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Signed total domination on Kronecker products of two complete graphs*

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Abstract

Given two graphs G_1 and G_2, the Kronecker product $G_1 \otimes G_2$ of G_1 and G_2 is a graph which has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$.

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In the present paper, we determine the exact value of the signed total domination number of $K_m \otimes K_n$ as a main result, and provide the exact values of the total domination number and the minus total domination number of $K_m \otimes K_n$.

1 Introduction

For terminology and notation on graph theory not given here, the reader is referred to [8]. In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For every vertex $v \in V$, the *neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. For an integer $r \geq 2$, a graph is *r-partite* if its vertex set can be partitioned into r subsets, or parts, $V_1, V_2, \ldots, V_r$, in such a way that no edge has both ends in the same part. An r-partite graph is called a *complete r-partite graph* if any two vertices in different parts are adjacent. If $|V_1| = n_1, |V_2| = n_2, \ldots, |V_r| = n_r$, then the complete r-partite graph is denoted by $K_{n_1,n_2,\ldots,n_r}$.

For a graph $G = (V, E)$, a function $f : V \to \{0, 1\}$ is called a *total dominating function* (TDF for short) of G if for every vertex $v \in V$, $\sum_{u\in N(v)} f(u) \geq 1$. The *weight* $w(f)$ of f is the sum of the function values of all vertices in G, that is, $w(f) = \sum_{v\in V} f(v)$. For a subset $S \subseteq V$, we define $f(S) = \sum_{v\in S} f(v)$, so $w(f) = f(V)$, $\sum_{u\in N(v)} f(u) = f(N(v))$. The *total domination number* $\gamma_t(G)$ of G is the minimum weight among all dominating functions of G, that is, $\gamma_t(G) = \min\{w(f) \mid f \text{ is a TDF of } G\}$. When a TDF f satisfies $w(f) = \gamma_t(G)$, f is also called a γ_t-function of G.

If in the above definitions we replace the set $\{0, 1\}$ by $\{-1, 0, 1\}$, we obtain the definitions of the *minus total dominating function* (MTDF) and of the *minus total domination number* $\gamma_t^-(G)$ of G. If we replace the set $\{0, 1\}$ by $\{-1, 1\}$, we obtain the definitions of the *signed total dominating function* (STDF) and of the *signed total domination number* $\gamma_{st}(G)$ of G. Total domination in graphs was introduced by Cokayne et al. in [2]. The minus (respectively, signed) total domination can be seen as a proper generalization of the classical total domination. All of the above parameters have been further studied in, for example, [1, 3, 4, 10, 13, 16].

Given two graphs G_1 and G_2, the *Kronecker product* $G_1 \otimes G_2$ of G_1 and G_2 is a graph which has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$. It is easy to see that the Kronecker product $K_m \otimes K_n$ of two complete graphs K_m and K_n can be considered as an m-partite graph with the size of each part n or as an n-partite graph with the size of each part m.

As stated in [12], the Kronecker product, in addition to its importance of constructing "bigger" graphs out of "small" ones, is useful in the recognition and decomposition of large graphs. It is also important in the sense that we can get an insight about its structural and topological properties from the factor graphs. The Kronecker product of graphs has other various applications; for instance, it can be used in modeling concurrency in multiprocessor system and in automata theory [6]. Moreover, the Kronecker product of graphs is the natural product in the category of graphs [7] and is also a widely used tool in the study of intersection networks. A number of parameters of the Kronecker product have been studied by various authors. Weichsel [15] studied the connectedness of the Kronecker product of two connected graphs. Miller [11] was interested in the connectivity of the Kronecker product. Fitina et al. [5] studied the edge connectivity of the Kronecker product. Waller [14] concentrated on an extension of the Kronecker product by a complete graph with two vertices. Some authors studied the planarity of the Kronecker product; see [9].

In [17], the authors have provided the values of some domination-related parameters of the Kronecker product of two complete graphs. In this paper, we concentrate on the computing of total-related domination parameters of the Kronecker product $K_m \otimes K_n$ of two complete graphs K_m and K_n. In Section 2 of the paper, we first show a lower bound of these parameters for $K_m \otimes K_n$. Then on the basis of this lower bound, we determine the exact value of the signed total domination number of $K_m \otimes K_n$ as a main result, and provide the exact values of the total domination number and the minus total domination number of $K_m \otimes K_n$.

For convenience, we give some definitions and notation. For the entire paper, let $V(K_m) = \{u_1, u_2, \ldots, u_m\}$, $V(K_n) = \{v_1, v_2, \ldots, v_n\}$, $V(K_m \otimes K_n) = \{v_{ij} = (u_i, v_j), i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$. Let $(V_1, V_2, \ldots, V_m)$ be the partitions of $K_m \otimes K_n$ as an m-partite graph, and let $(V'_1, V'_2, \ldots, V'_n)$ be the partitions of $K_m \otimes K_n$ as an n-partite graph. Suppose f is a function on the vertex set of $K_m \otimes K_n$, and v a vertex of $K_m \otimes K_n$. We denote $\sum_{s\in\{1,2,\ldots,m\}\setminus\{i\}} f(V_s)$ by $\sum_{s\neq i} f(V_s)$ for some $i \in \{1, 2, \ldots, m\}$, and denote $\sum_{t\in\{1,2,\ldots,n\}\setminus\{j\}} f(V'_t)$ by $\sum_{t\neq j} f(V'_t)$ for some $j \in \{1, 2, \ldots, n\}$. Also, we call the matrix $F = [f(v_{ij})]_{m\times n}$ the *matrix of* f, and call f the *function of* F, and denote by S_M the sum of all entries of a real matrix M.

2 Main result and the proof

First, we give a lower bound on the signed total domination number $\gamma_{st}(K_m \otimes K_n)$ as follows.

Lemma 1. *For two integers $m, n \geq 2$,*

$$\gamma_{st}(K_m \otimes K_n) \geq \frac{mn}{mn - m - n + 1}.$$

Proof. Let f be an STDF on $K_m \otimes K_n$. Then we have

$$f(N(v_{ij})) = \sum_{s=1}^{m}\sum_{t=1}^{n} f(v_{st}) - \sum_{t=1}^{n} f(v_{it}) - \sum_{s=1}^{m} f(v_{sj}) + f(v_{ij}) \geq 1.$$

Adding up these mn inequalities, we have

$$mn\sum_{s=1}^{m}\sum_{t=1}^{n}f(v_{st})-n\sum_{s=1}^{m}\sum_{t=1}^{n}f(v_{st})-m\sum_{s=1}^{m}\sum_{t=1}^{n}f(v_{st})+\sum_{s=1}^{m}\sum_{t=1}^{n}f(v_{st})\geq mn,$$

or equivalently

$$(mn-m-n+1)\sum_{s=1}^{m}\sum_{t=1}^{n}f(v_{st})\geq mn.$$

Therefore,

$$w(f)=\sum_{s=1}^{m}\sum_{t=1}^{n}f(v_{st})\geq\frac{mn}{mn-m-n+1}.$$

□

Now we give the signed total domination number of $K_m\otimes K_n$.

Theorem 2. *Let $m,n\geq 2$.*

(1) *When both m and n are even, $\gamma_{st}(K_m\otimes K_n)=4$.*

(2) *When both m and n are odd,*

$$\gamma_{st}(K_m\otimes K_n)=\begin{cases}7, & \text{if } \min\{m,n\}=3\\ 5, & \text{if } \min\{m,n\}\geq 5.\end{cases}$$

(3) *When m and n are of opposite parity,*

$$\gamma_{st}(K_m\otimes K_n)=\begin{cases}6, & \text{if } \min\{m,n\}\leq 3\\ 4, & \text{if } \min\{m,n\}\geq 4.\end{cases}$$

Proof. Suppose f is an STDF on $K_m\otimes K_n$. We consider the three parts of Theorem 2 respectively.

(1) First consider the case that both m and n are even. By Lemma 1, $w(f)\geq 2$. If $w(f)=2$, we deduce contradictions. Noting that $f(V_i)$ is even for all i, there must exist some $i\in\{1,2,\ldots,m\}$ such that $f(V_i)\geq 2$. Denote by F_i the matrix obtained from F by deleting the ith row of F and denote by l_j the sum of elements of the jth column of F_i. Suppose $f(V_i)=k\geq 2$. Then $S_{F_i}=w(f)-f(V_i)=2-k$. Since $S_{F_i}=\sum_{j=1}^{n}l_j$, there must exist some $j\in\{1,2,\ldots,n\}$ such that $l_j\geq\frac{2-k}{n}$. Therefore

$$f(N(v_{ij}))=\sum_{s\neq i}\sum_{t\neq j}f(v_{st})=S_{F_i}-l_j\leq\frac{2-k}{n}\leq 0,$$

contradicting the fact that f is an STDF on $K_m\otimes K_n$.

So $w(f)>2$. Noting that $w(f)$ is even, we have $w(f)\geq 4$, which implies that $\gamma_{st}(K_m\otimes K_n)\geq 4$. On the other hand, we define a function g on $K_m\otimes K_n$ as follows.

Let $g(v_{11}) = g(v_{12}) = 1$, $g(v_{1j}) = (-1)^{j+1}$ for $j = 3, 4, \ldots, n$; $g(v_{21}) = g(v_{22}) = 1$, $g(v_{2j}) = (-1)^{j+1}$ for $j = 3, 4, \ldots, n$; $g(v_{ij}) = (-1)^{i+j}$ for $i = 3, 4, \ldots, m$, $j = 1, 2, \ldots, n$. It is easy to see that g is an STDF on $K_m \otimes K_n$ with $w(g) = 4$. This means that $\gamma_{st}(K_m \otimes K_n) = 4$.

(2) Next consider the case that both m and n are odd. By Lemma 1, $w(f) \geq 2$. Noting that $w(f)$ is odd, we have $w(f) \geq 3$. If $w(f) = 3$, then there must exist some $i \in \{1, 2, \ldots, m\}$ such that $f(V_i) \geq 1$. Suppose $f(V_i) = k \geq 1$. Then $S_{F_i} = w(f) - f(V_i) = 3 - k$. Since $S_{F_i} = \sum_{j=1}^{n} l_j$, there must exist some $j \in \{1, 2, \ldots, n\}$ such that $l_j \geq \frac{3-k}{n}$. Therefore

$$f(N(v_{ij})) = \sum_{s \neq i} \sum_{t \neq j} f(v_{st}) = S_{F_i} - l_j \leq \frac{(n-1)(3-k)}{n}.$$

Note that k is odd. If $k = 1$, then $f(N(v_{ij})) < 2$. Noting that $f(N(v_{ij}))$ is even, we have $f(N(v_{ij})) \leq 0$, but this contradicts the fact that f is an STDF on $K_m \otimes K_n$. If $k \geq 3$, then $f(N(v_{ij})) \leq 0$, also a contradiction.

So $w(f) \geq 5$, which means that $\gamma_{st}(K_m \otimes K_n) \geq 5$.

If $\min\{m, n\} = 3$ and $w(f) = 5$, without loss of generality, suppose $m = 3$. Since $w(f) = f(V_1) + f(V_2) + f(V_3) = 5$ and $f(V_i)$ is odd for $i = 1, 2, 3$, there must exist some $i \in \{1, 2, 3\}$ such that $f(V_i) \geq 3$. Then by a similar argument to that in the case $w(f) = 3$, we can deduce contradictions. So $w(f) \geq 7$, which means that $\gamma_{st}(K_m \otimes K_n) \geq 7$ for $\min\{m, n\} = 3$. On the other hand, we define a function h on $K_m \otimes K_n$ as follows (we still suppose $m = 3$). Let $h(v_{11}) = h(v_{12}) = h(v_{13}) = 1$, $h(v_{1j}) = (-1)^j$ for $j = 4, \ldots, n$; $h(v_{21}) = h(v_{22}) = h(v_{23}) = 1$, $h(v_{2j}) = (-1)^j$ for $j = 4, \ldots, n$; $h(v_{3j}) = (-1)^{j+1}$ for $j = 1, 2, \ldots, n$. It is not difficult to see that h is an STDF on $K_m \otimes K_n$ with $w(h) = 7$. This means that $\gamma_{st}(K_m \otimes K_n) = 7$ for $\min\{m, n\} = 3$.

If $\min\{m, n\} \geq 5$, we define an STDF, say l, on $K_m \otimes K_n$ with $w(l) = 5$. Let matrix $A = [l(v_{ij})]$ for $i, j = 1, 2, 3, 4, 5$. First we define A as follows.

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \end{pmatrix}$$

Second, define $l(v_{ij}) = (-1)^{i+j}$ for all other vertices v_{ij}. It is not hard to verify that l is an STDF on $K_m \otimes K_n$ with $w(l) = 5$. This implies that $\gamma_{st}(K_m \otimes K_n) = 5$ for $\min\{m, n\} \geq 5$.

(3) Finally, consider the case that m and n are of opposite parity. Noting that $f(N(v_{ij}))$ is even for all $v_{ij} \in V(K_m \otimes K_n)$, the constraints $f(N(v_{ij})) \geq 1$ turn out

to be $f(N(v_{ij})) \geq 2$. Then by adding up these mn inequalities, we find that

$$w(f) = \sum_{s=1}^{m} \sum_{t=1}^{n} f(v_{st}) \geq \frac{2mn}{mn - m - n + 1} > 2.$$

Since $w(f)$ is even, we have $w(f) \geq 4$, which means that $\gamma_{st}(K_m \otimes K_n) \geq 4$.

If $\min\{m, n\} \leq 3$ and $w(f) = 4$, then by using a similar method as that used when $w(f) = 3$ in part (2), we can deduce contradictions. So $w(f) \geq 6$, which means that $\gamma_{st}(K_m \otimes K_n) \geq 6$ for $\min\{m, n\} \leq 3$. On the other hand, we define an STDF ϕ on $K_m \otimes K_n$ with $w(\phi) = 6$. Without loss of generality, suppose that m is even and n is odd. If $\min\{m, n\} = 2$ and thus $m = 2$, we define ϕ by: $\phi(v_{11}) = \phi(v_{12}) = \phi(v_{13}) = 1$, $\phi(v_{1j}) = (-1)^j$ for $j = 4, \ldots, n$; $\phi(v_{21}) = \phi(v_{22}) = h(v_{23}) = 1$, $\phi(v_{2j}) = (-1)^j$ for $j = 4, \ldots, n$. If $\min\{m, n\} = 3$ and thus $n = 3$, we define ϕ by: $\phi(v_{11}) = \phi(v_{12}) = \phi(v_{13}) = 1$; $\phi(v_{21}) = \phi(v_{22}) = h(v_{23}) = 1$; $\phi(v_{ij}) = (-1)^{i+j}$ for $i = 3, \ldots, m$, $j = 1, 2, 3$. One can easily verify that ϕ is an STDF on $K_m \otimes K_n$ with $w(\phi) = 6$. This means that $\gamma_{st}(K_m \otimes K_n) = 6$ for $\min\{m, n\} \leq 3$.

If $\min\{m, n\} \geq 4$, we still assume that m is even and n is odd. Now we define an STDF, say φ, on $K_m \otimes K_n$ with $w(\varphi) = 4$. Let matrix $B = [\varphi(v_{ij})]$ for $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4, 5$. First we define B as follows.

$$B = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Second define $\varphi(v_{ij}) = (-1)^{i+j}$ for all other vertices v_{ij}. It is not hard to verify that φ is an STDF on $K_m \otimes K_n$ with $w(\varphi) = 4$. This implies that $\gamma_{st}(K_m \otimes K_n) = 4$ when $\min\{m, n\} \geq 4$. We have now completed the proof of Theorem 2. □

Now we give the total and the minus total domination numbers of $K_m \otimes K_n$ as follows.

Theorem 3. *For $m, n \geq 2$,*

$$\gamma_t(K_m \otimes K_n) = \gamma_t^-(K_m \otimes K_n)) = \begin{cases} 4 & \text{if } \min\{m, n\} = 2 \\ 3 & \text{if } \min\{m, n\} \geq 3. \end{cases}$$

Proof. First, by a similar argument as in the proof of Lemma 1, we can prove that $\gamma_t^-(K_m \otimes K_n) \geq \frac{mn}{mn-m-n+1}$, and thus $\gamma_t(K_m \otimes K_n) \geq \gamma_t^-(K_m \otimes K_n) \geq 2$. Moreover, if there exists a TDF or an MTDF whose weight is equal to 2 or 3 when $\min\{m, n\} = 2$, or there exists a TDF or an MTDF whose weight is equal to 2 when $\min\{m, n\} \geq 3$, then we can deduce contradictions by similar arguments as in the proof for the case $w(f) = 3$ in part (2) of Theorem 2. On the other hand, we can give a function, which is a TDF and also an MTDF, say f, such that $w(f) = 4$

when $\min\{m,n\} = 2$, and $w(f) = 3$ when $\min\{m,n\} \geq 3$. Suppose without loss of generality that $\min\{m,n\} = m$. We define f as follows. When $\min\{m,n\} = 2$, let $f(v_{11}) = f(v_{12}) = f(v_{21}) = f(v_{22}) = 1$, and let $f(v_{ij}) = 0$ for all other vertices; when $\min\{m,n\} \geq 3$, let $f(v_{11}) = f(v_{22}) = f(v_{33}) = 1$, and let $f(v_{ij}) = 0$ for all other vertices. It is easy to verify that f is a TDF and also an MTDF on $K_m \otimes K_n$. Hence the desired result. □

At present, the study of the Kronecker product is mainly focused on the bounds or exact values of its several parameters. However, as far as we know, there is no published result on its computational complexity. For example, if there are polynomial algorithms for determining the total domination numbers of G_1 and G_2, then we hope to know whether there is a polynomial algorithm for determining the total domination numbers of $G_1 \otimes G_2$.

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