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On magicness and antimagicness of the union of 4-regular circulant graphs

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Abstract

Let $G = (V, E)$ be a graph of order n and size e. An (a, d)-vertex-antimagic total labeling is a bijection α from $V(G) \cup E(G)$ onto the set of consecutive integers $\{1, 2, \ldots, n + e\}$, such that the vertex-weights form an arithmetic progression with the initial term a and the common difference d. The vertex-weight of a vertex x is the sum of values $\alpha(xy)$ assigned to all edges xy incident to the vertex x together with the value assigned to x itself. In this paper we study the vertex-magicness and vertex-antimagicness of the union of 4-regular circulant graphs.

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1 Introduction

In the paper we consider finite, simple and undirected graphs. For a graph $G = (V, E)$, let $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively. We also denote $|V(G)| = n$ and $|E(G)| = e$.

A labeling of a graph G is a mapping from the set of vertices, edges, or both vertices and edges, to the set of labels. Depending on the domain, we distinguish vertex labeling, edge labeling and total labeling. The domain of the mappings in this paper is always $V(G) \cup E(G)$, that is, we deal with total labelings.

The *vertex-weight*, $wt(x)$, of a vertex $x \in V(G)$, under a labeling $\alpha\colon V(G) \cup E(G) \to \{1, 2, \dots, n + e\}$, is the sum of the labels assigned to all edges incident to a given vertex x, together with the value assigned to x itself:

$$wt(x) = \alpha(x) + \sum_{y \in N(x)} \alpha(xy),$$

where $N(x)$ is the set of the neighbors of x.

A bijection $\alpha\colon V(G) \cup E(G) \to \{1, 2, \dots, n + e\}$ is called an (a, d)-*vertex-antimagic total* (in short, (a, d)-VAT) *labeling* of G if the set of vertex-weights of all vertices in G is $\{a, a+d, a+2d, \dots, a+(n-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

An (a, d)-VAT labeling f is called *super* if the vertex labels are the integers $1, 2, \dots, n$, that is, the smallest possible labels. A graph which admits a (super) (a, d)-VAT labeling is said to be (super) (a, d)-VAT.

These labelings were introduced in [2] as a natural extension of the vertex-magic total labeling (VAT labeling for $d = 0$) defined by MacDougall et al. [13] (see also [17]).

Vertex-magic total labelings for K_n, for n odd, are given in [12], [13] and [14], and for K_n, n even, they can be found in [5] and [7]. In [8], it is completely determined which complete bipartite graphs have vertex-magic total labelings. Constructions of vertex-magic total labelings of certain regular graphs are given in [10], [11] and [16].

Basic properties of (a, d)-VAT labelings are investigated in [2]. In [15], it is shown how to construct super (a, d)-VAT labelings for certain families of graphs, including complete graphs, complete bipartite graphs, cycles, paths and generalized Petersen graphs.

In this paper we focus on disconnected graphs. Certain results on the vertex-magicness of disconnected graphs are known. Balbuena et al. [3] described super vertex-magic total labelings for disjoint union of cycles. Gray et al. [9] explored vertex-magic total labelings for a disjoint union of stars. Gómez [6] studied super vertex-magic total labeling for the disjoint union of regular graphs.

Ali et al. [1] studied properties of super (a, d)-VAT labelings and examined their existence for disjoint union of t copies of a regular graph.

In this paper we concentrate on (a, d)-VAT labeling of disjoint union of 4-regular circulant graphs.

2 Union of multiple copies of circulant graph

Circulant graphs are an important class of graphs which can be used in the design of local area networks, see [4]. Let $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq \lfloor \frac{n}{2} \rfloor$, where n and a_j, $j = 1, 2, \dots, k$, are positive integers. A *circulant graph* $C_n(a_1, a_2, \dots, a_k)$ is a regular graph with vertex set $V = \{v_0, v_1, \dots, v_{n-1}\}$ and edge set $E = \{(v_i v_{i+a_j}) \ (mod\ n) : i = 0, 1, 2, \dots, n-1$ and $j = 1, 2, \dots, k\}$.

In this section we study the super vertex-antimagicness of a disjoint union of t copies of circulant graph $C_n(1, m)$, denoted by $tC_n(1, m)$, for $m \in \{2, 3, \dots, \lfloor \frac{n}{2} \rfloor\}$.

Balbuena et al. [3] proved the following result:

Theorem 1. [3] *For odd $n \geq 5$ and $m \in \{2, 3, \dots, \frac{n-1}{2}\}$ circulant graphs $C_n(1, m)$ have a super vertex-magic total labeling with the magic constant $h = \frac{17n+5}{2}$.*

For super vertex-magic total labeling of disjoint union of t copies of a regular graph, there is a result of Gómez:

Theorem 2. [6] *Let t be a positive integer. If the graph G is an r-regular graph that admits a super vertex-magic total labeling and $\frac{(t-1)(r+1)}{2}$ is an integer, then the graph tG has a super vertex-magic total labeling.*

For 4-regular graph $tC_n(1, m)$, the expression $\frac{(t-1)(r+1)}{2}$ is an integer if and only if t is odd. The next corollary follows from Theorems 1 and 2.

Corollary 1. *For odd $n \geq 5$, odd $t \geq 1$ and $m \in \{2, 3, \dots, \frac{n-1}{2}\}$, a disjoint union of t copies of circulant graph $C_n(1, m)$ admits a super vertex-magic total labeling.*

With respect to Theorem 2, Ali et al. have proved the following theorem:

Theorem 3. [1] *Let t be a positive integer. If the graph G is an r-regular graph that admits a super vertex-magic total labeling and $\frac{(t-1)(r+1)}{2}$ is an integer, then the graph tG has a super $(a, 2)$-VAT labeling.*

The next corollary follows from Theorems 1 and 3.

Corollary 2. *For odd $n \geq 5$, odd $t \geq 1$ and $m \in \{2, 3, \dots, \frac{n-1}{2}\}$, a disjoint union of t copies of circulant graph $C_n(1, m)$ has a super $(a, 2)$-VAT labeling.*

Lemma 1. *For $tC_n(1, m)$, $m \in \{2, 3, \dots, \lfloor \frac{n-1}{2} \rfloor\}$ and $nt \not\equiv 1$ (mod 2), there is no super (a, d)-VAT labeling with $d \in \{0, 2\}$.*

Proof. Assume that $tC_n(1,m)$ has a super (a,d)-VAT labeling

$$\alpha : V(tC_n(1,m)) \cup E(tC_n(1,m)) \rightarrow \{1,2,\dots,3nt\}$$

with the set of vertex-weights $\{a, a+d, \dots, a+(nt-1)d\}$. The sum of all the vertex-weights is

$$\sum_{x \in V(tC_n(1,m))} wt(x) = nt\left(a + \frac{(nt-1)d}{2}\right). \tag{1}$$

In the computation of the vertex-weights of $tC_n(1,m)$, each vertex label is used once and the label of each edge is used twice. The sum of all the vertex labels and all the edge labels, used to calculate the vertex-weights, is thus equal to

$$\sum_{x \in V(tC_n(1,m))} \alpha(x) + 2 \sum_{xy \in E(tC_n(1,m))} \alpha(xy) = \frac{nt(17nt+5)}{2}. \tag{2}$$

Combining equations (1) and (2) for the minimum vertex-weight we obtain

$$a = \frac{17nt + 5 + (1-nt)d}{2}. \tag{3}$$

Clearly, for $d \in \{0,2\}$, the minimum edge-weight a is an integer if and only if t and n are both odd. □

From Corollaries 1 and 2 and Lemma 1, it follows that:

Theorem 4. *The disjoint union of t copies of circulant graph $C_n(1,m)$, $m \in \{2,3,\dots,\frac{n-1}{2}\}$, has a super (a,d)-VAT labeling for $d \in \{0,2\}$ if and only if n and t are both odd.*

In [1], the following result is given:

Theorem 5. [1] *Let G be an even regular Hamilton graph. Then tG is super $(a,1)$-VAT for every positive integer t.*

The circulant graph $C_n(1,m)$, $m \in \{2,3,\dots,\lfloor\frac{n-1}{2}\rfloor\}$, is 4-regular Hamilton graph. From (3) it follows that for $d=1$ the minimum vertex-weight is an integer. Thus, from Theorem 5 we get the following corollary:

Corollary 3. *For $m \in \{2,3,\dots,\lfloor\frac{n-1}{2}\rfloor\}$ a disjoint union of t copies of circulant graph $C_n(1,m)$ admits a super $(a,1)$-VAT labeling for every positive integer t.*

3 Union of circulant graphs

In this section we study an (a,d)-VAT labeling, for $d \in \{0,1,2\}$, of disjoint union of circulant graphs which are not necessarily isomorphic.

We shall use the following functions

$$odd(j) = \begin{cases} 1, & \text{if } j \text{ is odd,} \\ 0, & \text{otherwise,} \end{cases}$$

$$even(j) = \begin{cases} 1, & \text{if } j \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

$$a(i) = \begin{cases} 1, & \text{if } i \le m_j - 1, \\ 0, & \text{if } m_j \le i \le n-1, \end{cases}$$

to simplify later notations.

The next theorem gives a vertex-magic total labeling of disjoint union of circulant graphs $C_n(1,m_j)$, each of the same odd order but with arbitrary $m_j \in \{2,3,\dots,\frac{n-1}{2}\}$, for $j = 1,2,\dots,t$.

Theorem 6. *Let t be a positive integer and $n \ge 5$ be odd. If $m_j \in \{2,3,\dots,\frac{n-1}{2}\}$, for $j = 1,2,\dots,t$, then the graph $\bigcup_{j=1}^{t} C_n(1,m_j)$ has a vertex-magic total labeling with the magic constant $h = 8nt + \frac{n+5}{2}$.*

Proof. Let $\{C_n(1,m_j) : j = 1,2,\dots,t\}$ be the set of circulant graphs with n vertices each, $n \ge 5$ odd, and $m_j \in \{2,3,\dots,\frac{n-1}{2}\}$. Let $\{v_i^j : i = 0,1,\dots,n-1\}$ be the vertices of the graph $\bigcup_{j=1}^{t} C_n(1,m_j)$.

For $i = 0,1,\dots,n-1$ and $j = 1,2,\dots,t$, label the vertices and edges of $\bigcup_{j=1}^{t} C_n(1,m_j)$ as follows:

$$\alpha_0(v_i^j) = [jn - n(a(i))]odd(j) + [(2t-j)n + n(1-a(i))]even(j) + m_j - i,$$

$$\alpha_0(v_i^j v_{i+m_j}^j) = (2t-j)n(odd(j)) + (j-1)n(even(j)) + i + 1,$$

$$\alpha_0(v_i^j v_{i+1}^j) = \frac{1}{2}\left(4nt + jn(odd(j)) + (2t-j+1)n(even(j)) + n(even(i)) - i\right).$$

The vertex and edge labels under the labeling α_0 are $\alpha_0(V) = \{2pn+1, 2pn+2,\dots,(2p+1)n : p = 0,1,2,\dots,t-1\}$ and $\alpha_0(E) = \{(2p-1)n+1,(2p-1)n+2,\dots,2pn : p = 1,2,\dots,t\} \cup \{2nt+1, 2nt+2,\dots,3nt\}$. It means that the labeling α_0 is a bijection from the set

$V\left(\bigcup_{j=1}^{t} C_n(1,m_j)\right) \cup E\left(\bigcup_{j=1}^{t} C_n(1,m_j)\right)$ onto the set $\{1,2,\ldots,3nt\}$.

Now, we consider the vertex-weights of $\bigcup_{j=1}^{t} C_n(1,m_j)$ case by case.

Case 1. j odd

a) For $0 \le i \le m_j - 1$ and i even, we have

$$\begin{aligned} wt_{\alpha_0}(v_i^j) = \;& (jn-n+m_j-i)+\tfrac{1}{2}(4nt+jn+n-i)\\ &+\tfrac{1}{2}(4nt+jn-(i-1))+((2t-j)n+i+1)\\ &+((2t-j)n+(n-m_j+i)+1)\\ =\;& 8nt+\tfrac{n+5}{2}. \end{aligned}$$

b) For $0 \le i \le m_j - 1$ and i odd, we admit

$$\begin{aligned} wt_{\alpha_0}(v_i^j) = \;& (jn-n+m_j-i)+\tfrac{1}{2}(4nt+jn-i)\\ &+\tfrac{1}{2}(4nt+jn+n-(i-1))+((2t-j)n+i+1)\\ &+((2t-j)n+(n-m_j+i)+1)\\ =\;& 8nt+\tfrac{n+5}{2}. \end{aligned}$$

c) For $i \ge m_j$ and i even, we have

$$\begin{aligned} wt_{\alpha_0}(v_i^j) = \;& (jn+m_j-i)+\tfrac{1}{2}(4nt+jn+n-i)\\ &+\tfrac{1}{2}(4nt+jn-(i-1))+((2t-j)n+i+1)\\ &+((2t-j)n+(i-m_j)+1)\\ =\;& 8nt+\tfrac{n+5}{2}. \end{aligned}$$

d) For $i \ge m_j$ and i odd, we get

$$\begin{aligned} wt_{\alpha_0}(v_i^j) = \;& (jn+m_j-i)+\tfrac{1}{2}(4nt+jn-i)\\ &+\tfrac{1}{2}(4nt+jn+n-(i-1))+((2t-j)n+i+1)\\ &+((2t-j)n+(i-m_j)+1)\\ =\;& 8nt+\tfrac{n+5}{2}. \end{aligned}$$

Case 2. j even

a) For $0 \le i \le m_j - 1$ and i even, we admit

$$\begin{aligned} wt_{\alpha_0}(v_i^j) = \;& ((2t-j)n+m_j-i)+\tfrac{1}{2}(4nt+(2t-j+1)n+n-i)\\ &+\tfrac{1}{2}(4nt+(2t-j+1)n-(i-1))+((j-1)n+i+1)\\ &+((j-1)n+(n-m_j+i)+1)\\ =\;& 8nt+\tfrac{n+5}{2}. \end{aligned}$$

b) For $0 \le i \le m_j - 1$ and i odd, we have

$$\begin{aligned} wt_{\alpha_0}(v_i^j) = \;& ((2t-j)n+m_j-i)+\tfrac{1}{2}(4nt+(2t-j+1)n-i)\\ &+\tfrac{1}{2}(4nt+(2t-j+1)n+n-(i-1))+((j-1)n+i+1)\\ &+((j-1)n+(n-m_j+i)+1)\\ =\;& 8nt+\tfrac{n+5}{2}. \end{aligned}$$

c) For $i \geq m_j$ and i even, we get

$$\begin{aligned} wt_{\alpha_0}(v_i^j) = &\ ((2t-j)n+n+m_j-i)+\tfrac{1}{2}(4nt+(2t-j+1)n+n-i) \\ &+\tfrac{1}{2}(4nt+(2t-j+1)n-(i-1))+((j-1)n+i+1) \\ &+((j-1)n+(i-m_j)+1) \\ = &\ 8nt+\tfrac{n+5}{2}. \end{aligned}$$

d) For $i \geq m_j$ and i odd, we have

$$\begin{aligned} wt_{\alpha_0}(v_i^j) = &\ ((2t-j)n+n+m_j-i)+\tfrac{1}{2}(4nt+(2t-j+1)n-i) \\ &+\tfrac{1}{2}(4nt+(2t-j+1)n+n-(i-1))+((j-1)n+i+1) \\ &+((j-1)n+(i-m_j)+1) \\ = &\ 8nt+\tfrac{n+5}{2}. \end{aligned}$$

We obtain $wt_{\alpha_0}(v_i^j) = 8nt + \frac{n+5}{2}$ for all cases. This proves that α_0 is a vertex-magic total labeling for $\bigcup_{j=1}^{t} C_n(1, m_j)$ with the magic constant $h = 8nt + \frac{n+5}{2}$. □

The next theorem gives a super $(a, 1)$-VAT labeling for the disjoint union of t circulant graphs $C_{n_j}(1, m_j)$, each with an odd order $n_j \geq 5$, $j = 1, 2, \ldots, t$, and m_j, $j = 1, 2, \ldots, t$, an integer from the set $\{2, 3, \ldots, \frac{n_j-1}{2}\}$.

Theorem 7. *Let t be a positive integer and $n_j \geq 5$ be odd for all $j = 1, 2, \ldots, t$. If $m_j \in \{2, 3, \ldots, \frac{n_j-1}{2}\}$, for $j = 1, 2, \ldots, t$, then the graph $\bigcup_{j=1}^{t} C_{n_j}(1, m_j)$ admits a super $\left(8\sum_{k=1}^{t} n_k + 3, 1\right)$-VAT labeling.*

Proof. Let $\{C_{n_j}(1, m_j) : j = 1, 2, \ldots, t\}$ be the set of circulant graphs with n_j vertices each, $n_j \geq 5$ odd, and $m_j \in \{2, 3, \ldots, \frac{n_j-1}{2}\}$. Let $\{v_i^j : i = 0, 1, \ldots, n_j - 1$ and $j = 1, 2, \ldots, t\}$ be the vertices of the graph $\bigcup_{j=1}^{t} C_{n_j}(1, m_j)$.

Label the vertices and edges of $\bigcup_{j=1}^{t} C_{n_j}(1, m_j)$, for all $i = 0, 1, \ldots, n_j - 1$ and $j = 1, 2, \ldots, t$, as follows:

$$\alpha_1(v_i^j) = \begin{cases} \sum_{k=1}^{j-1} n_k + m_j - i, & \text{for } 0 \leq i < m_j, \\ \sum_{k=1}^{j} n_k + m_j - i, & \text{for } i \geq m_j, \end{cases}$$

$$\alpha_1(v_i^j v_{i+m_j}^j) = 2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + i + 1, \quad \text{for } i = 0, 1, \ldots, n_j - 1.$$

If $n_j \equiv 1 \pmod 4$ then $\alpha_1(v_i^j v_{i+1}^j)$

$$= \begin{cases} \sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \frac{n_j+2i+5}{4}, & \text{for } i = 1,3,5,\dots,n_j-2, \\ \sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \frac{2i-n_j+5}{4}, & \text{for } i = \frac{n_j-1}{2}, \frac{n_j-1}{2}+2,\dots,n_j-1, \\ \sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \frac{3n_j+2i+5}{4}, & \text{for } i = 0,2,4,\dots,\frac{n_j-1}{2}-2. \end{cases}$$

If $n_j \equiv 3 \pmod 4$ then $\alpha_1(v_i^j v_{i+1}^j)$

$$= \begin{cases} \sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \frac{n_j+2i+5}{4}, & \text{for } i = 0,2,4,\dots,n_j-1, \\ \sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \frac{2i-n_j+5}{4}, & \text{for } i = \frac{n_j-1}{2}, \frac{n_j-1}{2}+2,\dots,n_j-2, \\ \sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \frac{3n_j+2i+5}{4}, & \text{for } i = 1,3,5,\dots,\frac{n_j-1}{2}-2. \end{cases}$$

It is easy to see that the vertex labels, under the labeling α_1, are

$$\alpha_1(V) = \left\{1,2,\dots,\sum_{k=1}^{t} n_k\right\},$$

that is, the smallest possible labels. The edge labels are

$$\alpha_1(E) = \left\{\sum_{k=1}^{t} n_k + 1, \sum_{k=1}^{t} n_k + 2, \dots, 3\sum_{k=1}^{t} n_k\right\}.$$

This implies that the labeling α_1 is a bijection. Suppose that $n_j \equiv 1 \pmod 4$. Now, we consider the vertex-weights of the vertices of $\bigcup_{j=1}^{t} C_{n_j}(1,m_j)$, $i = 0,1,\dots,n_j-1$ and $j = 1,2,\dots,t$, case by case.

Case 1: For $0 \le i \le m_j - 1$ and i even, we have

$$wt_{\alpha_1}(v_i^j) = \left(\sum_{k=1}^{j-1} n_k + m_j - i\right) + \left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + i + 1\right)$$
$$+ \left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + (n_j - m_j + i) + 1\right) + \left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \frac{3n_j+2i+5}{4}\right)$$
$$+ \left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \frac{n_j+2(i-1)+5}{4}\right) = 8\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + 2i + 4.$$

Case 2: For $0 \le i \le m_j - 1$ and i odd, we get

$$wt_{\alpha_1}(v_i^j) = \left(\sum_{k=1}^{j-1} n_k + m_j - i\right) + \left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + i + 1\right)$$

$$+ \left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + (n_j - m_j + i) + 1\right) + \left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \tfrac{n_j+2i+5}{4}\right)$$

$$+ \left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \tfrac{3n_j+2(i-1)+5}{4}\right) = 8\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + 2i + 4.$$

Case 3: For $m_j \le i \le \frac{n_j-3}{2}$ and i even, we get

$$wt_{\alpha_1}(v_i^j) = \left(\sum_{k=1}^{j} n_k + m_j - i\right) + \left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + i + 1\right)$$

$$+ \left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + (i - m_j) + 1\right) + \left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \tfrac{3n_j+2i+5}{4}\right)$$

$$+ \left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \tfrac{n_j+2(i-1)+5}{4}\right) = 8\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + 2i + 4.$$

Case 4: For $m_j \le i \le \frac{n_j-3}{2}$ and i odd, we have

$$wt_{\alpha_1}(v_i^j) = \left(\sum_{k=1}^{j} n_k + m_j - i\right) + \left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + i + 1\right)$$

$$+ \left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + (i - m_j) + 1\right) + \left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \tfrac{n_j+2i+5}{4}\right)$$

$$+ \left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \tfrac{3n_j+2(i-1)+5}{4}\right) = 8\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + 2i + 4.$$

Case 5: For $i \ge \frac{n_j-1}{2}$ and i even, we have

$$wt_{\alpha_1}(v_i^j) = \left(\sum_{k=1}^{j} n_k + m_j - i\right) + \left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + i + 1\right)$$

$$+ \left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + (i - m_j) + 1\right) + \left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \tfrac{2i-n_j+5}{4}\right)$$

$$+ \left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \tfrac{n_j+2(i-1)+5}{4}\right) = 8\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l - n_j + 2i + 4.$$

Case 6: For $i \ge \frac{n_j-1}{2}$ and i odd, we get

$$wt_{\alpha_1}(v_i^j) = \left(\sum_{k=1}^{j} n_k + m_j - i\right) + \left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + i + 1\right)$$

$$+\left(2\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + (i-m_j)+1\right) + \left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \tfrac{n_j+2i+5}{4}\right)$$

$$+\left(\sum_{k=1}^{t} n_k + \sum_{l=j+1}^{t} n_l + \tfrac{2(i-1)-n_j+5}{4}\right) = 8\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l - n_j + 2i + 4.$$

We conclude that for $i \in \{0,1,2,\dots,\frac{n_j-3}{2}\}$, $wt_{\alpha_1}(v_i^j) = 8\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l + 2i + 4$ and for $i \in \{\frac{n_j-1}{2},\dots,n_j-1\}$, $wt_{\alpha_1}(v_i^j) = 8\sum_{k=1}^{t} n_k + \sum_{l=1}^{j-1} n_l - n_j + 2i + 4$. Therefore, the set of vertex-weights is

$$\{8\sum_{k=1}^{t} n_k+4, 8\sum_{k=1}^{t} n_k+6,\dots,9\sum_{k=1}^{t} n_k+1\}\cup\{8\sum_{k=1}^{t} n_k+3, 8\sum_{k=1}^{t} n_k+5,\dots,9\sum_{k=1}^{t} n_k+2\}.$$

The vertex-weights form a sequence of consecutive integers starting with $a = 8\sum_{k=1}^{t} n_k + 3$. Thus, $\bigcup_{j=1}^{t} C_{n_j}(1,m_j)$ admits a super $\left(8\sum_{k=1}^{t} n_k + 3, 1\right)$-VAT labeling.

Similarly, it can be proved that for $n_j \equiv 3 \pmod 4$, $j = 1,2,\dots,t$, the graph $\bigcup_{j=1}^{t} C_{n_j}(1,m_j)$ has a super $\left(8\sum_{k=1}^{t} n_k + 3, 1\right)$-VAT labeling. □

The following theorem concerns an $(a,2)$-VAT labeling of a disjoint union of an odd number of circulant graphs $C_n(1,m_j)$, each of odd order n and $m_j \in \{2,3,\dots,\frac{n-1}{2}\}$, $j = 1,2,\dots$

Theorem 8. *Let $t \geq 1$ and $n \geq 5$ both be odd positive integers. If $m_j \in \{2,3,\dots,\frac{n-1}{2}\}$, for $j = 1,2,\dots,t$, then the graph $\bigcup_{j=1}^{t} C_n(1,m_j)$ admits a $\left(7nt + \frac{n+1}{2} + 3, 2\right)$-VAT labeling.*

Proof. Let $\{C_n(1,m_j) : j = 1,2,\dots,t\}$ be the set of circulant graphs with n vertices each, $n \geq 5$ odd, and $m_j \in \{2,3,\dots,\frac{n-1}{2}\}$. Let $\{v_i^j : i = 0,1,\dots,n-1$ and $j = 1,2,\dots,t\}$, be the vertices of the graph $\bigcup_{j=1}^{t} C_n(1,m_j)$.

Label the vertices and edges of $\bigcup_{j=1}^{t} C_n(1,m_j)$, for $i = 0,1,\dots,n-1$ and $j = 1,2,\dots,t$, as follows:

$$\begin{aligned}
\alpha_2(v_i^j) &= (2j-1)n - (a(i))n + m_j - i,\\
\alpha_2(v_i^j v_{i+m_j}^j) &= 3nt - \frac{1}{2}nt(odd(j)) - \frac{jn}{2} + i + 1,\\
\alpha_2(v_i^j v_{i+1}^j) &= (t-j)n + nt(odd(j)) + \frac{n}{2}(odd(i)) + \frac{i+2}{2}.
\end{aligned}$$

The vertex and edge labels, under the labeling α_2, are: $\alpha_2(V) = \{2pn+1, 2pn+2, \dots, (2p+1)n : p \in \{0,1,2,\dots,t-1\}\}$ and $\alpha_2(E) = \{(2p-1)n+1, (2p-1)n+2, \dots, 2pn : p \in \{1,2,\dots,t\}\} \cup \{2nt+1, 2nt+2, \dots, 3nt\}$. This means that the labeling α_2 is a bijection from the set $V\left(\bigcup_{j=1}^{t} C_n(1,m_j)\right) \cup E\left(\bigcup_{j=1}^{t} C_n(1,m_j)\right)$ onto the set $\{1,2,\dots,3nt\}$.

Using a similar proof as in Theorem 6 (with the same cases), it is a matter of routine checking to see that the labeling α_2 is a required $\left(7nt + \frac{n+1}{2} + 3, 2\right)$-VAT labeling. □

4 Conclusion

In this paper we have shown that the graph $\bigcup_{j=1}^{t} C_n(1,m_j)$, $j = 1,2,\dots,t$, $m_j \in \{2,3,\dots,\frac{n-1}{2}\}$, has a $\left(7nt + \frac{n+1}{2} + 3, 2\right)$-VAT labeling for t and n odd. We have tried to find an $(a,2)$-VAT labeling also for nt even, but so far without success. So, we propose the following

Open Problem 1. *For the graph* $\bigcup_{j=1}^{t} C_n(1,m_j)$, $m_j \in \{2,3,\dots,\frac{n-1}{2}\}$, $j = 1,2,\dots,t$, *determine whether there is an* $(a,2)$*-VAT labeling for* nt *even,* $n \geq 5$, $t \geq 2$.

Assume that graph $\bigcup_{j=1}^{t} C_n(1,m_j)$ has a super (a,d)-VAT labeling $\alpha : V \cup E \to \{1,2,\dots,3nt\}$ with the set of the vertex-weights $\{a, a+d, \dots, a+(nt-1)d\}$. The minimum possible vertex-weight is at least $1 + (nt+1) + (nt+2) + (nt+3) + (nt+4) = 4nt + 11$. On the other hand, the maximum possible vertex-weight is no more than the sum of nt, the largest vertex label, and the four largest edge labels $3nt-3, 3nt-2, 3nt-1, 3nt$. Consequently, $a + (nt-1)d \leq 13nt - 6$ and then $d \leq 9 - \frac{8}{nt-1}$. Thus, we have obtained an upper bound on the feasible values of the difference d.

In the case when $4 \leq d < 9$ we do not have any answer for a super (a,d)-VAT labeling of a union of multiple copies of circulant graph $C_n(1,m)$ or $\bigcup_{j=1}^{t} C_n(1,m_j)$, $m_j \in \{2,3,\dots,\lfloor\frac{n-1}{2}\rfloor\}$ and $j = 1,2,\dots,t$. Therefore, for further investigation we propose the following open problem.

Open Problem 2. *For the graph* $tC_n(1,m)$ *or the graph* $\bigcup_{j=1}^{t} C_n(1,m_j)$, $m_j \in \{2,3,\dots,\lfloor\frac{n-1}{2}\rfloor\}$, $j = 1,2,\dots,t$, *determine if there is a super* (a,d)*-VAT labeling for the feasible values* t *and* n *and for the feasible values of the difference* $d \geq 4$.

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