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Edge reconstruction and the swapping number of a graph

Dalibor Froncek

Department of Mathematics and Statistics
University of Minnesota Duluth
1117, University Drive, Duluth, MN 55812
U.S.A.

Amy Hlavacek

Department of Mathematical Sciences
Saginaw Valley State University
7400 Bay Road, University Center, MI 48710
U.S.A.

Steven J. Rosenberg*

Department of Mathematics and Computer Science
University of Wisconsin – Superior
1610 Catlin Avenue, Superior, WI 54880
U.S.A.

Abstract

We define the swapping number of an arbitrary simple graph, which is related to edge reconstruction, and involves a weakening of the concept of a graph automorphism. We classify all 1-swappable trees and unicyclic graphs and prove that the expected value of the swapping number grows linearly with the order of the graph.

1 Introduction

In what follows, all graphs are finite, undirected, and simple unless otherwise specified. If $G = (V, E)$ is a graph and $S \subseteq V$, then we will understand $G - S$ to mean the graph induced by G on the vertex set $V - S$. By $\binom{V}{2}$, we mean the set of all 2-combinations of V.

* Corresponding author: `srosenbe@uwsuper.edu`

We recall that the *edge-deck* of G is the multiset $ED(G)$ of all isomorphism classes of graphs of the form $(V(G), E(G) - \{e\})$, where e is an edge of G. The *Edge-Reconstruction Conjecture* (Harary, 1964) states that every graph on 4 or more edges is uniquely reconstructible, up to isomorphism, from its edge-deck (see e.g. [3]).

If the Edge-Reconstruction Conjecture fails, then there are two non-isomorphic graphs G and H such that every member of $ED(G)$ can be extended to G by adding some edge e_1, and to H by adding some edge e_2, where $e_1 \neq e_2$. By requiring this latter situation to occur when $H = G$, we arrive at the definition of a 1-*swappable* graph: namely, a graph in which every edge can be replaced with a non-edge (which may depend on the edge chosen) to produce a graph isomorphic to the original graph. More generally, we have the following:

Definition 1 Let $G = (V, E)$ be a graph and let k be a positive integer. We say that G is k-*swappable* if for every edge e of G, the following condition is met:

$$\exists A \subseteq E \text{ and } B \subseteq \binom{V}{2} - E \text{ s. t. } e \in A, |A| \leq k, \text{ and } G \cong (V, (E \cup B) - A). \quad (1)$$

The *swapping number* of G is the minimum k such that G is k-swappable, if such a k exists. If such a k does not exist, we define the swapping number of G to be ∞.

When (1) is satisfied for a given edge $e \in E$, we will usually write $G' = (V, (E \cup B) - A)$ and denote the required isomorphism by $\sigma_e : G \to G'$. When $k = 1$, we refer to σ_e as a *swapping map*; in this case, we can write $B = \{e'\}$, and we call e' the *replacement* for e. We also say that e is *swappable* with e'.

We note that G is 1-swappable if and only if every graph in $ED(G)$ can be extended to G in at least two different ways. Note also that if G is 1-swappable and e is swappable with e', then e and e' are either similar or pseudosimilar in $H = G + e'$ (see [4]).

Example 2 Any path on three or more vertices is 1-swappable. The swapping number of an n-cycle is 2 when $n \geq 4$. However, any complete graph on two or more vertices (which includes the cases of a path on two vertices and a 3-cycle) is not k-swappable for any k (and thus has swapping number ∞), since there are no non-edges to serve as replacements.

The previous example generalizes to the following class of 1-swappable graphs:

Example 3 Let $T = (V, E)$ be an edge-transitive graph, and let e' be any edge of T. Then $G = (V, E - \{e'\})$ is 1-swappable. For if $e \in E(G)$, then we may replace e by e' to produce a graph G' with $G' \cong G$.

By taking T to be a cycle in Example 3, we get a path as in Example 2. Next we see that almost every 1-swappable tree is a path. Although the concept of swapping number was not explicit in earlier work, its close relationship to the edge-reconstruction problem allows us to classify all 1-swappable trees using a result of Harary and Lauri.

We introduce some terminology to facilitate our discussion:

Definition 4 A *comet* is a tree T in which exactly one vertex x has degree greater than 2. We call x the *central vertex* of T. An *arm* of a comet T is a connected component of $T - \{x\}$.

We denote a comet with central vertex of degree d by $\mathfrak{C}$ $(a_1,\ a_2,\ \ldots, a_d)$, where $a_1, a_2, \ldots, a_d$ are the numbers of vertices in the arms; thus $|V(\mathfrak{C}(a_1, a_2, \ldots, a_d))| = 1 + a_1 + a_2 + \cdots + a_d$.

Theorem 5 *Let T be a tree on two or more vertices. Then T is 1-swappable if and only if one of the following holds:*

1. *T is a path on at least three vertices;*
2. *$T \cong \mathfrak{C}(2, 1, 1)$;*
3. *$T \cong \mathfrak{C}(3, 2, 1)$.*

Proof: Theorem 4.4 of [2] gives that the graphs enumerated above (called *quasipaths* in that paper) are the only trees with the property that every edge meeting an endnode of the graph is swappable with some non-edge. Conversely, one sees easily that in fact every edge of the comets $\mathfrak{C}(2, 1, 1)$ and $\mathfrak{C}(3, 2, 1)$ is swappable with some non-edge. □

An alternate characterization of swappability comes from the following definition:

Definition 6 Let $G = (V, E)$ be a graph. A *near-automorphism* of G of *discrepancy* k is a bijection $\sigma : V \to V$ such that $|\sigma(E) - E| = k$.

Note that a near-automorphism of discrepancy 0 is an automorphism. (We use here our assumption that our graphs are finite.)

Lemma 7 *A graph G is k-swappable if and only if for every $e \in E$, there is a near-automorphism σ of G, of discrepancy at most k, such that $e \notin \sigma(E)$.*

Proof: Let $G = (V, E)$ be a graph. Both directions of the proof follow from the assignments $\sigma \leftrightarrow \sigma_e$, $A \leftrightarrow \sigma(E) - E$, $B \leftrightarrow E - \sigma(E)$. The details are left to the reader. □

The following lemma says that a complete collection of swapping maps for the leaf-edges of a tree cannot have a common fixed point.

Lemma 8 *Let T be a tree with more than one vertex, and let v be any vertex of T. Suppose that for each leaf $\lambda \neq v$ of T, we are given a corresponding isomorphism $\sigma_e : T \to (V(T), (E(T) - \{e\}) \cup \{e'\})$, where e is the leaf-edge of T containing λ and $e' \in \binom{V}{2} - E(T)$. Then there is some leaf-edge e such that $\sigma_e(v) \neq v$.*

Proof: By Theorem 4.4 of [2], we need only prove the result for paths and for the two comets $\mathfrak{C}(2, 1, 1)$ and $\mathfrak{C}(3, 2, 1)$. The details are straightforward and left to the reader. □

In Section 2, we explore the notion of 1-swappable weighted cycles, and we use these concepts to classify 1-swappable unicyclic graphs. In Section 3, we give a linear upper bound on the swapping number of a graph, and we compare this worst-case bound to the average swapping number.

2 1-swappable Unicyclic Graphs

Let G be a connected unicyclic graph having an n-cycle C as an induced subgraph, with $n \geq 3$. Let $v_1, \ldots, v_n$ be the vertices of C with subscripts in $\mathbf{Z}/n\mathbf{Z}$, and with v_i adjacent to v_{i+1} in C. Let G_i be the union of the components of $G - \{v_i\}$ disjoint from C, and let T_i be the subgraph of G induced by $V(G_i) \cup \{v_i\}$. Thus T_i is the tree pendant from v_i in G.

We first focus on the possibilities for the ordered sequence $(t_1, \ldots, t_n)$, where $t_i = |V(T_i)|$. For each swappable edge e of G, fix a corresponding swapping map $\sigma_e : G \to G'$ (note that G' depends on e). If $e \notin E(C)$, then $E(C) \subseteq E(G')$, so, as G is unicyclic, we must have $\sigma_e(V(C)) = V(C)$; that is, σ_e restricts to an automorphism of C, namely, an element of the dihedral group D_{2n}. For such edges e, we will consider σ_e to act on the elements of $\mathbf{Z}/n\mathbf{Z}$ via $\sigma_e(i) = j$ where $\sigma_e(v_i) = v_j$. Recall that D_{2n} consists of n rotations (shifts) and n reflections.

Our first result on unicyclic graphs is a corollary of Lemma 8.

Corollary 9 *Suppose that each leaf-edge of G is swappable. Then for every $i \in \mathbf{Z}/n\mathbf{Z}$ with $t_i > 1$, there is a leaf λ of T_i meeting an edge e of T_i such that the associated swap replaces e with an edge meeting $V(T_j)$ for some $i \neq j$, and we have $t_j = t_i - 1$. In particular, σ_e acts non-trivially on C.*

Proof: Fix $i \in \mathbf{Z}/n\mathbf{Z}$ with $t_i > 1$. For a leaf-edge e of T_i, if $\sigma_e(v_i) = v_i$ then σ_e acts on T_i as a swapping map. By Lemma 8, this cannot happen for every leaf-edge e of T_i, so there is an e whose replacement does not lie in $\binom{V(T_i)}{2}$. The result follows. $\square$

From now on, we assume that each leaf-edge of G is swappable, and for each $i \in \mathbf{Z}/n\mathbf{Z}$ with $t_i > 1$, we fix a swapping map σ_i with the property of Corollary 9. (Note that here we use an element i of $\mathbf{Z}/n\mathbf{Z}$ as an index, instead of an edge, to emphasize the dependence on the vertex v_i. This should cause no confusion.) We also let ρ_i denote the restriction of σ_i to C, and $\phi(i)$ denote the value j of Corollary 9. Thus, ϕ is a function from $\{i \in \mathbf{Z}/n\mathbf{Z} \,:\, t_i > 1\}$ to $\mathbf{Z}/n\mathbf{Z}$ with $t_{\phi(i)} = t_i - 1$.

To emphasize that the next few results only depend on the ordered sequence of t_i's, we introduce the notion of an isomorphism of vertex-weighted graphs. If H is a graph in which each vertex v has a weight $w(v)$, and H' is another such vertex-weighted graph, with weight function w', then by an isomorphism from H to H', we mean a graph isomorphism $\sigma : H \to H'$ such that $w(v) = w'(\sigma(v))$ for every v in $V(H)$. We consider the vertices of C to be weighted, with $w(v_j) = t_j$. Then for $i, j \in \mathbf{Z}/n\mathbf{Z}$, we let t^i_j denote the order of the tree pendant from v_j in the range G^i of σ_i, and we let C^i denote C with the vertex weights $w^i(v_j) = t^i_j$. We note that for every $i \in \mathbf{Z}/n\mathbf{Z}$, the map $\rho_i : C \to C^i$ is an isomorphism of vertex-weighted graphs.

Corollary 10 *If $i, y \in \mathbf{Z}/n\mathbf{Z}$ with $t_i, t_y > 1$ and $i \neq y$, then $\rho_i \neq \rho_y$. Further, if ρ_i is a reflection, then $\phi(i) = \rho_i(i)$.*

Proof: Choose $i \in \mathbf{Z}/n\mathbf{Z}$ with $t_i > 1$. Then for $j \in \mathbf{Z}/n\mathbf{Z}$, we have

$$t^i_j = \begin{cases} t_j - 1, & \text{if } j = i; \\ t_j + 1, & \text{if } j = \phi(i); \\ t_j, & \text{otherwise.} \end{cases}$$

Since we have $t_j = t^i_{\rho_i(j)}$ for all j, we see that

$$t_{\rho_i(j)} - t_j = \begin{cases} 1, & \text{if } \rho_i(j) = i; \\ -1, & \text{if } \rho_i(j) = \phi(i); \\ 0, & \text{otherwise.} \end{cases}$$

Thus i is uniquely determined by ρ_i. If ρ_i is a reflection, then ρ_i has order 2, so, taking $j = \rho_i^{-1}(i) = \rho_i(i)$, we have $t_i - t_{\rho_i(i)} = 1$, or $t_{\rho_i(i)} - t_i = -1$, which implies $\rho_i(i) = \phi(i)$, as desired. □

Definition 11 Let $d \in \mathbf{Z}/n\mathbf{Z}$. A *d-segment* in $\mathbf{Z}/n\mathbf{Z}$ is an arithmetic progression in $\mathbf{Z}/n\mathbf{Z}$ with common difference d.

Definition 12 Let j, f, k be non-negative integers with $f|n$, let

$$S_k = \{y \in \mathbf{Z}/n\mathbf{Z} \ : \ t_y = k\},$$

and let

$$B_{j,f} = \{y \in \mathbf{Z}/n\mathbf{Z} \ : \ y \equiv j \pmod{f}\}.$$

Let

$$O_{j,f,k} = B_{j,f} \cap S_k.$$

We say that $O_{j,f,k}$ is full, empty, or partial if the cardinality of $O_{j,f,k}$ is $|B_{j,f}|$, 0, or neither, respectively. We call $B_{j,f}$ an *f-band*. For f dividing n, let $P_{f,k}$ be the union over j of the partial $O_{j,f,k}$ sets.

When a particular swapping map $\sigma_i : G \to G^i$ is under consideration, we make use of the "i-superscript" notation throughout: thus, we take $S^i_k = \{y \in \mathbf{Z}/n\mathbf{Z} : t^i_y = k\}$; $O^i_{j,f,k} = B_{j,f} \cap S^i_k$; and $P^i_{f,k}$ is the union over j of the partial $O^i_{j,f,k}$ sets. We note that for $y \in S_k$, the only way we could have $t_j \neq t_{\rho_y(j)}$ is when one of these values is k and the other is $k - 1$.

Lemma 13 *Suppose that $k > 1$ and S_k is non-empty. Then $P_{f,k}$ is non-empty if $f < n$. If ρ_i is a rotation for some $i \in S_k$, then $P_{f,k}$ is a d-segment, where d is the amount of this rotation and $f = \gcd(d, n)$. Otherwise, there exists $d \in \mathbf{Z}/n\mathbf{Z} - \{0\}$ such that $P_{f,k}$ is the disjoint union of at most two d-segments, where $f = \gcd(d, n)$.*

Proof: Choose $k > 1$ such that S_k is non-empty. Suppose that $f|n$ and $f < n$. The f-bands $B_{j,f}$, for $j : 0 \leq j < f$, partition $\mathbf{Z}/n\mathbf{Z}$, so the sets $O_{j,f,k}$ partition S_k. Suppose that $O_{j,f,k}$ is full. Then $|O_{j,f,k}| = |B_{j,f}| = n/f \geq 2$, so we can choose distinct elements $a, b \in O_{j,f,k}$. Then we have the isomorphism $\sigma_a : G \to G^a$, with $t_a^a = t_a - 1 = k - 1$ and $t_b^a = t_b = k$. Therefore, the set $O_{j,f,k}^a$ is partial. Since σ_a is an isomorphism and ρ_a permutes the f-bands in C, then, for some q, the graph G must contain a partial $O_{q,f,k}$-set, namely $\rho_a^{-1}(O_{j,f,k}^a)$.

For a rotation $\rho \in D_{2n}$, set

$$\Lambda_k(\rho) = \{i \in \mathbf{Z}/n\mathbf{Z} \;:\; t_i = k \text{ and } t_{\rho(i)} \neq k\}.$$

Let d be the amount of rotation; i.e., $\rho(i) = i + d$ for all i. Let $f = \gcd(d, n)$. Then the orbits of ρ on $\mathbf{Z}/n\mathbf{Z}$ are the bands $B_{j,f}$ for $j : 0 \leq j < f$. We have that $\Lambda_k(\rho)$ intersects $B_{j,f}$ non-trivially if and only if $O_{j,f,k}$ is partial, and we see that $|\Lambda_k(\rho)|$ is the smallest number μ such that $P_{f,k}$ can be written as the disjoint union of μ distinct d-segments.

Let $a \in S_k$. First suppose that ρ_a is a rotation, and let $\rho = \rho_a$. If $i \in \mathbf{Z}/n\mathbf{Z}$, then $t_i \neq t_{\rho(i)}$ if and only if $\rho(i) \in \{a, \phi(a)\}$; if $\rho(i) = a$, then $t_i = t_{\rho(i)} - 1 = t_a - 1 = k - 1$, while if $\rho(i) = \phi(a)$, then $t_i = t_{\rho(i)} + 1 = t_{\phi(a)} + 1 = t_a = k$. Thus we have $\Lambda_k(\rho) = \{\rho^{-1}(\phi(a))\}$, so $|\Lambda_k(\rho)| = 1$.

So we may assume that ρ_a is a reflection for all $a \in S_k$. If $|S_k| = 1$ then the result is trivial, so assume $|S_k| > 1$, and choose distinct $a, b \in S_k$. Let $\sigma = \rho_a \circ \rho_b$. Then σ is a rotation, and by Corollary 10 and the fact that reflections have order 2, σ is not the identity. Let $i \in \Lambda_k(\sigma)$. Then $t_{\sigma(i)} \neq t_i$, so either $t_{\rho_b(i)} \neq t_i$ or $t_{\rho_a\rho_b(i)} \neq t_{\rho_b(i)}$. Therefore, $i \in \{b, \rho_b(b), \rho_b(a), \rho_b\rho_a(a)\}$. Now $t_a = t_b = t_i = k$, so $i \neq \rho_b(b)$. If $i \neq b$, then we must have $t_{\rho_b(i)} = t_i = k$ and $t_{\rho_a\rho_b(i)} = k - 1$; so $\rho_b(i) = a$ and $i = \rho_b(a)$. Therefore, $\Lambda_k(\sigma) \subseteq \{b, \rho_b(a)\}$, so $|\Lambda_k(\sigma)| \leq 2$. □

Lemma 14 *Let $k > 1$ with S_k non-empty.*

(i) If for some $d \in \mathbf{Z}/n\mathbf{Z} - \{0\}$, with $f = \gcd(d, n)$, we have that $P_{f,k}$ is a single d-segment, then we have $|P_{f,k}| \in \{1, 2, n/f - 1\}$.

(ii) Otherwise, there exists $d \in \mathbf{Z}/n\mathbf{Z} - \{0\}$ such that, with $f = \gcd(d, n)$, we have that $P_{f,k}$ is the disjoint union of two d-segments I_1 and I_2 with $|I_1| \geq |I_2|$, and $(|I_1|, |I_2|) \in \{(1, 1), (2, 1)\}$.

Proof: First note that for every $i \in \mathbf{Z}/n\mathbf{Z}$, we have $\rho_i(S_k) = S_k^i$ and ρ_i permutes the f-bands, so for each j, we have $\rho_i(O_{j,f,k}) = O_{u,f,k}^i$ for some u; in particular, $\rho_i(P_{f,k}) = P_{f,k}^i$.

(i) Suppose that $P_{f,k}$ is a single d-segment. Then we can write $P_{f,k} = \{i, i + d, i + 2d, \ldots, i + \ell d\}$ with $t_j = k$ for $j = i + md$ with $0 \leq m \leq \ell$ and $t_{i-d} \neq k$, $t_{i+(\ell+1)d} \neq k$. We may suppose that $|P_{f,k}| > 2$. Choose an interior point $j = i + md$ with $1 \leq m \leq \ell - 1$, and consider the swapping map $\sigma_j : G \to G^j$. Since $\rho_j(P_{f,k}) = P_{f,k}^j$ and ρ_j takes d-segments to d-segments, we must have that $P_{f,k}^j$ is a single d-segment. Now the only way to fill the gap in $P_{f,k}$ left by lowering t_j to $t_j - 1$ is by having $\phi(j) = i - d = i + (\ell + 1)d$, so we must have $|P_{f,k}| = |B_{j,f}| - 1 = n/f - 1$.

(ii) By Lemma 13, ρ_i is a reflection for all $i \in S_k$, and there exists $d \in \mathbf{Z}/n\mathbf{Z} - \{0\}$ such that $P_{f,k}$ is the disjoint union of two d-segments $I_1 = \{i_1, i_1 + d, \ldots, i_1 + \ell_1 d\}$ and $I_2 = \{i_2, i_2+d, \ldots, i_2+\ell_2 d\}$, where $f = \gcd(d, n)$ and, without loss of generality, $\ell_1 \geq \ell_2 \geq 0$.

Suppose for a contradiction that $|I_1| \geq 3$. Choose an interior point j of I_1. Then $I_1 - \{j\} \subseteq O^j_{i_1,f,k}$ and $t^j_j = k - 1$, so $O^j_{i_1,f,k}$ is partial; so $O^j_{i_1,f,k} \subseteq P^j_{f,k}$.

We claim that $O^j_{i_1,f,k}$ cannot be written as a single d-segment. For suppose otherwise. Then it is not hard to see that $O^j_{i_1,f,k} = B_{i_1,f} - \{j\}$ and $I_1 = O_{i_1,f,k} = B_{i_1,f} - \{\phi(j)\}$, with $\phi(j) = \rho_j(j) \in B_{i_1,f}$. If $\phi(y) \in B_{i_1,f}$ for all $y \in I_1$, then we would have $\rho_y(I_2) = I_2$, so $\{\rho_y(i_2) \ : \ y \in I_1\}$ has size $|I_1| = n/f - 1$ and is a subset of I_2; so $|I_2| = n/f - 1$, and $I_2 = B_{i_2,f} - \{c\}$ for some c; but then we would have $\rho_y(c) = c$ for all $y \in I_1$, which is impossible, since only one reflection fixes any given point. Therefore, we must have $\phi(y) \in B_{i_2,f}$ for some $y \in I_1$. This forces $|I_2| = |I_1| - 1 \geq 2$. Now we must have $\rho_w(B_{i_2,f}) = B_{i_2,f}$ for all $w \in I_2$. This ensures that ρ_w also sends $B_{i_1,f}$ to itself for $w \in I_2$, which in turn implies that ρ_w fixes $\phi(j)$. Again, this is impossible, since only one reflection fixes $\phi(j)$.

Now since $P^j_{f,k}$ is the disjoint union of two d-segments, while $O^j_{i_1,f,k}$ cannot be written as a single d-segment, we must have $P^j_{f,k} = O^j_{i_1,f,k}$. Since $\sigma_j(P_{f,k}) = P^j_{f,k}$, then I_1 and I_2 must lie in the same f-band, with $\phi(j) \in X := \{i_1 - d, i_2 - d, i_1 + (\ell_1 + 1)d, i_2 + (\ell_2 + 1)d\}$. We see further that $|X| \leq 3$ and $|I_2| \leq |I_1| - 2$. Also, we cannot have $i_1 - d = i_1 + (\ell_1 + 1)d$ since in that case, I_1 would occupy all but one element of $B_{i_1,f}$. Similarly, $i_2 - d \neq i_2 + (\ell_2 + 1)d$. Now considering $j = i_1$ when $i_1 + (\ell_1 + 1)d = i_2 - d$ and $j = i_1 + \ell_1 d$ when $i_2 + (\ell_2 + 1)d = i_1 - d$, we find that $|I_2| + 1 = |I_1|$, a contradiction.

Finally, we eliminate the possibility that $|I_1| = |I_2| = 2$. So suppose that $I_1 = \{i_1, i_1 + d\}$ and $I_2 = \{i_2, i_2 + d\}$. To preserve the two d-segments of length 2, we must have $\phi(i_1) = i_1 + 2d$. Now ρ_{i_1} is a reflection, so by Corollary 10, we have $\phi(i_1) = \rho_{i_1}(i_1)$. Since a reflection in D_{2n} has the form $x \mapsto r - x$ for some fixed $r \in \mathbf{Z}/n\mathbf{Z}$, we find that $\rho_{i_1}(x) = 2(i_1 + d) - x$ for all $x \in \mathbf{Z}/n\mathbf{Z}$, so ρ_{i_1} fixes $i_1 + d$. Now we must have $t_{i_2} = t^{i_1}_{i_2} = t_{i_2+d} = t^{i_1}_{i_2+d} = k$, so we can say that ρ_{i_1} acts as a 2-cycle on $\{i_2, i_2 + d\}$ (a non-identity reflection in D_{2n} cannot have 3 fixed points). Similarly, we find that ρ_{i_1+d} switches i_2 with $i_2 + d$. But only one reflection in D_{2n} switches any given pair of elements, so we have reached a contradiction. □

Lemma 15 *Let $k \in \mathbf{Z}$, $k > 1$, and suppose that S_k is non-empty. Set*

$$F_k = \{\sigma \in D_{2n} \ : \ \sigma \neq e \text{ and } \sigma(S_k) = S_k\},$$

where e is the identity element of D_{2n}; that is, F_k is the set of all non-identity elements of D_{2n} which fix S_k as a set. Then $|F_k| \leq 1$.

Proof: Let d be as in the statement of Lemma 14 and let $f = \gcd(d, n)$. Let $\sigma \in F_k$. We proceed using the five cases listed in Lemma 14, writing I_1 and I_2 for the (up to) two d-segments comprising $P_{f,k}$, with $|I_1| \geq |I_2|$, and possibly with $|I_2| = 0$.

Case 1: $|I_1| = 1$ and $|I_2| = 0$.

Then there is a unique j such that $O_{j,f,k}$ is partial, and we have $O_{j,f,k} = \{a\}$ for some $a \in \mathbf{Z}/n\mathbf{Z}$. Because σ permutes the f-bands of $\mathbf{Z}/n\mathbf{Z}$ and $\sigma(S_k) = S_k$, this forces $\sigma(B_{j,f}) = B_{j,f}$ and $\sigma(a) = a$. Thus σ is the unique reflection about a in D_{2n}.

Case 2: $|I_1| = 2$ and $|I_2| = 0$.

Then there is a unique j such that $O_{j,f,k}$ is partial, and we can write $O_{j,f,k} = \{a, b\}$ for some $a, b \in \mathbf{Z}/n\mathbf{Z}$ with $a \neq b$ and $b = a + d$. Again we must have $\sigma(B_{j,f}) = B_{j,f}$, and so σ acts on $\{a, b\}$. There is a unique reflection τ in D_{2n} which sends a to b. Suppose for a contradiction that $\sigma \neq \tau$. Then either σ is a reflection which fixes a and b, or else σ is a rotation which sends a to b and b to a. In both cases, n must be even, say $n = 2m$, and we must have $b = a + m$. Thus $m = d$. But this forces $B_{a,f} = \{a, a + d\}$, so $O_{a,f,k}$ is full, contradicting that $a \in P_{f,k}$.

Case 3: $|I_1| = n/f - 1 \geq 3$ and $|I_2| = 0$.

Then we can write $P_{f,k} = O_{j,f,k}$ for a unique j, and we have $B_{j,f} - O_{j,f,k} = \{a\}$ for some $a \in \mathbf{Z}/n\mathbf{Z}$. It follows that $\sigma(B_{j,f}) = B_{j,f}$ and $\sigma(a) = a$. Thus σ is the unique reflection about a. (Notice that the assumption $n/f - 1 \geq 3$, which we introduced for the sake of mutual exclusivity of cases, was never used and the argument here holds even for $n/f \in \{2, 3\}$.)

Case 4: $|I_1| = |I_2| = 1$.

Then write $I_1 = \{a\}$ and $I_2 = \{b\}$. As in Case 2, we have that σ acts on $\{a, b\}$, so we proceed as in that case and assume for a contradiction that σ is not the reflection which switches a with b. Then we must have $b = a + m$ where $n = 2m$. Since we are in Case (ii) of Lemma 14, we know that ρ_a and ρ_b are reflections. Further, we have $\rho_a(a), \rho_a(b) \in P^a_{f,k} = (P_{f,k} - \{a\}) \cup \{\phi(a)\} = \{\phi(a), b\}$, and $\rho_a(a) = \phi(a)$, so $\rho_a(b) = b$. Similarly, $\rho_b(a) = a$. But the reflection which fixes a is the same reflection which fixes b, since $b = a + m$. This gives $\rho_a = \rho_b$, contradicting Corollary 10.

Case 5: $|I_1| = 2$ and $|I_2| = 1$.

Then write $I_2 = \{a\}$. We have $t_a = k$, $t_{a-d} \neq k$, $t_{a+d} \neq k$. Further, a is the unique element of $\mathbf{Z}/n\mathbf{Z}$ with this property. Since $\sigma(S_k) = S_k$ and σ sends d-segments to d-segments, then σ fixes a, so σ is the reflection about a. $\square$

Corollary 16 *We have $|S_k| = 0$ if $k \geq 4$. Further, if $|S_3| > 0$, then we have $|S_3| = |S_2| = 1$.*

Proof: Let $k \in \mathbf{Z}$, $k \geq 3$, and suppose that S_k is non-empty. Let $H = \{\rho_i \ : \ 2 \leq t_i < k\}$ and choose $\rho \in H$. Set $c = \sum_{2 \leq j < k} |S_j|$. By Corollary 10, we have $|H| = c$. Also, for $i \in \mathbf{Z}/n\mathbf{Z}$, we have that $t_i = t_{\rho(i)}$ unless $t_i < k$. Therefore, $\rho(S_k) = S_k$. It follows from Lemma 15 that $c \leq 1$. Since, for $j > 1$, $|S_j| > 0 \implies |S_{j-1}| > 0$ (by Corollary 9), we also have $c \geq k - 2$, so $k \leq 3$.

Suppose $k = 3$. Then $c = 1 = |S_2|$. Assume for a contradiction that $|S_3| \geq 2$. Set $U = S_2 \cup S_3$. Then for all $i \in S_3$, we have $\rho_i(U) = U$. Now there exist $i, j \in S_3$ with $i \neq j$, and at least one of ρ_i, ρ_j, $\rho_i \circ \rho_j$ must be a non-identity rotation in D_{2n} which fixes U setwise. It follows that U is a union of f-bands in $\mathbf{Z}/n\mathbf{Z}$, where $f = \gcd(d, n)$ and d is the amount of this rotation. There is a unique $\ell \in \mathbf{Z}/n\mathbf{Z}$ such that $t_\ell = 2$. We have $\rho_\ell(U) =: U' = (U - \{\ell\}) \cup \{\ell'\}$ for some ℓ' with $t_{\ell'} = 1$. So U' must also be a union of f-bands. But each f-band has size $n/f \geq 2$, so we must have $\ell = \ell'$, which is impossible. $\square$

In the following theorem, $\Delta(G)$ is, as usual, the maximum degree of the graph G.

Theorem 17 *Let G be a connected unicyclic graph with an n-cycle C, $V(C) = \{v_1, v_2, \ldots, v_n\}$, $\{v_i, v_{i+1}\} \in E(C)$, with subscripts taken modulo n. Let t_i be the number of vertices in the tree T_i pendant from v_i in G. Then G is 1-swappable if and only if it is isomorphic to one of the following types:*

(i) $n = 2m$ *where* $m \geq 2$, *and* $t_j = 2$ *for* $j \in \{4, 6, \ldots, 2m\}$, $t_j = 1$ *otherwise;*

(ii) $n = 3$, $t_0 = 2$, $t_1 = 1$, $t_2 = 1$;

(iii) $n = 3$, $\Delta(G) = 3$, $t_0 = 3$, $t_1 = 2$, $t_2 = 1$;

(iv) $n = 5$, $\Delta(G) = 3$, $t_0 = 3$, $t_1 = 2$, $t_j = 1$ *otherwise;*

(v) $n = 5$, $\Delta(G) = 3$, $t_0 = 3$, $t_2 = 2$, $t_j = 1$ *otherwise.*

We prove Theorem 17 using four claims, leaving the reader to check that the graphs listed above are indeed 1-swappable. Some notation: $H(x, y)$ is the graph G with the edge $\{x, y\}$ removed. Once $\{x, y\}$ is specified, we may use only H. We denote by $G' = G'(x, y)$ the graph isomorphic to G arising from $H(x, y)$, and $\sigma = \sigma_{\{x,y\}} : G \to G'$ is the associated swapping map. The vertices on the cycle C are taken to be the integers modulo n, and a vertex of T_i with degree one in G is denoted by i' or i''. A vertex of T_i with degree two in G is denoted by i^*. By $\mathrm{dist}_{G'}(T_i, T_k)$ we mean $\mathrm{dist}_{G'}(\sigma(i), \sigma(k))$. As usual, C_k denotes a k-cycle.

Claim 1 *Suppose* $|S_k| = 0$ *for* $k > 3$, $|S_3| = |S_2| = 1, |S_1| = n - 2$, $t_0 = 3$, $t_i = 2$. *Then* $\Delta(G) = 3$.

Proof: Without loss of generality, we assume that $n \geq 2i$. Assume for a contradiction that $\Delta(G) = 4$. We observe that if $n = 2i$, then we cannot find $G'(i, i')$ since $\sigma(0) = 0$ and $\sigma(i)$ cannot be equal to i. Hence, $n > 2i$ and $\mathrm{dist}_G(0, i) = i$.

To show that $i \leq 2$, we proceed by contradiction. Suppose $i \geq 3$. Then $n \geq 7$. Observe that in $H(n-1, 0)$ there are two vertices of degree 3, namely 0 and i. If we want to complete $H(n-1, 0)$ to G', we must join one of $0, i$ to a vertex of degree one. Because $\mathrm{ecc}_H(i) \leq n - 2$, joining i to any vertex will produce a cycle C_k for $k < n$, a contradiction. Hence we must join 0 to a vertex of degree one. The edges $\{n-1, 0\}, \{0, 0'\}, \{0, 0''\}$ all belong to G and $\{0, i'\}$ produces a cycle C_k for $k < n$, a contradiction.

Now suppose $i = 2$, so that $n \geq 5$. Then in $H(n-1, 0)$ we need to join one of the vertices of degree three, namely 0 or 2, to a vertex of degree one. All edges $\{n-1, 0\}, \{0, 0'\}, \{0, 0''\}$ belong to G while adding $\{0, 2'\}$ produces C_4, which is not allowed. Joining 2 to $0'$ or $0''$ produces C_4, joining 2 to $n-1$ produces C_{n-2}, and $\{2, 2'\}$ belongs to G. Therefore, the edge $\{n-1, 0\}$ has no replacement.

Finally, we need to show that i cannot be equal to one. Suppose that $i = 1$, so $n \geq 3$. Then $H(0, 1)$ has one vertex of degree three, namely 0, and all other vertices of degree less than three. To complete $H(0, 1)$ to G', we need to join 0 to a vertex of degree two other than 1. This produces C_k for $k < n$ or a multiple edge, a contradiction. □

Claim 2 *Suppose $|S_k| = 0$ for $k > 3$, $|S_3| = |S_2| = 1, |S_1| = n-2$, $t_0 = 3$, $t_i = 2$, and $\Delta(G) = 3$. Then G is isomorphic to one of the following three graphs: $n = 3$ and $i = 1$, or $n = 5$ and $i \in \{1, 2\}$.*

Proof: As in Claim 1, we can say $n > 2i$ and $\mathrm{dist}_G(0, i) = i$.

Assume $i \geq 4$ and observe that to complete $H(0, 1)$ to G', we need to join a vertex of degree one, namely i', 1, or $0'$, to a vertex of degree two to obtain C_n in G'. The only vertex at distance $n-1$ from 1 is 0, but $\{0, 1\} \in E(G)$. At the same time, $\mathrm{ecc}_H(i') = \max\{n-i+3, i\} = n-i+3$. For $i \geq 5$, this is less than $n-1$, hence adding an edge adjacent to i' produces C_k for $k < n$. For $i = 4$ the unique vertex at distance $n-1$ from i' is $0'$, which is not of degree two. Finally, consider $0'$. The only possibility here is to join $0'$ to 3, but then $\mathrm{dist}_{G'}(T_0, T_i) = i-3$, which is not allowed.

If $i = 3$, we proceed similarly and investigate $H(0, 1)$. We need to join one of $3', 1, 0'$ to a vertex of degree two to close C_n. The only vertex at distance $n-1$ from 1 is 0, but $\{0, 1\} \in E(G)$. The only vertex at distance $n-1$ from $0'$ is 3, which is of degree three. The only vertex at distance $n-1$ from $3'$ is 0^*, and in the graph G' arising from $H(0, 1)$ by adding the edge $\{3', 0^*\}$ we have $\mathrm{dist}_{G'}(T_0, T_i) = 2 < \mathrm{dist}_G(T_0, T_i) = i = 3$. This is impossible.

Therefore, $i < 3$. Suppose $i = 2$. One can check that if $n = 5$, then G is 1-swappable. If $n > 5$, look at $H(1, 2)$. We again need to add an edge joining a vertex of degree one, namely $2', 1$, or $0'$, to a vertex of degree two. The only vertex at distance $n-1$ from 1 is 2, but $\{1, 2\} \in E(G)$. The only vertex at distance $n-1$ from $2'$ is 0, but it is of degree three. The only vertex at distance $n-1$ from $0'$ is 3, but by adding the edge $\{0', 3\}$ to $H(1, 2)$ we obtain G' in which $\mathrm{dist}_{G'}(T_0, T_2) = \min\{3, n-3\}$. Because $\mathrm{dist}_G(T_0, T_i) = i = 2$, we must have $n-3 = 2$ and $n = 5$, which contradicts our assumption that $n > 5$.

Finally, suppose $i = 1$. For $n = 3$, G is 1-swappable. If $n = 4$, then $H(n-1, 0)$ cannot be completed, so G with these parameters is not 1-swappable.

For $n = 5$, G is again 1-swappable. For $n \geq 6$, we check $H(n-1, 0)$ and try to join one of the vertices of degree one, namely $0', 1', n-1$, to a vertex of degree two at distance $n-1$. There is no such vertex for $1'$, since the only vertex at distance $n-1$ is $n-1$. The only vertex at distance $n-1$ from $0'$ is $n-3$. Adding the edge $\{0', n-3\}$ we obtain G'. But then $\mathrm{dist}_{G'}(T_0, T_1) = \min\{4, n-4\}$, which must equal $\mathrm{dist}_G(T_0, T_i) = i = 1$. So we must have $n = 5$, which contradicts our assumption that $n \geq 6$. The only vertex of degree 2 at distance $n-1$ from $n-1$ is 0, and because $\{n-1, 0\} \in E(G)$, this edge cannot be added. This completes the proof. □

Claim 3 *Suppose* $|S_k| = 0$ *for* $k > 2$ *and* $|S_2| \geq 2$. *Then* $n = 2m$ *for some* $m \geq 3$, *and* G *is isomorphic to the unicyclic graph with* $t_j = 2$ *for* $j \in \{4, 6, \ldots, 2m\}$ *and* $t_j = 1$ *otherwise.*

Proof: We prove this Claim in four steps.

Subclaim 1. Suppose $t_n = t_j = 2$ and $t_1 = t_2 = \cdots = t_{j-1} = 1$. Then $j \leq 4$.

Assume that $j > 4$ and look at $H(2,3)$. We need to join two vertices of degree one to obtain C_n in G'. However, for $i' \in \{n', j'\}$ we have $\mathrm{ecc}_H(i') < n-1$ and hence neither of them can be used to produce C_n. Thus we are left only with 2 and 3, but $\{2,3\} \in E(G)$. Therefore, $j \leq 4$.

Subclaim 2. Suppose $t_n = t_1 = 2$. Then G is not 1-swappable.

In $H(n,1)$ the number of vertices of degree one is the same as in G. Therefore, we need to join two vertices of degree greater than one to produce C_n. However, the only such pair of vertices with eccentricities at least $n-1$ is $n, 1$, and $\{n,1\} \in E(G)$. Hence, G is not 1-swappable.

Subclaim 3. Suppose $t_n = t_3 = 2$ and $t_1 = t_2 = 1$. Then G is not 1-swappable.

In $H(2,3)$ the number of vertices of degree one is one more than in G. Therefore, we need to join a vertex of degree one to a vertex of degree two to produce C_n. There are only two vertices of degree two with eccentricity at least $n-1$ in $H(2,3)$, namely 1 and 3. Vertex 3 would have to be joined back to 2, which is not allowed. The only vertex at distance $n-1$ from 1 in $H(2,3)$ is $3'$, but adding the edge $\{1,3'\}$ would produce two neighboring vertices of degree three in G'. By Subclaim 2, G' cannot contain such vertices, since $G' \cong G$.

Subclaim 4. Suppose $t_n = t_4 = 2$, $t_1 = t_2 = t_3 = 1$, and there exists $j \geq 4$ such that $t_{j+1} = t_{j+2} = t_{j+3} = 1$. Then G is not 1-swappable.

By Subclaim 1 we have $t_{j+4} = 2 = t_j$. We can assume without loss of generality that these two segments of three consecutive vertices of degree two are "closest" to each other in G in the sense that if there are $k \neq s$ such that $t_k = t_{k+4} = 2, t_{k+1} = t_{k+2} = t_{k+3} = 1$ and $t_s = t_{s+4} = 2, t_{s+1} = t_{s+2} = t_{s+3} = 1$, then $\mathrm{dist}_G(k,s) \geq j$.

As before, in $H(3,4)$ the number of vertices of degree one is one more than in G. Therefore, we need to join a vertex of degree one to a vertex of degree two to produce C_n. There are only two vertices of degree two with eccentricity at least $n-1$ in $H(3,4)$, namely 2 and 4. Vertex 4 would have to be joined back to 3, which is not allowed. The only vertex at distance $n-1$ from 2 in $H(3,4)$ is $4'$. If $j = 4$, then adding the edge $\{2,4'\}$ would produce five consecutive vertices of degree two in G', namely $4', 4, 5, 6, 7$, which is impossible by Subclaim 1. So $j > 4$, and by Subclaim 2 we have $j \geq 6$. But then we have two segments of three consecutive vertices of degree two at distance $j-2$ in G', namely $4', 4, 5$ and $j+1, j+2, j+3$. This contradicts our choice of j as the minimum distance between them.

Therefore, we can conclude that there is only one segment of three consecutive vertices of degree two, and Claim 3 follows immediately. □

Claim 4 *If* $|S_k| = 0$ *for* $k > 2$ *and* $|S_2| = 1$*, then* $n \in \{3, 4\}$.

Proof: We may suppose that $t_0 = 2$. Assume for a contradiction that $n \geq 5$, and consider $H(2, 3)$. We must join two vertices of degree one to produce G'. The three vertices of degree one in $H(2, 3)$ are $0'$, 2, and 3, and we have $\mathrm{ecc}_H(0') = \max\{3, n-2\} < n-1$. So the only choice is to join 2 with 3; but $\{2, 3\} \in E(G)$, a contradiction. □

3 Asymptotic Growth of the Swapping Number

In this section, we give a linear upper bound on the swapping number of a graph in terms of the order of its vertex set, and then we show that the average swapping number comes close to this worst-case bound, in the sense that it grows linearly.

Lemma 18 *Let* $G = (V, E)$ *be a graph on* n *vertices* ($n \geq 5$)*, not isomorphic to* K_n*. Then the swapping number* k *of* G *satisfies* $k \leq 2n - 7$.

Proof: Let $e = \{x, y\}$ be an edge of G.

Case 1: $\deg(x) < n-1$ or $\deg(y) < n-1$. Say without loss of generality that $\deg(x) < n-1$, and let a be a vertex not adjacent to x. Let $\sigma : V \to V$ be the permutation (y, a) switching y with a. The discrepancy of σ is $|\sigma(E) - E| = |E - \sigma(E)| = |\{q \in V : q \text{ is adjacent to exactly one of } a \text{ or } y\}| \leq n - 2$.

Case 2: $\deg(x) = \deg(y) = n - 1$. Let $\{a, b\} \in \binom{V}{2} - E$. Let $\sigma : V \to V$ be the permutation $(x, a)(y, b)$ switching x with a and y with b. Let $Q = V - \{a, b, x, y\}$. The only edge in $E - \sigma(E)$ contained in $\{a, b, x, y\}$ is $\{x, y\}$. All other edges in $E - \sigma(E)$ meet $\{a, b, x, y\}$ in one vertex, and the discrepancy of σ is $1 + |\{q \in Q : q \text{ is not adjacent to } a\}| + |\{q \in Q : q \text{ is not adjacent to } b\}| \leq 1 + 2|Q| = 2n - 7$. □

We note that this bound is achieved when G is the complete tripartite graph $K_{1,1,r}$ ($r > 1$).

Theorem 19 *There is a positive constant* ε *such that* $\Pr(\mathrm{sw}(G_n) > \varepsilon n \text{ or } \mathrm{sw}(G_n) \text{ does not exist}) \to 1$ *as* $n \to \infty$*, where* G_n *is a graph chosen uniformly at random from the set of all graphs on* n *vertices* $\{1, 2, \ldots, n\}$*,* $\mathrm{sw}(G)$ *is the swapping number of* G*, and* $\Pr$ *is the probability function.*

Proof: Fix a positive integer n and set $V = [n] = \{1, 2, \ldots, n\}$. Let k be a positive integer, $k \leq \binom{n}{2}/2$. Let σ be a permutation of V. Let us count the number of graphs G with $V(G) = V$ for which σ has discrepancy k.

Form a directed graph Γ (with loops allowed) as follows. Set $V(\Gamma) = \binom{[n]}{2}$, the set of all potential edges of G. Note that σ acts on $V(\Gamma)$ in a natural way, namely $\sigma(\{a, b\}) = \{\sigma(a), \sigma(b)\}$. Set $E(\Gamma) = \{(e, \sigma(e)) : e \in V(\Gamma)\}$. Thus, Γ is a collection of disjoint cycles, namely, the orbits of σ on $\binom{[n]}{2}$.

Let $\Gamma_1, \ldots, \Gamma_c$ be the components of Γ, and choose a representative $m_i \in V(\Gamma_i)$ for each i. Let f be the number of fixed points of σ acting on V. Let c_1 be the number of components of Γ of size 1. Note that $c_1 = \binom{f}{2} + t$, where t is the number of 2-cycles of σ acting on V. We have $t \le (n-f)/2$ and $c \le c_1 + \left(\binom{n}{2} - c_1\right)/2 \le \left(\binom{f}{2} + \binom{n}{2} + \frac{n-f}{2}\right)/2$. Writing $r = n - f$ and simplifying, we find $c \le \binom{n}{2} - \frac{r(n-2)}{4}$.

We next designate the elements of the symmetric difference $S = (E - \sigma(E)) \cup (\sigma(E) - E)$, by selecting a set of $2k$ vertices of Γ. The only restriction is that there must be an *even* number of elements of S in each component of Γ. In particular, we cannot choose elements of S to be edges both of whose endpoints are fixed by σ, and so the number of choices for S is at most $\binom{\binom{n}{2}-\binom{f}{2}}{2k}$. Since $\binom{n}{2} - \binom{f}{2} = \frac{r(2n-r-1)}{2} < nr$, the number of choices for S is at most $\binom{nr}{2k}$.

Finally, we determine G completely by specifying, for each i, whether or not m_i is an edge of G. The number of choices here is 2^c.

Therefore, the number of graphs G for which σ has discrepancy k is at most

$$2^{\binom{n}{2} - \frac{r(n-2)}{4}} \binom{nr}{2k}.$$

The number of permutations of V with f fixed points is at most $\binom{n}{f}(n-f)!$ (first choose the f fixed points, then choose a permutation of the remaining $n-f$ points) $= \binom{n}{r} r! = \frac{n!}{(n-r)!} \le n^r$. Let $g(n,k)$ denote the total number of graphs G such that $V(G) = V$ and G has swapping number k. Summing over all σ, we find

$$g(n,k) \le \sum_{r=2}^{n} n^r \cdot 2^{\binom{n}{2} - \frac{r(n-2)}{4}} \binom{nr}{2k} = 2^{\binom{n}{2}} \sum_{r=2}^{n} \left(2^{\frac{1}{2} + \log_2(n) - \frac{n}{4}}\right)^r \binom{nr}{2k}.$$

For n large enough, we have $\frac{1}{2} + \log_2(n) - \frac{n}{4} < -\frac{n}{5}$, so we can write

$$g(n,k) \le 2^{\binom{n}{2}} \sum_{r=2}^{n} 2^{-rn/5} \binom{nr}{2k}.$$

From Stirling's formula, $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$, we deduce

$$\binom{nr}{\varepsilon nr} \le \left(\varepsilon^{-\varepsilon}(1-\varepsilon)^{\varepsilon-1}\right)^{nr}$$

for any fixed ε with $0 \le \varepsilon < 0.5$, $r \ge 2$, and n large enough. Since

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-\varepsilon}(1-\varepsilon)^{\varepsilon-1} = 1,$$

we can choose an ε small enough that $\varepsilon^{-\varepsilon}(1-\varepsilon)^{\varepsilon-1} \le 2^{1/10}$ and $\varepsilon < 0.1$.

Let $\mathfrak{G}(n,k)$ denote the number of all graphs G such that $V(G) = V$ and G has swapping number *at most* k. Fix a k with $k \le \varepsilon n$. Then we have $\mathfrak{G}(n,k) = \sum_{\ell=1}^{k} g(n,\ell)$. If $1 \le \ell \le k$, then

$$\binom{nr}{2\ell} \Big/ \binom{nr}{2(\ell-1)} = \frac{(nr-2\ell+2)(nr-2\ell+1)}{2\ell(2\ell-1)} = \frac{nr-2\ell+2}{2\ell}\left(\frac{nr}{2\ell-1} - 1\right)$$

$$> \frac{nr-2\ell}{2\ell}\left(\frac{nr}{2\ell}-1\right) = \left(\frac{nr-2\ell}{2\ell}\right)^2 \geq \left(\frac{1-\varepsilon}{\varepsilon}\right)^2 > \left(\frac{1-0.1}{0.1}\right)^2 = 81.$$

So we have

$$\sum_{\ell=1}^{k}\binom{nr}{2\ell} \leq \sum_{q=0}^{k-1}\binom{nr}{2k}81^{-q} < \binom{nr}{2k}/\left(1-1/81\right) = \frac{81}{80}\binom{nr}{2k}.$$

Therefore,

$$\begin{aligned} \mathfrak{G}\left(n,k\right) &\leq 2^{\binom{n}{2}}\sum_{r=2}^{n}2^{-rn/5}\frac{81}{80}\binom{nr}{2k} \leq 2^{\binom{n}{2}}\sum_{r=2}^{n}2^{-rn/5}\frac{81}{80}2^{nr/10} \\ &= \frac{81}{80}2^{\binom{n}{2}}\sum_{r=2}^{n}2^{-rn/10} < \frac{81}{80}2^{\binom{n}{2}}\frac{2^{-2n/10}}{1-2^{-n/10}}. \end{aligned}$$

Thus,

$$\lim_{n\to\infty}\mathfrak{G}\left(n,k\right)/2^{\binom{n}{2}} = 0,$$

which completes the proof. □

4 Conclusion

The classification of k-swappable trees for $k \geq 2$ requires further study. In addition, we are unaware of any infinite families of 1-swappable graphs other than those described by Example 3 and Case (i) of Theorem 17.

Until now, the prevailing definitions of network reliability have focussed on preserving connectivity of the network, and not on preserving its logical structure [1]. By changing this focus, a possible application of 1-swappable graphs to network design is as follows. Assuming that the physical location of nodes in a network is immaterial, but that maintaining the logical topology of the network is critical, and that connections between nodes are expensive to add or remove, then a 1-swappable graph gives an optimal design for the problem of correcting for an arbitrary faulty connection. This is because a 1-swappable graph provides a network topology in which a faulty connection may be replaced by a single new connection (as the faulty one is being repaired, say) to produce an equivalent logical topology.

It is possible to define several natural variations of the notion of swappability. To define *vertex swappability*, we require that for every $v \in V(G)$, there is an element $w \notin V(G)$ and an isomorphism $G \to G' = (V', E')$, where $V' = (V(G) - \{v\}) \cup \{w\}$, $E(G-v) \subseteq E'$, and the neighborhoods $N_G(v)$ and $N_{G'}(w)$ are distinct. In view of this definition, we use the term *edge-swappable* to refer to the notion which we have employed in the preceding sections of this paper.

Stronger than vertex swappability but weaker than 1-edge-swappability is the condition that for every $v \in V(G)$, there is an edge $e \in E(G)$ meeting v such that e can be replaced by a non-edge of G to produce a graph isomorphic to G. We may term this condition *weak edge-swappability*.

To strengthen the notion of edge-swappability, we define G to be r*-fold* 1-edge-swappable if for every $e \in E(G)$, there are at least r distinct elements $e' \in \binom{V}{2} - E(G)$ such that e' is swappable with e. A somewhat trivial example of an r-fold 1-edge-swappable graph is the graph on n vertices with only one edge, where we may take $r = \binom{n}{2} - 1$.

Finally[1], we mention a notion of 1-swappability for the action of a group Γ on a set S with a given weight function $w : S \to \{0, 1, 2, \dots\}$. Namely, the weighting w will be called *1-swappable* with respect to this group action if for every $x \in S$ with $w(x) > 0$, there is a way to transfer a single unit of weight away from x to another element $y \in S$ so that the new weighting w' is obtained from the original weighting w via an element of Γ: that is, there exists $\sigma \in \Gamma$ such that $w(v) = w'(\sigma(v))$ for every $v \in S$. This notion already appears implicitly in our treatment of isomorphisms of vertex-weighted cycles: in light of Corollary 10, the results of Lemma 13 through Corollary 16 may be interpreted as statements about the possible weight functions on $\mathbf{Z}/n\mathbf{Z}$ which make this set 1-swappable under the action of the dihedral group D_{2n}. For example, after shifting down the t_i's to weights $w(i) = t_i - 1$, Corollary 16 translates to the following statement.

Corollary 20 *If w is a* 1*-swappable weighting of $\mathbf{Z}/n\mathbf{Z}$ with respect to D_{2n}, then $w(x) \le 2$ for all $x \in \mathbf{Z}/n\mathbf{Z}$; and if $w(x) = 2$ for some x, then this x is unique, and there is a unique y such that $w(y) = 1$.*

These results are not enough to completely characterize the 1-swappable weightings of $\mathbf{Z}/n\mathbf{Z}$ with respect to D_{2n}, and further work is needed to realize this goal.

Furthermore, the whole notion of 1-edge-swappability is the special case of a 1-swappable weighting where $\Gamma = \mathrm{Sym}(V)$ acts on the set $\binom{V}{2}$ in the natural way, with each edge $e \in E(G)$ having weight 1 and each non-edge having weight 0.

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