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On Boole's formula for factorials

Horst Alzer

Morsbacher Str. 10
51545 Waldbröl
Germany
h.alzer@gmx.de

Robin Chapman

Mathematics
University of Exeter
Exeter EX4 4QF
U.K.
R.J.Chapman@exeter.ac.uk

Abstract

We present a simple new proof and a new generalization of Boole's formula

$$n! = \sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} j^n \quad (n \in \mathbf{N}).$$

1 Introduction

The elegant formula

$$n! = \sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} j^n \quad (n \in \mathbf{N}) \tag{1.1}$$

is given in Boole's classical book "Calculus of Finite Differences" [3, p.5, p.19]. In 2005, Anglani and Barile [2] used methods from real analysis and combinatorics to provide two proofs. An interesting extension of Boole's identity was published in 2008 by Pohoata [3]. He applied Lagrange's interpolating polynomial theorem to establish

$$a_0 \cdot b^n \cdot n! = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} P(a+jb) \quad (n \in \mathbf{N}),$$

where a and b are real numbers with $b \neq 0$ and P is a real polynomial of degree n with leading coefficient a_0. The special case $a = 0$, $b = 1$, $P(x) = x^n$ leads to (1.1).

The aim of this note is twofold. In Section 2, we present a simple new proof of (1.1), and in Section 3, we offer a new generalization of (1.1).

2 A new proof

Here, we apply the method of induction to obtain a short and elementary proof of Boole's identity. We need the following well-known formulas:

$$j\binom{n+1}{j} = (n+1)\binom{n}{j-1}, \tag{2.1}$$

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}, \tag{2.2}$$

$$\sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}j^{\nu} = 0 \quad (\nu = 0, 1, ..., n). \tag{2.3}$$

Using the formula

$$\sum_{j=0}^{n+2}(-1)^{n+2-j}\binom{n+2}{j}j^{\nu} = (n+2)\sum_{k=0}^{\nu-1}\binom{\nu-1}{k}\sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}j^{k} \quad (\nu = 1, 2, ..., n+1)$$

it follows by induction on n that (2.3) is valid; see also [1, chapter 2.4].

Proof of (1.1). If $n = 1$, then both sides of (1.1) are equal to 1. Next, we assume that (1.1) holds. Applying (2.1), (2.2), (2.3) with $\nu = n$, and the induction hypothesis we obtain

$$\sum_{j=1}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}j^{n+1} = (n+1)\sum_{j=1}^{n+1}(-1)^{n+1-j}\binom{n}{j-1}j^{n}$$

$$= (n+1)\sum_{j=1}^{n+1}(-1)^{n+1-j}\left[\binom{n+1}{j} - \binom{n}{j}\right]j^{n} = (n+1)\left[0 + \sum_{j=1}^{n}(-1)^{n-j}\binom{n}{j}j^{n}\right]$$

$$= (n+1)\cdot n! = (n+1)!.$$

This reveals that (1.1) is valid if we replace n by $n+1$.

3 A new generalization

We prove the following

Theorem. *Let* $\big(f(j)\big)_{j=0}^{\infty}$ *be a sequence of complex numbers with* $f(0)=1$. *Then,*

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{\substack{c_1+\cdots+c_j=m\\ c_1,\ldots,c_j\geq 0}} f(c_1)\cdots f(c_j)=\sum_{\substack{d_1+\cdots+d_n=m\\ d_1,\ldots,d_n\geq 1}} f(d_1)\cdots f(d_n)\quad (m,n\in\mathbf{N}). \tag{3.1}$$

Proof. We define the formal power series

$$F(X)=\sum_{j=0}^{\infty} f(j)X^j.$$

On the one hand we have

$$\begin{aligned}\big(F(X)-1\big)^n &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}F(X)^j\\ &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{c_1,\ldots,c_j=0}^{\infty} f(c_1)\cdots f(c_j)X^{c_1+\cdots+c_j}\\ &= \sum_{m=0}^{\infty}X^m\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{\substack{c_1+\cdots+c_j=m\\ c_1,\ldots,c_j\geq 0}} f(c_1)\cdots f(c_j).\end{aligned}$$

On the other hand we obtain

$$\big(F(X)-1\big)^n=\Big(\sum_{k=1}^{\infty}f(k)X^k\Big)^n=\sum_{m=n}^{\infty}X^m\sum_{\substack{d_1+\cdots+d_n=m\\ d_1,\ldots,d_n\geq 1}} f(d_1)\cdots f(d_n).$$

Comparing the coefficients of X^m gives (3.1).

Remark

The special case $m=n$ leads to

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\sum_{\substack{c_1+\cdots+c_j=n\\ c_1,\ldots,c_j\geq 0}} f(c_1)\cdots f(c_j)=f(1)^n. \tag{3.2}$$

Since

$$\sum_{\substack{c_1+\cdots+c_j=n\\ c_1,\ldots,c_j\geq 0}}\frac{1}{c_1!\cdots c_j!}=\frac{j^n}{n!},$$

we conclude that formula (3.2) with $f(c)=1/c!$ implies (1.1).

References

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