v$ paths is at least $2\xi^2 \ge 2\ell_0^2 > 2(\ell_0^2 - 1) \ge k$. ∎

We know that the graph in Proposition 2.5 is a spanning subgraph of the complete graph $K_n$ where $n = \ell r + p$, $\ell \ge \ell_0^2$, $0 \le p < r$. On the other hand, we choose

$r = r_0 = \lceil\sqrt{2k}\rceil$, and then

$$\begin{aligned}\ell_0 &= \left\lceil \max\{\sqrt{\frac{k}{2}+1}, \frac{k-1}{r}+2\} \right\rceil \\ &= \left\lceil \max\{\sqrt{\frac{k}{2}+1}, \frac{k-1}{\lceil\sqrt{2k}\rceil}+2\} \right\rceil \\ &= \left\lceil \frac{k-1}{\lceil\sqrt{2k}\rceil} \right\rceil + 2.\end{aligned}$$

Let $G''$ be the complete graph $K_n$ where $n = \ell_3 r_0 + p$, $\ell_3 \geq \ell_0^2$, $0 \leq p < r_0$. Then $G''$ contains a spanning subgraph $G'$, where $G'$ is a complete $(\ell_3+1)$-partite graph with $\ell_3$ parts of order $r_0$ and one part of order $p$ where $0 \leq p < r_0$. From Proposition 2.5, Theorem 1.2 follows.

## Acknowledgement

The authors are very grateful to the referees for their helpful comments and suggestions.

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(Received 11 June 2010; revised 9 Aug 2010)