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The sandpile group of a bilateral regular tree*

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Abstract

Let T_n be a $(d-1)$-ary tree of height n. A bilateral regular tree is the multigraph $T^d_{n,n}$ that is defined by : (i) juxtaposing two T_n's, (ii) drawing a new edge between two roots and (iii) drawing $d-1$ edges between each leaf and a new vertex, called the sink. In this paper the sandpile group of $T^d_{n,n}$ is determined.

1 Introduction

The abelian sandpile model was introduced by Bak et al. in [2]. Since then, the sandpile group has been widely considered in various domains, including statistical physics [2, 7], algebraic combinatorics [3, 4, 5, 6] and arithmetic geometry [9, 10]. A sandpile group is also called a *critical group* ([4]) or a *Jacobian* ([1]) in the combinatorics literature. It is well-known that the order of the sandpile group of a graph is the number of spanning trees in the graph.

Let G be a simple connected graph; we single out one vertex, s, called the *sink*. A nonnegative vector $u \in \mathbb{Z}^{|V(G)|-1}$ may be considered as a (chip) configuration on G with u_i chips on vertex i. A toppling rule is defined as follows: a toppling occurs when a vertex has a number of chips not less than its degree; in that case it transfers a chip to each of its neighbors. A vertex i (not the sink) is *stable* if the number of chips u_i on it satisfies $u_i < d_i$, where d_i is the degree of the vertex i. If i is not stable, then by successively toppling unstable vertices in finitely many steps, a stable configuration u° can be achieved, and we call u° the stabilization of u. A stable configuration u is called *recurrent* if there is a nonzero configuration v such that $(u+v)^\circ = u$.

The set of all stable configurations on a graph G forms a commutative semigroup under the operation $(u; v) \to (u+v)^\circ$ and the set of all recurrent configurations forms

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a subgroup of the above semigroup; this group is called the *sandpile group* (denoted by $SP(G)$) of the graph G. See [7] for more details.

Let T_n be a $(d-1)$-ary tree of height n. Adding an edge between the roots of two T_n's, and connecting each leaf to a single vertex (called the sink) by $d-1$ edges, we get a *bilateral d-regular tree* $T_{n,n}^d$. For convenience, we also regard the roots of two T_n's as the roots of $T_{n,n}^d$. The spectra of bilateral regular trees has been discussed in [11].

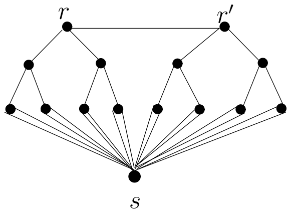

Fig. 1. A bilateral d-regular tree with $d=3, n=2$ and the roots r, r'

The aim of this paper is to explore the structure of the sandpile group $SP(T_{n,n}^d)$ on $T_{n,n}^d$. The remainder of this paper is organized as follows. In Section 2 we enumerate the number of spanning trees of $T_{n,n}^d$ and the orders of some elements of $SP(T_{n,n}^d)$. In Section 3 we characterize the recurrent configurations on $T_{n,n}^d$ in terms of critical vertices. In Section 4 we obtain the direct sum decomposition of $SP(T_{n,n}^d)$.

2 The order of the sandpile group of $T_{n,n}^d$

Let $T_n^\star$ be the graph obtained by drawing $d-1$ edges between each leaf of T_n and the sink. Adding an edge between the root and the sink of $T_n^\star$, we obtain a wired d-regular tree $\bar{T}_n$ of height n. The sandpile group of $\bar{T}_n$ and $T_n^\star$ was discussed in [8, 12].

Throughout this paper, we let $a = d-1$ and $[a]_k = 1 + a + a^2 + \cdots + a^{k-1}$.

Lemma 2.1 *([8]) Let t_n be the number of spanning trees of $\bar{T}_n$ with $n \geq 3$. Then*

$$t_n \;=\; t_{n-1}^{d-1} + (d-1)^{n+1}\prod_{k=1}^{n-1} t_k^{d-2} \;=\; t_{n-1}^{d-2}(dt_{n-1} - (d-1)t_{n-2}^{d-1}). \tag{2.1}$$

More explicitly,

$$t_n \;=\; [a]_{n+2}\prod_{k=1}^{n}[a]_{k+1}^{a^{n-k}(a-1)}. \tag{2.2}$$

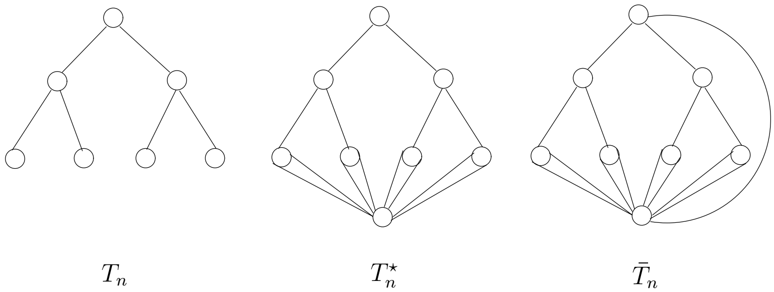

Fig. 2. Graphs $T_n, T_n^\star, \bar{T}_n$ with $d = 3, n = 2$.

The number of spanning trees of $T_{n,n}^d$ is given by the following lemma.

Lemma 2.2 *Let $\kappa(T_{n,n}^d)$ be the number of spanning trees of $T_{n,n}^d$ and $n \geq 3$. Then*

$$\kappa(T_{n,n}^d) \;=\; ([a]_{n+2} - [a]_{n+1})([a]_{n+2} + [a]_{n+1})(\prod_{k=1}^{n} [a]_{k+1}^{a^{n-k}(a-1)})^2.$$

That is,

$$\kappa(T_{n,n}^d) \;=\; a^{n+1}([a]_{n+2} + [a]_{n+1}) \prod_{k=1}^{n} [a]_{k+1}^{2(a-1)a^{n-k}}.$$

Proof. Let r, r' be the roots of $T_{n,n}^d$. We divide the spanning trees of $T_{n,n}^d$ into two classes:

Case 1. The edge rr' is included in the spanning tree. Then there is a path from the sink to the root r (also to the root r') in the spanning tree. Without loss of generality, we assume that the path exists in the first $T_n^\star$ with root r. Regarding this path from r' to the sink s as an edge $r's$, it is not hard to get the number of spanning trees of $\bar{T}_n$ including edge $r's$ is t_{n-1}^{d-1}, here t_n is the number of spanning trees of $\bar{T}_n$. The number of spanning trees of $T_n^\star$ with root r is $\prod_{k=1}^{n-1} t_k^{d-2}$. Since there are $2(d-1)^{n+1}$ possible paths from roots r to the sink s, the number of spanning trees in the first case is

$$2(d-1)^{n+1} t_{n-1}^{d-1} \prod_{k=1}^{n-1} t_k^{d-2} = 2t_{n-1}^{d-1}(t_n - t_{n-1}^{d-1})$$

by Eq. (2.1).

Case 2. The edge rr' is not included in the spanning tree. Since there are $(d-1)^{n+1}\prod_{k=1}^{n-1} t_k^{d-2}$ spanning trees on $\bar{T}_n$, the number of spanning trees of $T_{n,n}^d$ in the second case is

$$((d-1)^{n+1}\prod_{k=1}^{n-1} t_k^{d-2})^2 \;=\; (t_n - t_{n-1}^{d-1})^2.$$

Therefore the number of spanning trees of $T_{n,n}^d$ is

$$2t_{n-1}^{d-1}(t_n - t_{n-1}^{d-1}) + (t_n - t_{n-1}^{d-1})^2 \;=\; ((t_n - t_{n-1}^{d-1})(t_n + t_{n-1}^{d-1}).$$

Substituting t_n by Eq. (2.2), the result follows immediately. □

Let G be a tree with root r and let j $(j \neq r)$ be a vertex of G. Let p_j be the parent of vertex j, C_j be the set of children of j, V_j be the set of descendants of j, and V_j^l be the set of descendants of j in depth l. Denote the distance of vertices i and j by $\text{dist}(i,j)$. Now S_m is the sphere of radius m about the root of G, i.e., $S_m = \{k \in V(G) : \text{dist}(r,k) = m\}$. We say that the root r (respectively, r') is neither a parent nor a child of root r' (respectively, r) in $T_{n,n}^d$. The Laplacian matrix of $T_{n,n}^d$ is $L = D - A$, where A and D are the adjacency matrix and the degree matrix of $T_{n,n}^d$, respectively. Deleting the row and column corresponding to the sink s in L, we obtain a matrix Δ. Let $V = V(T_{n,n}^d) \setminus \{s\}$, δ_i $(i \in V)$ be the row vectors of Δ, $\{x_i : i \in V\}$ be the standard basis in $\mathbb{Z}^V$. That is, the i-th coordinate of x_i is 1 and the others are 0. It is clear that

$$\delta_i = \begin{cases} dx_i - x_{p_j} - \sum_{j\in C_i} x_j, & \text{if } \text{dist}(r,i) < n, \\ dx_i - x_{p_j}, & \text{otherwise.} \end{cases} \tag{2.3}$$

Let Λ be a lattice of $\mathbb{Z}^V$ spanned by $\{\delta_i : i \in V\}$. Then the sandpile group of $T_{n,n}^d$ is isomorphic to $\mathbb{Z}^V/\Lambda$ ([1]). Obviously, the image of δ_i in $SP(T_{n,n}^d)$ (denoted by $\bar{\delta}_i$) is the identity element e.

By Lemma 2.2, we know the order of the sandpile group $SP(T_{n,n}^d)$. Next we will find out some elements with orders $(d-1)^{n+1}$, $([a]_{n+2}+[a]_{n+1})$ and $(d-1)^{n+1}([a]_{n+2}+[a]_{n+1})$ in $SP(T_{n,n}^d)$, respectively.

Lemma 2.3 *([12]) Let $\mu_1,\dots,\mu_t \in \mathbb{Z}^t$ be linearly independent over $\mathbb{Q}$ and let $\Lambda = \Sigma_{n=1}^t \mathbb{Z}\mu_n$ be the lattice spanned by $\mu_1,\dots,\mu_t$. Consider the finite abelian group $K = \mathbb{Z}^V/\Lambda$. Assume that $\mu \in \mathbb{Z}^t$ satisfies*

$$r\mu = \sum_{n=1}^{t} r_n\mu_n,$$

where r, $r_n \in \mathbb{Z}$, $r > 0$ and $\gcd(r_1,\dots,r_t) = 1$. Then the order of $\bar{\mu}$ in K is r.

Lemma 2.4 *([12]) Let G be an abelian group and let $z_n \in G$ $(0 \le n \le t)$. Let $\text{ord}(z_0) = s$. Assume $r_n z_{n+1} = r_{n+1} z_n$ $(0 \le n \le t-1)$, where $r_n \in \mathbb{Z}$ and $\gcd(r_n, r_{n+1}) = 1$ $(0 \le n \le t-1)$. Then $\text{ord}(z_n) | \text{lcm}\{sr_0, r_1, \dots, r_{n-1}\}$ $(1 \le n \le t)$.*

Lemma 2.5 *([12]) Let $j \in S_m$ $(1 \le m \le n)$. Then*

$$\sum_{l=m}^{n}[a]_{n+1-l}\sum_{k\in V_j^l}\delta_k \;=\; [a]_{n+2-m}x_j - [a]_{n+1-m}x_{p_j}. \tag{2.4}$$

Furthermore,

$$[a]_{n+2-m}\bar{x}_j \;=\; [a]_{n+1-m}\bar{x}_{p_j}. \tag{2.5}$$

By the same argument, Lemma 2.5 is true for the graph $T_{n,n}^d$.

Proposition 2.6 *Let x_0 and x_0' be the generators corresponding to roots r and r', respectively. Then*

$$\operatorname{ord}(\bar{x}_0 + \bar{x}_0') \;=\; [a]_{n+2} - [a]_{n+1} \;=\; (d-1)^{n+1},$$
$$\operatorname{ord}(\bar{x}_0 - \bar{x}_0') \;=\; [a]_{n+2} + [a]_{n+1}.$$

Proof. Let S_l be the set of descendants of both r and r' in depth l. That is, $S_l = V_r^l + V_{r'}^l$. Consider the sum

$$((d-1)^{n+1}-1)\delta_r + \sum_{l=1}^{n}((d-1)^{n+1-l}-1)\sum_{k\in S_l}\delta_k. \tag{2.6}$$

For $l > 1$, $k \in S_l$, by Eq. (2.3), the coefficient of x_k in (2.6) is

$$-(d-1)((d-1)^{n-l}-1) + d((d-1)^{n+1-l}-1) - ((d-1)^{n+2-l}-1) \;=\; 0.$$

Similarly, the coefficient of x_k is also 0 for $l = 1$, $k \in V_r^l$. For $l = 1$, $k \in V_{r'}^l$, the coefficient of x_k is

$$d((d-1)^n - 1) - (d-1)((d-1)^{n-1}-1) = (d-1)^{n+1} - 1.$$

The coefficient of x_0 is

$$d((d-1)^{n+1} - 1) - (d-1)((d-1)^{n}-1) = (d-1)^{n+2} - 1.$$

The coefficient of x_0' is

$$-((d-1)^{n+1} - 1) - (d-1)((d-1)^{n}-1) = -2(d-1)^{n+1} + d.$$

Therefore we have

$$\begin{aligned}&((d-1)^{n+1}-1)\delta_r + \sum_{l=1}^{n}((d-1)^{n+1-l}-1)\sum_{k\in S_l}\delta_k\\ &= ((d-1)^{n+1}-1)\sum_{k\in C_{r'}}x_k + ((d-1)^{n+2}-1)x_0 - (2(d-1)^{n+1}-d)x_0'. \end{aligned} \tag{2.7}$$

Dividing by $d-2$ in both sides of Eq. (2.7) , since

$$[a]_n = 1 + (d-1) + \cdots + (d-1)^{n-1} = \frac{(d-1)^n - 1}{d-2},$$

we obtain

$$[a]_{n+1}\delta_r + \sum_{l=1}^{n}[a]_{n+1-l}\sum_{k\in S_l}\delta_k = [a]_{n+1}\sum_{k\in C_{r'}} x_k + [a]_{n+2}x_0 - (2[a]_{n+1}-1)x_0'. \quad (2.8)$$

Let $j \in C_{r'}$. By a similar argument, we get

$$\sum_{l=1}^{n}[a]_{n+1-l}\sum_{k\in v_{r'}^l}\delta_k = [a]_{n+1}\sum_{k\in C_{r'}} x_k - (d-1)[a]_n x_0'. \quad (2.9)$$

Substituting Eq. (2.9) into Eq. (2.8), since $(d-1)[a]_n = [a]_{n+1} - 1$, we have

$$[a]_{n+2}x_0 - [a]_{n+1}x_0' = \sum_{l=0}^{n}[a]_{n+1-l}\sum_{k\in V_r^l}\delta_k. \quad (2.10)$$

Similarly, we obtain

$$[a]_{n+2}x_0' - [a]_{n+1}x_0 = \sum_{l=0}^{n}[a]_{n+1-l}\sum_{k\in V_{r'}^l}\delta_k. \quad (2.11)$$

By Eqs. (2.10) and (2.11), we get

$$([a]_{n+2} + [a]_{n+1})(x_0 - x_0') = \sum_{l=0}^{n}[a]_{n+1-l}(\sum_{k\in V_r^l}\delta_k - \sum_{k\in V_{r'}^l}\delta_k), \quad (2.12)$$

$$([a]_{n+2} - [a]_{n+1})(x_0 + x_0') = \sum_{l=0}^{n}[a]_{n+1-l}(\sum_{k\in V_r^l}\delta_k + \sum_{k\in V_{r'}^l}\delta_k). \quad (2.13)$$

Since $[a]_1 = 1$, by Lemma 2.3, the result follows. □

The orders of more elements in $SP(T_{n,n}^d)$ are given in the next two propositions.

Proposition 2.7 *The order of* $\bar{x}_0$ *(* $\bar{x}_0'$*) is* $([a]_{n+2} - [a]_{n+1})([a]_{n+2} + [a]_{n+1})$.

Proof. Since

$$2x_0 = (x_0 + x_0') + (x_0 - x_0'), \quad (2.14)$$

multiplying $([a]_{n+2} - [a]_{n+1})([a]_{n+2} + [a]_{n+1})$ on both sides of Eq. (2.14), we obtain

$2([a]_{n+2}-[a]_{n+1})([a]_{n+2}+[a]_{n+1})x_0 = ([a]_{n+2}-[a]_{n+1})([a]_{n+2}+[a]_{n+1})(x_0+x_0')+([a]_{n+2}-[a]_{n+1})([a]_{n+2}+[a]_{n+1})(x_0-x_0')$. By Eqs. (2.12) and (2.13), we obtain

$$([a]_{n+2}-[a]_{n+1})([a]_{n+2}+[a]_{n+1})x_0 = \sum_{l=0}^{n}[a]_{n+1-l}([a]_{n+2}\sum_{k\in V_r^l}\delta_k + [a]_{n+1}\sum_{k\in V_{r'}^l}\delta_k).$$

Since $[a]_1=1$, $\gcd([a]_{n+2},[a]_{n+1})=1$, by Lemma 2.3,

$$\mathrm{ord}(\bar{x}_0) = ([a]_{n+2}-[a]_{n+1})([a]_{n+2}+[a]_{n+1}).$$

Similarly, we can obtain the order of $\bar{x}_0'$. □

Proposition 2.8 *Let $j_1, j_2 \in S_m$, $1\le m\le n$. Then the order of $\bar{x}_{j_1}-\bar{x}_{j_2}$ is $[a]_{n+2-m}$ if they are siblings (that is, they have the same parents);* $\mathrm{ord}(\bar{x}_{j_1}-\bar{x}_{j_2})|\mathrm{lcm}([a]_{n+1}([a]_{n+2}+[a]_{n+1}),[a]_n,\ldots,[a]_{n+2-m})$ *if they are not siblings.*

Proof. The proof of the first part is similar to the proof of Proposition 7.6 in [12]. Let c_{j_1} (c_{j_2}) be a child of j_1 (j_2), by Eq. (2.5), $[a]_{n-m+1}(\bar{x}_{c_{j_1}}-\bar{x}_{c_{j_2}}) = [a]_{n-m}(\bar{x}_{j_1}-\bar{x}_{j_2})$. Let $r_m=[a]_{n-m+1}$, $z_m=\bar{x}_{j_1}-\bar{x}_{j_2}$; by Proposition 2.6 and Lemma 2.4, the second conclusion holds. In particular, for $m=1$, and with j_1, j_2 not siblings, $\mathrm{ord}(\bar{x}_{j_1}-\bar{x}_{j_2})=[a]_{n+1}([a]_{n+2}+[a]_{n+1})$. □

Similar to the proof of Proposition 7.7 in [12], we also have the following.

Proposition 2.9 *Let $y_m=\sum_{k\in S_m}x_k$. Then the order of $\bar{y}_m$ is $(d-1)^{n+1-m}$, $0\le m\le n$.*

3 The recurrent configurations on $T_{n,n}^d$

In this section, we discuss the recurrent configurations on $T_{n,n}^d$ in terms of critical vertices.

Lemma 3.1 *([7], Burning algorithm) Let $\beta(i)$ be the number of edges in graph G from vertex i to the sink. A stable configuration u on G is recurrent if and only if adding $\beta(i)$ chips at each vertex i cause every vertex to topple exactly once.*

Let T_n (respectively, T_n') be a $(d-1)$-ary (respectively, $(d'-1)$-ary) tree of depth n and root r (respectively, r'). The graph Γ is obtained by joining the two roots by an edge and connecting each leaf of T_n (respectively, T_n') to the sink s by $(d-1)$ (respectively, $(d'-1)$) edges.

Definition 3.2 ([8]) A vertex $i\in V(\Gamma)$ is *critical* for a configuration u if $i\neq s$ and

$$u(i) \;\le\; \#\{j\in C_i \mid j \text{ is critical}\}. \tag{3.1}$$

From the definition, we know that the leaf vertex i is critical if and only if $u(i) = 0$.

Proposition 3.3 *A stable configuration u on Γ is recurrent if and only if at least one of the two roots is non-critical and equality holds in (3.1) for every critical vertex i.*

Proof. If a stable configuration is recurrent, then by Lemma 3.1, adding $\beta(i)$ chips at each vertex i cause every vertex to topple exactly once. If i is critical, then

$$u(i) + \#\{j \in C_i \mid j \text{ is not critical}\} \ \leq \ d_i - 1.$$

Obviously, for every critical vertex i, equality holds in (3.1); otherwise i never topples. Assume that both r and r' are critical. Then after chips are added, the two roots never topple because they can receive at most $(d_r - 1)$ and $(d_{r'} - 1)$ chips, respectively, so u is not recurrent.

If at least one of the roots is non-critical and equality holds in (3.1) for each critical vertex i, similar to the proof of Proposition 3.1 in [8], we begin toppling vertices in order of decreasing distance from the roots. A non-critical vertex i satisfies

$$u(i) + \#\{j \in C_i \mid j \text{ is not critical}\} \ \geq \ d_i - 1.$$

Inducting upwards, every non-critical vertex topples once. Since equality holds in (3.1) for every critical vertex i, the critical vertex i has either toppled (if its parents toppled) or is left with exactly $d_i - 1$ chips (if its parents did not topple). If two roots are both non-critical, then they toppled and all vertices toppled just once. If one of the two roots is critical, say r, then the number of chips on it is $d_r - 1$ by the above discussion. When r' toppled, r receives a chip and then it begins to topple. Thus all vertices topple just once. By Lemma 3.1, u is recurrent. □

By the Burning algorithm, for a graph G with $|V(G)|$ vertices,

$$\beta = (\beta(i))_{i \in V(G)} = \Delta = -\Sigma_{j=1}^{|V(G)|-1} \Delta_j \in \Delta\mathbb{Z}^{|V(G)|-1}.$$

Its recurrent representative is $\hat{\beta} = e$. Since β is constant on each of the levels on $T^d_{n,n}$, one can denote β by $(0, 0, \ldots, 0, d-1)$ which means that the vertices on the $(n+1)$-st level have $d-1$ chips and all the other vertices have no chip on them. It is not hard to obtain $e = (d-1, d-2, \ldots, d-2)$ from $\hat{\beta} = e = (\beta + e)^{\circ}$.

The principal branches of $\bar{T}_n$ are the subtrees rooted at the children of the root. Let T^i, $i = 1, \ldots, d-1$ (respectively, T'^i, $i = 1, \ldots, d'-1$) be the principal branches of $\bar{T}_n$ (respectively, $\bar{T}'_n$). If s_i (respectively, t_j) is a configuration on T^i (respectively, T'^i), a, b are two nonnegative integers. Denote $\binom{a \qquad\qquad b}{s_1,\ldots,s_{d-1},t_1,\ldots,t_{d-1}}$ the configuration on Γ with a chips on r and b chips on r'. By Proposition 3.3, we have the following lemma.

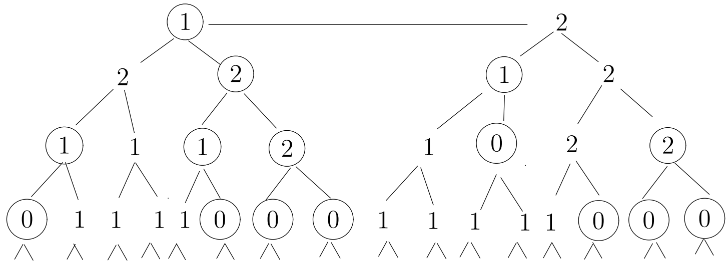

Fig. 3. A recurrent configuration on the bilateral 3-regular tree of height 3. The bottom edges lead to the sink and the critical vertices are circled.

Lemma 3.4 *Let* $u = \binom{a \qquad b}{s_1,\ldots,s_{d-1},t_1,\ldots,t_{d-1}}$.

(i) If u is a recurrent configuration on Γ, then $\binom{a}{s_1,\ldots,s_{d-1}}$ (respectively, $\binom{b}{t_1,\ldots,t_{d-1}}$) is a recurrent configuration in $SP(\bar{T}_n)$ (respectively, $SP(\bar{T}'_n)$).

(ii) If $\binom{a}{s_1,\ldots,s_{d-1}}$ (respectively, $\binom{b}{t_1,\ldots,t_{d-1}}$) is a recurrent configuration on $\bar{T}_n$ (respectively, $\bar{T}'_n$), and at least one of the two roots is non-critical, then u is recurrent on Γ.

Let $R(T^d_{n,n})$ be the subgroup $\langle \bar{x}_0, \bar{x}'_0 \rangle$ generated by $\bar{x}_0, \bar{x}'_0$ in $SP(T^d_{n,n})$. Then $R(T^d_{n,n})$ is a cyclic subgroup from the next proposition.

Proposition 3.5 *Let $R(T^d_{n,n}) = \langle \bar{x}_0, \bar{x}'_0 \rangle$. Then $R(T^d_{n,n})$ is a cyclic subgroup with generator $\bar{x}_0$ (or $\bar{x}'_0$).*

Proof. Obviously, since e is constant on each of the levels of $T^d_{n,n}$, in the process of stabilizing configuration $(k_1x_0+k_2x'_0)$ (k_1, k_2 are non-negative integers), the property of being constant on each of the levels of T_n is preserved. Enumerate the number of recurrent configurations of $T^d_{n,n}$ with constant on each levels of T_n now. Let $u = \binom{a \qquad b}{s_1,\ldots,s_{d-1},t_1,\ldots,t_{d-1}} = (w \vee v)$ be a recurrent configuration, constant on each level of T_n, where $w = \binom{a}{s_1,\ldots,s_{d-1}}$, $v = \binom{b}{t_1,\ldots,t_{d-1}}$ and the symbol "$\vee$" is a connector. Then both w and v are recurrent configurations, constant on each level of T_n, by Lemma 3.4. Conversely, let $w = \binom{a}{s_1,\ldots,s_{d-1}}$, $v = \binom{b}{t_1,\ldots,t_{d-1}}$ be recurrent configurations, constant on each level of T_n. We can represent w (v) as a vector $(a_1, \ldots, a_{n+1})$.

If neither w nor v contains coordinate 0, then there are $(d-1)^{2(n+1)}$ recurrent configurations.

If both w and v contain a coordinate 0, then all vertices between that level and the root are critical and each of them must have $d-1$ chips. By Proposition 3.3, $u = (w \vee v)$ is not a recurrent configuration. This means the configuration $u = (w \vee v)$ is recurrent, constant on each of the levels of T_n, if and only if both w and v are recurrent configurations constant on each of the levels of $\bar{T}_n$ and coordinate 0 cannot occur in w and v at the same time.

If there is a 0 coordinate in either w or v, the number of recurrent configurations is

$$2((d-1)^{(n+1)+n} + (d-1)^{(n+1)+(n-1)} + \cdots + (d-1)^{n+1}) = 2(d-1)^{n+1}[a]_{n+1}.$$

So the number of all recurrent configurations constant on each level of T_n is

$$\begin{aligned}(d-1)^{2(n+1)} + 2(d-1)^{n+1}[a]_{n+1} &= (d-1)^{n+1}([a]_{n+2} + [a]_{n+1}) \\ &= ([a]_{n+2} - [a]_{n+1})([a]_{n+2} + [a]_{n+1}).\end{aligned}$$

Since $\mathrm{ord}(\bar{x}_0) = \mathrm{ord}(\bar{x}'_0) = ([a]_{n+2} - [a]_{n+1})([a]_{n+2} + [a]_{n+1})$, it follows that $\langle \bar{x}_0 \rangle = R(T^d_{n,n})$. □

The following proposition gives the relation between $\bar{x}_0$ and $\bar{x}'_0$.

Proposition 3.6

$$\bar{x}_0 = (d(d-1)^{n+1}[a]_{n+1} - (d-2)[a]_{n+2})\bar{x}'_0.$$

Proof. By Eqs. (2.10) and (2.11), we have

$$\begin{aligned}[a]_{n+2}\bar{x}_0 &= [a]_{n+1}\bar{x}'_0; \\ [a]_{n+2}\bar{x}'_0 &= [a]_{n+1}\bar{x}_0. \end{aligned} \tag{3.2}$$

Then

$$[a]_{n+2}\bar{x}_0 = \bar{x}_0 + (d-1)[a]_{n+1}\bar{x}_0 = \bar{x}_0 + (d-1)[a]_{n+2}\bar{x}'_0 = [a]_{n+1}\bar{x}'_0.$$

So we obtain

$$\bar{x}_0 = ([a]_{n+1} - (d-1)[a]_{n+2})\bar{x}'_0.$$

Since $[a]_{n+1} - (d-1)[a]_{n+2} < 0$,

$$\mathrm{ord}(\bar{x}_0) = ([a]_{n+2} - [a]_{n+1})([a]_{n+2} + [a]_{n+1}) = (d-1)^{n+1}([a]_{n+2} + [a]_{n+1}),$$

and we have

$$\begin{aligned}\bar{x}_0 &= ((d-1)^{n+1}([a]_{n+2} + [a]_{n+1}) + [a]_{n+1} - (d-1)[a]_{n+2})\bar{x}'_0 \\ &= ((d-1)^{n+1}(1 + (d-1)[a]_{n+1} + [a]_{n+1}) + [a]_{n+1} \\ &\qquad - (d-1)([a]_{n+1} + (d-1)^{n+1}))\bar{x}'_0 \\ &= ((d-1)^{n+1}(1 + d[a]_{n+1}) - (d-2)[a]_{n+1} - (d-1)^{n+2})\bar{x}'_0 \\ &= (d(d-1)^{n+1}[a]_{n+1} - (d-1)^{n+2} + 1)\bar{x}'_0 \\ &= (d(d-1)^{n+1}[a]_{n+1} - (d-2)[a]_{n+2})\bar{x}'_0.\end{aligned}$$

□

4 The direct sum decomposition of $SP(T^d_{n,n})$

The following theorem plays an important role in the proof of our main results.

Theorem 4.1 *Let $T^d_{n,n}$ be a bilateral d-regular tree, $R(T^d_{n,n})$ be the subgroup generated by $\bar{x}_0, \bar{x}'_0$ and $R(\bar{T}_n)$ be the subgroup generated by $\bar{x}^\star_0$ in $SP(\bar{T}_n)$, where $\bar{x}^\star_0$ is the generator of root r in $\bar{T}_n$. Then*

$$SP(T^d_{n,n})/R(T^d_{n,n}) \simeq SP(\bar{T}_n)/R(\bar{T}_n) \oplus SP(\bar{T}_n)/R(\bar{T}_n). \tag{4.1}$$

Proof. Define $\varphi : SP(T^d_{n,n}) \to SP(\bar{T}_n) \oplus SP(\bar{T}_n)$ by

$$\begin{pmatrix} a & b \\ s_1, \ldots, s_{d-1}, & t_1, \ldots, t_{d-1} \end{pmatrix} \mapsto \left(\begin{pmatrix} a \\ s_1, \ldots, s_{d-1} \end{pmatrix}, \begin{pmatrix} b \\ t_1, \ldots, t_{d-1} \end{pmatrix} \right).$$

By Lemma 3.4, the map φ is well-defined. From a combinatorial viewpoint, x_0 (or x'_0) can be regarded as a configuration with a single chip on r (or r') and no chip on other vertices. For any recurrent configuration u, $(u + \bar{x}_0)^\circ = (u + x_0)^\circ$. Consider the recurrent configuration

$$u = \left(\begin{pmatrix} a & b \\ s_1, \ldots, s_{d-1}, & t_1, \ldots, t_{d-1} \end{pmatrix} + \bar{x}_0 \right)^\circ = ((w \vee v) + \bar{x}_0)^\circ.$$

If $a < d-1$, since r does not topple after adding a chip on it, then $u \mapsto ((w + \bar{x}_0)^\circ, v)$.

If $a = d-1$, $b < d-1$, after r topples, then r' receives a chip from r but does not topple, and then $u \mapsto ((w + \bar{x}^\star_0)^\circ, (v + \bar{x}^\star_0)^\circ)$.

If $a = b = d-1$, then there are two positive integers k_1, k_2, such that $u \mapsto ((w + k_1\bar{x}_0)^\circ, (v + k_2\bar{x}'_0)^\circ)$. So we can get a map of quotients

$$\bar{\varphi} : SP(T^d_{n,n})/R(T^d_{n,n}) \to SP(\bar{T}_n)/R(\bar{T}_n) \oplus SP(\bar{T}_n)/R(\bar{T}_n).$$

Now we prove that $\bar{\varphi}$ is an isomorphism. First, we assert that $\bar{\varphi}$ is a bijection. For elements $w = \binom{a}{s_1,\ldots,s_{d-1}}, v = \binom{b}{t_1,\ldots,t_{d-1}} \in SP(\bar{T}_n)$, if either r or r' is non-critical, then by Lemma 3.4, $(w \vee v)$ is the inverse image. If both r and r' are critical, we assert that there is a recurrent configuration w' (respectively, v') in $SP(\bar{T}_n)$ whose root is non-critical, which is in the same equivalence class of w (respectively, v) modulo $R(\bar{T}_n)$. We prove the assertion by discussing two cases.

Case 1. There is a child of r (r') which is non-critical. Then $a < d-1$ ($b < d-1$). Adding a chip on r (r') does not cause r (r') to topple; then r (r') becomes a non-critical vertex. Thus $w' = \binom{a+1}{s_1,\ldots,s_{d-1}}$ (respectively, $v' = \binom{b+1}{t_1,\ldots,t_{d-1}}$) is in the same equivalence class of w (respectively, v) in $SP(\bar{T}_n)/R(\bar{T}_n)$.

Case 2. All the children of r and r' are critical. Then $a = b = d-1$. Without loss of generality, assume that $v_1 \in T_n$ is the first vertex whose chips number is less than $d-1$ but greater than 0. Then v_1 is not a leaf since all the children of r are critical. Adding a chip on r makes r topple. Now v_1 receives a chip from its parent and becomes a non-critical vertex which never topples. If the number of chips on r is less than $d-1$ after stabilization and r is non-critical now, then $w' = \left(\binom{a}{s_1,\ldots,s_{d-1}} + \bar{x}_0^\star\right)^\circ$ is what we required. If r is critical, after adding a chip on r again, then r is now non-critical. So $w' = \left(\binom{a}{s_1,\ldots,s_{d-1}} + 2\bar{x}_0^\star\right)^\circ$ is in the same equivalence class of w in $SP(\bar{T}_n)/R(\bar{T}_n)$. If the number of chips on r is $d-1$ after stabilization, then r must be a non-critical vertex by induction on the vertices of the path from v_1 to r during the stabilization. For $(d-1, d-1, \ldots d-1, 0)$, adding $n+1$ chips on r, by toppling successively, we get a recurrent configuration $(1, 1, \ldots, 1, 1, 1)$ and r is not critical now. Thus we find the class of recurrent configurations $(w' \vee v)$ in $SP(T^d_{n,n})$ modulo $R(\bar{T}^d_{n,n})$ corresponding to (w, v). So $\bar{\varphi}$ is a surjection.

Since the order of $SP(T^d_{n,n})/R(T^d_{n,n})$ is $\prod_{k=1}^{n}[a]_{k+1}^{2(a-1)a^{n-k}}$, which is the same as that of $SP(\bar{T}_n)/R(\bar{T}_n) \oplus SP(\bar{T}_n)/R(\bar{T}_n)$, the map $\bar{\varphi}$ is an injection.

Next, we prove that $\bar{\varphi}$ is a group homomorphism. For

$$u = \begin{pmatrix} & a & & b & \\ s_1, \ldots, s_{d-1}, t_1, \ldots, t_{d-1} \end{pmatrix}$$

$$u' = \begin{pmatrix} & a' & & b' & \\ s'_1, \ldots, s'_{d-1}, t'_1, \ldots, t'_{d-1} \end{pmatrix}$$

we have

$$(u+u')^\circ = ((w+w') \vee (v+v'))^\circ \mapsto$$

$$\left(\left(\begin{pmatrix} a+a' \\ s_1+s'_1, \ldots, s_{d-1}+s'_{d-1} \end{pmatrix}^\circ + j_1\bar{x}_0^\star\right)^\circ, \left(\begin{pmatrix} b+b' \\ t_1+t'_1, \ldots, t_{d-1}+t'_{d-1} \end{pmatrix}^\circ + j_2\bar{x}_0^\star\right)^\circ\right)$$

$$= ((w+w'+j_1\bar{x}_0^\star)^\circ, (v+v'+j_2\bar{x}_0^\star)^\circ)$$

for some nonnegative integers j_1, j_2. Thus $\bar{\varphi}$ is a group homomorphism. The proof is now complete. □

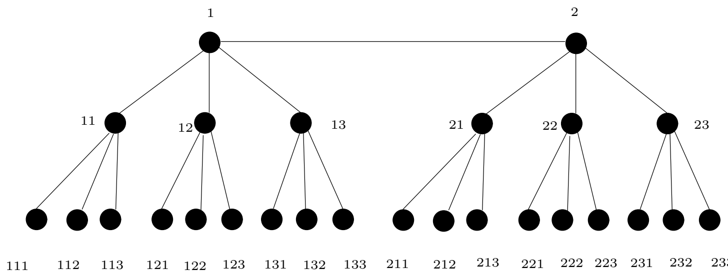

Fig. 4. An example of labelling of vertices of $T^d_{n,n}$ with $d = 4, n = 2$.

Index the non-sink vertices of $\bar{T}_n$ by words of length no more than $n+1$ in the alphabet $[d-1] = \{1, 2, \ldots, d-1\}$ (see Fig. 4). For $i = 1, \ldots, n+1$, let σ_i be an automorphism of $T^d_{n,n}$ given by $\sigma_i(w_1 \ldots w_k) = w_1 \ldots (w_{i+1}+1) \ldots w_k$ and $\sigma_i(v_1 \ldots v_k) = v_1 \ldots (v_{i+1}+1) \ldots v_k$, with the sum taken modulo $d-1$, where $(w_1 \ldots w_k), (v_1 \ldots v_k)$ are the labels of vertices on depth $k-1$ of two T_n's, respectively. If $k \leq i$, then $\sigma_i(w_1 \ldots w_k) = w_1 \ldots w_k, \sigma_i(v_1 \ldots v_k) = v_1 \ldots v_k$. For a map $\alpha : [n] \to [d-1]$, let σ_α be the composition $\prod_{i=1}^n \sigma_i^{\alpha(i)}$. Then σ_α is an automorphism of the sandpile group ([8]).

Lemma 4.2 *Let $T^d_{n,n}$ be a bilateral d-regular tree. Then*

$$SP(T^d_{n,n}) \simeq R(T^d_{n,n}) \oplus SP(\bar{T}_n)/R(\bar{T}_n) \oplus SP(\bar{T}_n)/R(\bar{T}_n). \tag{4.2}$$

Proof. Let $u = (w \vee v)$ be a recurrent configuration of $SP(T^d_{n,n})$ and

$$\rho(x) = ((d-1)^2 \sum_{\alpha:[n]\to[d-1]} \sigma_\alpha(x))^\circ, x \in \{w, v\}.$$

Then ρ is a map from $SP(\bar{T}_n)$ to $SP(\bar{T}_n)$ and is a projection onto $R(\bar{T}_n)$ (Proposition 4.2 in [8]). Let $y = (\rho(w) \vee (\rho(v))$. Then $\bar{y} = (y+e)^\circ$ is the recurrent representative of y in $SP(T^d_{n,n})$. Define a map $\tau : SP(T^d_{n,n}) \to SP(T^d_{n,n})$ by $\tau(u) = \bar{y}$. Obviously, $\tau(u)$ is constant on each level of each subgraph T_n by construction, and is a recurrent configuration by definition. So the image of τ lies in $R(T^d_{n,n})$. Conversely, let $u = (w \vee v) \in R(T^d_{n,n})$; since u is constant on each level of each subgraph T_n, we have $\sigma(w) = w, \sigma(v) = v$. Then $y = (((d-1)^{n+2}w)^\circ \vee ((d-1)^{n+2}v)^\circ) = (w \vee v) = u = \bar{y}$ since the order of $R(\bar{T}_n)$ is $\frac{(d-1)^{n+2}-1}{d-2}$. So $\tau(u) = u$. Thus $R(T^d_{n,n})$ is a summand of $SP(T^d_{n,n})$. $\square$

By Theorem 1.2 of [8],

$$SP(\bar{T}_n)/R(\bar{T}_n) \simeq \mathbb{Z}^{d-2}_{[a]_{n+1}} \oplus \mathbb{Z}^{(d-1)(d-2)}_{[a]_n} \oplus \cdots \oplus \mathbb{Z}^{(d-1)^{n-1}(d-2)}_{[a]_2}.$$

We obtain the abstract structure of $SP(T^d_{n,n})$ immediately by Lemma 4.1 and the above equality.

Theorem 4.3 *The sandpile group of a bilateral d-regular tree of height n is given by*

$$SP(T^d_{n,n}) \simeq \mathbb{Z}_{(d-1)^{n+1}([a]_{n+2}+[a]_{n+1})} \oplus \mathbb{Z}^{2(d-2)(d-1)^{n-1}}_{[a]_2} \oplus \mathbb{Z}^{2(d-2))(d-1)^{n-2}}_{[a]_3} \oplus \cdots \oplus \mathbb{Z}^{2(d-2)}_{[a]_{n+1}}.$$

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