On shifted intersecting families with respect to posets

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Abstract. In this paper, we show that for a shifted complex $\mathcal{F} \subsetneq 2^P$ with respect to a poset P with minimum element 0 and an intersecting subfamily $\mathcal{G} \subsetneq \mathcal{F}$, $\#\mathcal{G} \leqq \#\{ F \in \mathcal{F} ; 0 \in F \}$.

We denote the set $\{ 1,2,..,n \}$ by $[n]$, the family of all subsets of a set X by 2^X. $\#F$ denotes the number of elements of a set F. Let $\mathcal{F}$ be a family of subsets of $[n]$, i.e., $\mathcal{F} = \{ F_1,..,F_m \}$ where $F_1, ..., F_m$ are distinct subsets of $[n]$. A family $\mathcal{F}$ is *intersecting* if for every F_i, $F_j \in \mathcal{F}$, $F_i \cap F_j \neq \emptyset$. For families $\mathcal{G}$, $\mathcal{F} \subsetneq 2^{[n]}$, $\mathcal{G}$ and $\mathcal{F}$ are *cross-intersecting* if $G \cap F \neq \emptyset$ for $\forall$ $G \in \mathcal{G}$ and $\forall$ $F \in \mathcal{F}$. A family $\mathcal{F} \subsetneq 2^{[n]}$ is called a *complex* if $G \subsetneq F \in \mathcal{F}$ implies $G \in \mathcal{F}$. We already know the following results. For an intersecting family $\mathcal{F} \subsetneq 2^{[n]}$, $\#\mathcal{F} \leqq 2^{n-1}$ ([1]) and for a complex $\mathcal{F} \subsetneq 2^{[n]}$ and cross-intersecting subfamilies $\mathcal{G}$, $\mathcal{H} \subsetneq \mathcal{F}$, $\#\mathcal{G} + \#\mathcal{H} \leqq \#\mathcal{F}$ ([4]).

For F, $G \subsetneq [n]$, if there exists a one-to-one mapping $f : F \to G$ with $x \leqq f(x)$ for each $x \in F$, then we write $F \leqq G$. $\mathcal{F} \subsetneq 2^{[n]}$

is *V-hereditary* if $G \leqq F \in \mathcal{F}$ implies $G \in \mathcal{F}$. V.Chvátal introduced this notion and proved the next result.

Theorem A (1974 [2]). Let $\mathcal{F} \subsetneq 2^{[n]}$ be a V-hereditary family and $\mathcal{G}$ be an intersecting subfamily of $\mathcal{F}$. Then $\#\mathcal{G} \leqq \#\{ F \in \mathcal{F} ; 1 \in F \}$. ■

H.Era extended the notion of V.Chvátal and also showed the following result. Let P be a finite ranked poset with the minimum element **0**. For F, $G \subsetneq P$, if there exists a one-to-one mapping $f : \mathrm{F} \dashrightarrow G$ with $x \leqq f(x)$ in P or x and $f(x)$ are incomparable for each $x \in \mathrm{F}$, then we write $F \leqq_P G$. $\mathcal{F} \subsetneq 2^P$ is *P-hereditary* if $G \leqq_P F \in \mathcal{F}$ implies $G \in \mathcal{F}$.

Theorem B ([3]). Let P be a finite ranked poset with the minimum element **0** and $\mathcal{F} \subsetneq 2^P$ be a P-hereditary family. For an intersecting subfamily $\mathcal{G}$ of $\mathcal{F}$, $\#\mathcal{G} \leqq \#\{ F \in \mathcal{F} ; \mathbf{0} \in F \}$. ■

Let P be a finite poset with the minimum element **0**. For a family $\mathcal{F} \subsetneq 2^P$ and $\alpha \leqq \beta$ in P, we define

$$S_{\alpha,\beta}(F) = \begin{cases} (F-\{\beta\}) \cup \{\alpha\} & \text{if } \alpha \notin F,\ \beta \in F,\ (F-\{\beta\}) \cup \{\alpha\} \notin \mathcal{F} \\ F & \text{otherwise} \end{cases}$$

for each $F \in \mathcal{F}$ and $S_{\alpha,\beta}(\mathcal{F}) = \{ S_{\alpha,\beta}(F) ; F \in \mathcal{F} \}$. Then $\#S_{\alpha,\beta}(\mathcal{F}) = \#\mathcal{F}$ and if $\mathcal{F}$ is complex and intersecting, then $S_{\alpha,\beta}(\mathcal{F})$ is also complex and intersecting.

Proposition 1. For a finite poset P and $\alpha \leqq \beta$ in P, if $\mathcal{F} \subsetneq 2^P$ is complex, then $S_{\alpha,\beta}(\mathcal{F})$ is also complex.

Proof. We suppose that there exist G, $F \subsetneq P$ such that $G \subsetneq F \in S_{\alpha,\beta}(\mathcal{F})$ and $G \notin S_{\alpha,\beta}(\mathcal{F})$.

Case 1. $F \in \mathcal{F}$.

Since $\mathcal{F}$ is complex, $G \in \mathcal{F}$. So $\alpha \notin G$, $\beta \in G$, $(G-\{\beta\})\cup\{\alpha\} \notin \mathcal{F}$ and $\beta \in F$. If $\alpha \in F$, then $(G-\{\beta\})\cup\{\alpha\} \subsetneq F$, which contradicts the property that $\mathcal{F}$ is complex. If $\alpha \notin F$, then $(F-\{\beta\})\cup\{\alpha\} \in \mathcal{F}$ and $(G-\{\beta\})\cup\{\alpha\} \subsetneq (F-\{\beta\})\cup\{\alpha\}$, which contradicts the property that $\mathcal{F}$ is complex.

Case 2. $F \notin \mathcal{F}$.

Then $\alpha \in F$, $\beta \notin F$ and $(F-\{\alpha\})\cup\{\beta\} \in \mathcal{F}$. If $G \in \mathcal{F}$, then $\alpha \notin G$, $\beta \in G$ and $(G-\{\beta\})\cup\{\alpha\} \notin \mathcal{F}$. So $G \not\subseteq F$, which is a contradiction. If $G \notin \mathcal{F}$, then $(G-\{\alpha\})\cup\{\beta\} \subsetneq (F-\{\alpha\})\cup\{\beta\}$ and $G' = (G-\{\alpha\})\cup\{\beta\} \in \mathcal{F}$. Since $G'\cap\{\alpha,\beta\} = \{\beta\}$, $(G'-\{\beta\})\cup\{\alpha\} = G \in S_{\alpha,\beta}(\mathcal{F})$, which is a contradiction. ■

Proposition 2. For a finite poset P and $\alpha \leqq \beta$ in P, if $\mathcal{F} \subsetneq 2^P$ is intersecting, then $S_{\alpha,\beta}(\mathcal{F})$ is also intersecting.

Proof. We suppose that there exist G , $F \in S_{\alpha,\beta}(\mathcal{F})$ such that $G\cap F = \emptyset$. Since $\mathcal{F}$ is intersecting, both of G and F do not belong to $\mathcal{F}$. We assume that $F \notin \mathcal{F}$. Thus there exists $H \in \mathcal{F}$ such that $S_{\alpha,\beta}(H) = F$ and $H \neq F$. By the definition of (α,β)-shifting, $H = (F-\{\alpha\})\cup\{\beta\} \in \mathcal{F}$, $\alpha \in F$ and $\beta \notin F$. If $G \notin \mathcal{F}$, then $\alpha \in G$ and $F\cap G \neq \emptyset$, which is a contradiction. Thus $G \in \mathcal{F}$, $\beta \in G$ and $\alpha \notin G$. Since $S_{\alpha,\beta}(G) = G$, $(G-\{\beta\})\cup\{\alpha\} \in \mathcal{F}$ by the definition of (α,β)-shifting. Then $((F-\{\alpha\})\cup\{\beta\}) \cap ((G-\{\beta\})\cup\{\alpha\}) = ((F-\{\alpha\})\cap(G-\{\beta\})) \cup (\{\beta\}\cap(G-\{\beta\})) \cup ((F-\{\alpha\})\cap\{\alpha\}) \cup (\{\alpha\}\cap\{\beta\}) = (F-\{\alpha\})\cap(G-\{\beta\}) = \emptyset$, contradicting the fact that $\mathcal{F}$ is an intersecting family. ■

A family $\mathcal{F}$ is *shifted* if $S_{\alpha,\beta}(\mathcal{F}) = \mathcal{F}$ for all α, β such that $\alpha < \beta$ in P. We obtain the following result which is concerned with shifted complexes and intersecting families.

Theorem 3. Let P be a finite poset with the minimum element $\mathbf{0}$ and $\mathcal{F} \subsetneq 2^P$ be a shifted complex. For an intersecting subfamily $\mathcal{G}$ of $\mathcal{F}$, $\#\mathcal{G} \leqq \#\{ F \in \mathcal{F} ; \mathbf{0} \in F \}$.

Proof. Let $\mathcal{F}(\mathbf{0}) = \{ F-\{\mathbf{0}\} ; \mathbf{0} \in F \in \mathcal{F} \}$ and $\mathcal{F}_0 = \{ F \in \mathcal{F} ; \mathbf{0} \notin F \}$. By Proposition 2, we can assume that $\mathcal{G}$ is shifted. Then we define the family $\mathcal{G}_* = \{H ; H \subsetneq \exists G \in \mathcal{G}\}$, that is, if $G \in \mathcal{G}$ and $H \subsetneq G$, then $H \in \mathcal{G}_*$. In the following we show that $\mathcal{G}_* = \{H ; H \subsetneq \exists G \in \mathcal{G}\}$ is a shifted complex.

Suppose that $\mathcal{G}_*$ is not a shifted complex. Then there exist α, $\beta \in P$ and $H \in \mathcal{G}_*$ such that $\alpha \leqq \beta$, $H \cap \{\alpha, \beta\} = \{\beta\}$ and $(H - \{\beta\}) \cup \{\alpha\} \notin \mathcal{G}_*$. By definition of $\mathcal{G}_*$, there exists $G \in \mathcal{G}$ such that $H \subsetneq G$. If $G \cap \{\alpha, \beta\} = \{\beta\}$, then $(G - \{\beta\}) \cup \{\alpha\} \in \mathcal{G}$ because $\mathcal{G}$ is shifted. Since $(H - \{\beta\}) \cup \{\alpha\} \subsetneq (G - \{\beta\}) \cup \{\alpha\} \in \mathcal{G}$, $(H - \{\beta\}) \cup \{\alpha\} \in \mathcal{G}_*$, which is a contradiction. If $G \cap \{\alpha, \beta\} \neq \{\beta\}$, then $\alpha, \beta \in G$. Since $(H - \{\beta\}) \cup \{\alpha\} \subsetneq (G - \{\beta\}) \cup \{\alpha\} \subsetneq G \in \mathcal{G}$, $(H - \{\beta\}) \cup \{\alpha\} \in \mathcal{G}_*$, which is a contradiction.

Thus $\mathcal{G}_* = \{H ; H \subsetneq \exists G \in \mathcal{G}\}$ is a shifted complex and $\mathcal{G} \subsetneq \mathcal{G}_* \subsetneq \mathcal{F}$. So for $\mathcal{G}_*(\mathbf{0}) = \{ G-\{\mathbf{0}\} ; \mathbf{0} \in G \in \mathcal{G}_*\}$, $\#\mathcal{G}_*(\mathbf{0}) \leqq \#\mathcal{F}(\mathbf{0})$. Therefore without loss of generality we can assume that $\mathcal{G}_* = \mathcal{F}$. For $\forall H \in \mathcal{G}_* - \mathcal{G}$, $H \subset \exists G \in \mathcal{G}$. Since $\mathcal{G}_*$ is shifted, $\mathbf{0} \notin H$ implies $H \cup \{\mathbf{0}\} \in \mathcal{G}_*$. Let $\mathcal{G}_0 = \{ G \in \mathcal{G} ; \mathbf{0} \notin G \}$ and $\mathcal{C} = \{ C \in \mathcal{F}_0 ; \exists G \in \mathcal{G}_0, C \cap G = \emptyset\}$. Since $\mathcal{G}_0$ and $\mathcal{F}_0 - \mathcal{C}$ are cross-intersecting, $\#\mathcal{G}_0 + \#(\mathcal{F}_0 - \mathcal{C}) \leqq \#\mathcal{F}_0$ and therefore $\#\mathcal{C} \geqq \#\mathcal{G}_0$. For $\mathcal{C}^+ = \{ C \cup \{\mathbf{0}\} ; C \in \mathcal{C}\}$, $\#\mathcal{C}^+ = \#\mathcal{C}$. For $C \in \mathcal{C}$ and $G \in \mathcal{G}_0$, since $\mathbf{0} \notin G$ and $C \cap G = \emptyset$, $(\{\mathbf{0}\} \cup C) \cap G = \emptyset$. By the fact that $\mathcal{G}$ is intersecting, $\{\mathbf{0}\} \cup C \notin \mathcal{G}$. So $\mathcal{C}^+ \cap \mathcal{G} = \emptyset$. Since every element of $(\mathcal{G} - \mathcal{G}_0) \cup \mathcal{C}^+$ contains $\mathbf{0}$ and $\mathcal{C}^+ \subsetneq \mathcal{F}$, $\#\mathcal{G} \leqq \#\mathcal{G} - \#\mathcal{G}_0 + \#\mathcal{C} = \#((\mathcal{G} - \mathcal{G}_0) \cup \mathcal{C}^+) \leqq \#\mathcal{F}(\mathbf{0})$. ■

Proposition 4. Let P be a finite poset with the minimum element $\mathbf{0}$. If

$\mathcal{F} \subsetneq 2^P$ is a P-hereditary family, then $\mathcal{F}$ is a shifted complex.

Proof. We assume that $G \subsetneq 2^P$ and $G \subsetneq \exists F \in \mathcal{F}$. Since the mapping f from G to F such that $f(x) = x$ is a one-to-one mapping, $G \leqq_P F$. By the property that $\mathcal{F}$ is a P-hereditary family, $G \in \mathcal{F}$. Thus $\mathcal{F}$ is complex.

We assume that $\mathcal{F}$ is not shifted. Then there exist α and β such that $\alpha, \beta \in P$ and $\alpha \leqq \beta$ and $F \in \mathcal{F}$ such that $F \cap \{\alpha, \beta\} = \{\beta\}$ and $(F - \{\beta\}) \cup \{\alpha\} \notin \mathcal{F}$. We define the mapping f from $(F - \{\beta\}) \cup \{\alpha\}$ to F as follows:

$$f(x) = \begin{cases} x & \text{if } x \neq \alpha \\ \beta & \text{if } x = \alpha. \end{cases}$$

Since $\alpha \leqq \beta$ in P, $x \leqq f(x)$ for $\forall x \in (F - \{\beta\}) \cup \{\alpha\}$. Thus f is a one-to-one mapping and $(F - \{\beta\}) \cup \{\alpha\} \leqq_P F$. By the property that $\mathcal{F}$ is a P-hereditary family, $(F - \{\beta\}) \cup \{\alpha\} \in \mathcal{F}$, which is a contradiction.
■

By Proposition 4 and Theorem 3, we also obtain Theorem B. However the converse of Proposition 4 does not hold. For example, for the poset of Figure 1, $\mathcal{F} = \{ \{0,1,2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0\}, \{1\}, \{2\} \}$ is a shifted complex. Since $\{3\} \leqq_P \{2\}$ and $\{3\} \notin \mathcal{F}$, $\mathcal{F}$ is not P-hereditary. So we do not obtain Theorem 3 from Theorem B.

We can easily see that $\mathcal{F}$ is a V-hereditary family if and only if $\mathcal{F}$ is a shifted family with respect to a linear order set. Let P be a poset with the minimum element and $l(P)$ be a liner extension of P. If $\mathcal{F}$ is a shifted family with respect to $l(P)$, then $\mathcal{F}$ is a shifted family with respect to P. So we also obtain Theorem A by Theorem 3. But the converse does not hold. For example, $\mathcal{F} = \{ \{0,1,2\}, \{0,3,4\} \}$ is a shifted family with respect to the poset of Figure 1 and is not a shifted family with respect to the liner extension $0 \leqq 1 \leqq 2 \leqq 3 \leqq 4$.

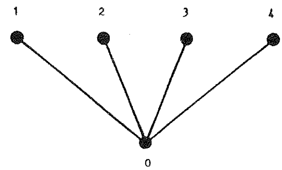

Figure 1.

References.

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