Matching Extensions of Strongly Regular Graphs

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Abstract: Let β be the number of vertices commonly adjacent to any pair of non-adjacent vertices. It is proved that every strongly regular graph with even order and $\beta \geq 1$ is 1-extendable. We also show that every strongly regular graph of degree at least 3 and cyclic edge connectivity at least $3k-3$ is 2-extendable. Strongly regular graphs of even order and of degree k at least 3 with $\beta \geq \frac{k}{3}$ are 2-extendable, except the Petersen graph and one other graph.

1. Introduction and terminology

All graphs considered are finite, undirected, connected and simple.

A graph G is called <u>strongly regular</u> if G is k-regular and there are two integers $\alpha, \beta \geq 0$ such that for each pair of vertices u and v, $u \neq v$, the number of the common neighbours of u and v is α or β according as u and v are adjacent or non-adjacent. A strongly regular graph with v vertices is called a (v, k, α, β)-graph. These general parameters will be assumed unless stated otherwise.

A graph G is called <u>cyclically m-edge-connected</u> if $|S| \geq m$ for each edge cutset S of G such that there are two components in $G - S$ each of which has a cycle. The set S is called a cyclic edge cutset. The size of a minimal cyclic edge cutset is called the <u>cyclic edge connectivity</u>, and is denoted by $c\lambda(G)$.

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Suppose G has a perfect matching. A graph G is called n-extendable if for the given integer $n \leq (\upsilon-2)/2$, G has n independent edges and any n independent edges are contained in a perfect matching of G.

In [1], the n-extendability of edge (vertex) transitive graphs is discussed. When the cyclic edge connectivity is large enough, an edge transitive graph is n-extendable. We show here that there is a similar relation between cyclic edge connectivity and n-extendability in strongly regular graphs. We also find some n-extendable strongly regular graphs for arbitrary n.

All terminology and notation not defined in this paper can be found in [2].
Reference [4] provides a strong background for matching theory and [6] gives a survey of results in strongly regular graphs.

Lemma 1 If G is a strongly regular graph, then $k(k-\alpha-1) = (\upsilon-k-1)\beta$.

Proof See Theorem 2.2 in [3]. ❑

If G is a cubic strongly regular graph, then by Lemma 1, $3(2-\alpha) = (\upsilon-4)\beta$. For $\alpha = 2$, we find $G = K_4$. If $\alpha = 1$, then υ is odd which is not possible since G is cubic. If $\alpha = 0$, then $\beta = 1$ and $\upsilon = 10$ or $\beta = 3$ and $\upsilon = 6$. For $\beta = 1$, we obtain the Petersen graph P, while for $\beta = 3$ we get $K_{3,3}$.

Lemma 2 Let G be a graph with even order. If $\delta(G) \geq \frac{\upsilon}{2} + n$, then G is n-extendable.

Proof See [5]. ❑

2. Matching of strongly regular graphs

In this section, we show that every connected strongly regular graph with even order has a perfect matching. Furthermore, every edge of such a graph lies in a perfect matching.

Let G be a strongly regular graph with even order and $\beta = 0$. By Lemma 1, $\alpha = k-1$. Let u,v be adjacent in G. Then we may suppose that u,v are both adjacent to $w_1, w_2, \ldots, w_{k-1}$. Let $A = \{u,v,w_1,w_2,\ldots,w_{k-1}\}$. If w_i,w_j are not adjacent then they are both adjacent to u and v. This contradicts the fact that $\beta = 0$. Hence $G[A] \cong K_{k+1}$, where $G[A]$ is the graph induced by the vertex set A.

Let $x \in V(G) - A$. Now x cannot be adjacent to any vertex in A since each of these vertices has degree k already. Hence x must be in another component isomorphic to K_{k+1}.

So $G \cong rK_{k+1}$. Such graphs are 1- and 2-extendable if and only if k is odd. We therefore assume that $\beta \geq 1$ in the rest of this paper.

Let G be a strongly regular graph with even order and let $C_1, C_2, \ldots, C_t$ be the components of $G - S$.

Lemma 3 Each vertex of C_i sends at least β edges to S $(i=1,2,\ldots,t)$.

Proof Let u be a vertex of C_i and v be a vertex of C_j $(i \neq j)$. Since u and v are non-adjacent, u and v have β common neighbours. Those neighbours can only be in S. So every vertex of C_i sends at least β edges to S. ❑

Lemma 4 There are at least k edges from C_i to S $(i=1,2, \ldots, t)$.

Proof Let m be the number of vertices of C_i. Let γ be the minimum number of edges from a vertex of C_i to S. By Lemma 3, $\gamma \geq \beta \geq 1$. There are at least $m\gamma$ edges from C_i to S.

Suppose $m\gamma < k$. Then $m < k/\gamma$. Suppose $m \geq k - \gamma + 1$. Then $k/\gamma > k - \gamma + 1$. Hence $k\gamma - \gamma^2 + \gamma < k$. So $\gamma^2 - k\gamma + k - \gamma > 0$ and $(\gamma-k)(\gamma-1) > 0$.

But $\gamma \leq k$ and $\gamma \geq 1$. This contradiction shows that $m < k - \gamma + 1$. But now each vertex u of C_i is adjacent to at most $k - \gamma - 1$

vertices in C_i. So there are at least $\gamma + 1$ edges from u to S by the k-regularity. This contradicts the assumption on the minimality of γ. So $m\gamma \geq k$. ❑

Theorem 1 Every strongly regular graph of even order ($\beta \geq 1$) has a perfect matching.

Proof Assume that G has no perfect matching, that $S \subseteq V(G)$ such that $o(G-S) > |S|$, where $o(G-S)$ is the number of odd component of G-S, and that $C_1, C_2, \ldots, C_t$ are the components of G-S.

By the k-regularity of G, S accepts at most $k|S|$ edges from $C_1, C_2, \ldots, C_t$. By Lemma 4, $C_1, C_2, \ldots, C_t$ send at least $kt > k|S|$ edges to S. This is a contradiction. ❑

Theorem 2 Every strongly regular graph G with even order is 1-extendable ($\beta \geq 1$).

Proof Suppose G is not 1-extendable. There is an edge e = uv such that G - {u,v} does not have a perfect matching. Let G' = G - {u,v}. By Tutte's Theorem, there is a set $S' \subseteq V(G')$ such that $o(G'-S') > |S'|$. By parity, $o(G'-S') \geq |S'| + 2$. Let $S = S' \cup \{u,v\}$. $o(G-S) = o(G'-S') \geq |S'| + 2 = |S|$. By Theorem 1, $o(G-S) \leq |S|$. So $o(G-S) = |S|$. Let $C_1, C_2, \ldots, C_{|S|}$ be the odd components of G - S.

By k-regularity, S can accept at most $k|S| - 2$ edges from $C_1, C_2, \ldots, C_{|S|}$. By Lemma 4, there are at least $k|S|$ edges from $C_1, C_2, \ldots, C_{|S|}$ to S. This is a contradiction. ❑

Not every strongly regular graph is 2-extendable. The Petersen graph is a counterexample.

3. Relation between cyclic edge connectivity and 2-extendability

In [1], it was proved that an edge (vertex) transitive graph is n-extendable when the cyclic edge connectivity is large enough. For strongly regular graphs, we have a similar relation between cyclic edge connectivity and 2-extendability.

Theorem 3 Let G be a strongly regular graph with even order and $k \geq 4$. If $c\lambda(G) \geq 3k - 3$, G is 2-extendable.

Proof Suppose G is not 2-extendable. There are two edges $e_i = u_iv_i$ $(i=1,2)$ such that $G - \{u_1,v_1,u_2,v_2\}$ does not have a perfect matching. Let $G' = G - \{u_1,v_1,u_2,v_2\}$. By Tutte's Theorem, there is a set $S' \subseteq V(G')$ such that $o(G'-S') > |S'|$. By parity, $o(G'-S') = |S'| + 2m$ $(m \geq 1)$. Let
$S = S' \cup \{u_1,v_1,u_2,v_2\}$. $o(G-S) = o(G'-S') = |S'| + 2m = |S| - 4 + 2m$. By Theorem 1, $o(G-S) \leq |S|$. So $1 \leq m \leq 2$.

If $o(G-S) = |S|$, there are at least $k|S|$ edges from the components of $G - S$ to S by Lemma 4. By the k-regularity, S can accept at most $k|S| - 4$ edges, a contradiction. So $o(G-S) = |S| - 2$ and $m = 1$. Let $C_1, C_2, \ldots, C_{|S|-2}$ be the odd components of $G - S$.

Let N be the number of edges from the components of $G - S$ to S. By k-regularity, $N \leq k|S| - 4$. By Lemma 4, $N \geq k(|S|-2)$. So there are at most $k + k|S| - 4 - k(|S|-2) = 3k - 4$ edges from a component of $G - S$ to S. Hence every component of $G - S$ is a tree or else the fact the $c\lambda(G) \geq 3k-3$ is contradicted.

Claim 1 Every component of $G - S$ has order at most three.

Let b be the order of a component C of $G - S$. But C is a tree. So $kb - 2(b-1) \leq 3k - 4$. Hence $(k-2)\, b \leq 3k - 6$. So $b \leq 3$.

Claim 2 $\alpha = 0$ and so no triangle exists.

If there is a triangle T, and the edge cut $(T, G-T)$ is a cyclic edge cutset, it has size $3k - 6$, contradicting $c\lambda(G) \geq 3k - 3$. Suppose $(T, G-T)$ is not a cyclic edge cutset. Let c be the order of $G - T$. As $G - T$ is a forest, $kc - 2(c-1) \leq 3k - 6$. So $c < 3$. By hypothesis G has even order. So G has order four. But this is not possible since $k \geq 4$.

We now show that $G-S$ contains at least three singletons.

(1) If there is a component C of $C_1, C_2, \ldots, C_{|S|-2}$ with order three, then there are at least three singleton components of $G - S$.

Without loss of generality, assume C_1 has order three. As C_1 is a tree, there are $k \times 3 - 2 \times (3-1) = 3k - 4$ edges from C_1 to S. By counting the edges from the components of G - S to S, there are exactly k edges from each of $C_2, C_3, \ldots, C_{|S|-2}$ to S. But $C_2, C_3, \ldots, C_{|S|-2}$ are trees. So, since $k \geq 4$, $C_2, C_3, \ldots, C_{|S|-2}$ are singletons. As there are $k|S| - 4$ edges from $C_1, C_2, \ldots, C_{|S|-2}$ to S, G[S] has exactly two edges.

If there are at most two singleton components of G - S the number of odd components of G - S is at most three and $|S| \leq 5$. Now G[S] contains two independent edges and C_2 is a singleton. As there are at least four edges from C_2 to S, there is always a triangle containing C_2, contradicting Claim 2.

(2) If each of $C_1, C_2, \ldots, C_{|S|-2}$ is a singleton, then there are at least three singleton components of G - S.

Since $|S| \geq 4$, $o(G-S) = |S| - 2 \geq 2$. Suppose $|S| - 2 = 2$. Then $|S| = 4$ and S contains two independent edges. But there are at least three edges from C_1 to S. So there is a triangle, contradicting Claim 2.

We may therefore assume that v_1, v_2, v_3 are singletons in o(G-S).

Let $r_1, r_2, \ldots, r_k$ be the neighbours of v_1 and $s_1, s_2, \ldots, s_k$ be the neighbours of v_2. By the definition of strongly regular graphs, as v_1 and v_2 are non-adjacent, $|\{r_1, r_2, \ldots, r_k\} \cap \{s_1, s_2, \ldots, s_k\}| = \beta$. Without loss of generality, assume $\{r_1, r_2, \ldots, r_k\} \cap \{s_1, s_2, \ldots, s_k\} = \{r_1, r_2, \ldots, r_\beta\} = \{s_1, s_2, \ldots, s_\beta\}$. Now $r_{\beta+1}, \ldots, r_k$ are not adjacent to v_2, so each of $r_{\beta+1}, \ldots, r_k$ sends β edges to $s_{\beta+1}, \ldots, s_k$. There are $(k-\beta)\beta$ edges from $\{r_{\beta+1}, \ldots, r_k\}$ to $\{s_{\beta+1}, \ldots, s_k\}$. When $\beta = k$ and $\alpha = 0$, G is $K_{k,k}$, and hence is 2-extendable. For $1 \leq \beta \leq k-1$, the quadratic $(k-\beta)\beta$ is greater than or equal to $k-1$.

Let $t_1, t_2, \ldots, t_k$ be the neighbours of v_3 in S. Now v_3 has β common neighbours with v_2.

Suppose $\{t_1, t_2, \ldots, t_k\} \cap \{s_1, s_2, \ldots, s_k\} = \{s_1, s_2, \ldots, s_\beta\} = \{t_1, t_2, \ldots, t_\beta\} = \{r_1, r_2, \ldots, r_\beta\}$.

Then $\{t_{\beta+1}, t_{\beta+2}, \ldots, t_k\} \cap \{s_{\beta+1}, s_{\beta+2}, \ldots, s_k\} = \emptyset$ and $\{t_{\beta+1}, t_{\beta+2}, \ldots, t_k\} \cap \{r_1, r_2, \ldots, r_k\} = \emptyset$. Otherwise, v_3 and v_2, or v_3 and v_1, have more than β common neighbours. None of $t_{\beta+1}, t_{\beta+2}, \ldots, t_k$ is adjacent to v_2. Hence each of $t_{\beta+1}, \ldots, t_k$ sends β edges to $s_{\beta+1}, s_{\beta+2}, \ldots, s_k$. There are $(k-\beta)\beta \geq k - 1$ edges from $\{t_{\beta+1}, \ldots, t_k\}$ to $\{s_{\beta+1}, \ldots, s_k\}$. So $G[S]$ has at least $2(k-1) \geq k + 1$ edges for $k \geq 4$. By Lemma 4, there are at least $k(|S|-2)$ edges from $C_1, C_2, \ldots, C_{|S|-2}$ to S. By k-regularity, S can accept at most $k|S| - 2(k+1) = k(|S|-2) - 2$ edges, a contradiction.

We may therefore suppose that $\{t_1, t_2, \ldots, t_k\} \cap \{s_1, s_2, \ldots, s_k\} \neq \{s_1, s_2, \ldots, s_\beta\}$.

Without loss of generality, assume $\{t_1, t_2, \ldots, t_k\} \cap \{s_1, \ldots, s_\beta\} = \{s_1, s_2, \ldots, s_i\} = \{t_1, t_2, \ldots, t_i\}$ $(i < \beta)$.

If $\{t_1, \ldots, t_k\} \setminus \{s_1, \ldots, s_k\}$ is contained in $\{r_1, \ldots, r_k\}$, as s_β is not adjacent to v_3, s_β sends β edges to $\{r_1, \ldots, r_k\}$. So there is a triangle containing v_1, a contradiction.

So there is a neighbour u of v_3 which is not in $\{r_1, \ldots, r_k\} \cup \{s_1, \ldots, s_k\}$.

Assume $\beta > 1$.

u is not adjacent to v_2 and sends β edges to $\{s_1, \ldots, s_k\}$. So $G[S]$ contains at least $(k-\beta)\beta + \beta \geq k - 1 + \beta \geq k - 1 + 2 = k + 1$ edges. By Lemma 4, there are at least $k(|S|-2)$ edges from $C_1, C_2, \ldots, C_{|S|-2}$ to S. By k-regularity, S can accept at most $k|S| - 2(k+1) = k(|S|-2) - 2$ edges, a contradiction.

So assume that $\beta = 1$.

v_3 is adjacent to one vertex in $\{r_1, r_2, \ldots, r_k\}$ and one vertex in $\{s_1, \ldots, s_k\}$. But $k \geq 4$ so v_3 is adjacent to at least two vertices which are not in $\{r_1, \ldots, r_k\} \cup \{s_1, \ldots, s_k\}$. Let u, v be such two vertices, u

and v are not adjacent to v_2. Both u and v send an edge to $\{s_1, \ldots, s_k\}$. So G[S] has at least $(k-\beta)\beta + 2 = k - 1 + 2 = k + 1$ edges. By Lemma 4, there are at least $k(|S|-2)$ edges from $C_1, \ldots, C_{|S|-2}$ to S. By k-regularity, S can accept at most $k|S| - 2(k+1) = k(|S|-2) - 2$ edges, a contradiction.
This contradiction proves the theorem. □

Corollary 1 Let G be a strongly regular graph with even order and $k \geq 3$. If $c\lambda(G) \geq 3k - 3$, G is 2-extendable.

Proof The cubic strongly regular graphs are K_4, $K_{3,3}$ and the Petersen graph. It is easy to verify the result holds for these graphs. □

Since the girth of a strongly regular graph is at most 5, $c\lambda(G) \leq 5(k-2) = 5k - 10$. If we were to try to prove the 3-extendability of stronger regular graphs by increasing the cyclic edge connectivity, we would need $c\lambda(G) \geq 5k - 6$. Hence it is necessary to look in another direction to find results of 3-extendable strongly regular graphs.

4. Some 2-extendable strongly regular graphs

In this section, we give some 2-extendable strongly regular graphs.

Theorem 4 A strongly regular graph with even order is 2-extendable when $\frac{k}{3} \leq \beta \leq k - 1$ and $k \geq 4$.

Proof Let G be a strongly regular graph of even order with $\frac{k}{3} \leq \beta \leq k - 1$.

Suppose G is not 2-extendable. By the arguments of Theorem 3, if N is the number of edges from the components of G-S to S, then $N \leq k|S| - 4$ and $N \geq k(|S|-2)$.

Claim 1 There are at least $\frac{3}{2}k$ edges to S from a component C_i of order at least three $(1 \leq i \leq |S|-2)$.

Let C_i be an odd component of order at least three. If C_i has order three, each vertex u of C_i is adjacent to at most two other vertices of C_i. There are therefore at least $k - 2 \geq \frac{k}{2}$ edges from u to S as $k \geq 4$. So there are $3 \times \frac{k}{2}$ edges from C_i to S.

If C_i has order at least five, by Lemma 3 and $\beta \geq \frac{k}{3}$, there are at least $\frac{5k}{3} \geq \frac{3k}{2}$ edges from C_i to S.

(1) Suppose there are at least three singleton components of G-S. Let these three vertices be v_1, v_2, v_3. Let $\{r_1, r_2, \ldots, r_k\}$, $\{s_1, s_2, \ldots, s_k\}$, $\{t_1, t_2, \ldots, t_k\}$, be the neighbours of v_1, v_2, v_3, respectively.

Since v_1, v_2 are not adjacent, $|\{r_1, r_2, \ldots, r_k\} \cap \{s_1, s_2, \ldots s_k\}| = \beta$. Without loss of generality we may assume that $\{r_1, r_2, \ldots, r_k\} \cap \{s_1, s_2, \ldots, s_k\} = \{r_1, r_2, \ldots, r_\beta\} = \{s_1, s_2, \ldots, s_\beta\}$.

Since $r_{\beta+1}, r_{\beta+2}, \ldots, r_k$ are not adjacent to v_2, each of $r_{\beta+1}, r_{\beta+2}, \ldots, r_k$ is adjacent to β of the vertices $s_1, s_2, \ldots, s_k$. Hence there are $(k-\beta)\beta$ edges from $\{r_{\beta+1}, r_{\beta+2}, \ldots, r_k\}$ to $\{s_1, s_2, \ldots, s_k\}$. But $\frac{k}{3} \leq \beta \leq k - 1$, so $(k-\beta)\beta \geq k - 1$.

We also note that since v_1 and v_3 are not adjacent, $|\{r_1, r_2, \ldots, r_k\} \cap \{t_1, t_2, \ldots, t_k\}| = \beta$.

(1.1) Suppose $\{r_1, r_2, \ldots, r_k\} \cap \{t_1, t_2, \ldots, t_k\} = \{r_1, r_2, \ldots, r_\beta\} = \{t_1, t_2, \ldots, t_\beta\}$. Then $\{r_1, r_2, \ldots, r_k\} \cap \{t_{\beta+1}, t_{\beta+2}, \ldots, t_k\} = \emptyset$ and $\{s_1, s_2, \ldots, s_k\} \cap \{t_{\beta+1}, t_{\beta+2}, \ldots, t_k\} = \emptyset$.

Since none of $t_{\beta+1}, t_{\beta+2}, \ldots, t_k$ is adjacent to v_1, there are β edges from each of $t_{\beta+1}, t_{\beta+2}, \ldots, t_k$ to $\{r_1, r_2, \ldots, r_k\}$. Hence there are $(k-\beta)\beta \geq k - 1$ edges from $\{t_{\beta+1}, t_{\beta+2}, \ldots, t_k\}$ to $\{r_1, r_2, \ldots, r_k\}$. So G[S] has at least $2(k-1)$ edges. Since $k \geq 4$, $2(k-1) \geq k + 1$. By Lemma

4, there are at least $k(|S|-2)$ edges from $C_1, C_2, \ldots, C_{|S|-2}$ to S. However, by the k-regularity, the number of edges going into S is at most $k|S| - 2(k+1) = k(|S|-2) - 2$. This gives a contradiction.

(1.2) Suppose $\{r_1, r_2, \ldots, r_k\} \cap \{t_1, t_2, \ldots, t_k\} \neq \{r_1, r_2, \ldots, r_\beta\}$. Without loss of generality we may assume that $\{r_1, r_2, \ldots, r_\beta\} \cap \{t_1, t_2, \ldots, t_k\} = \{r_1, r_2, \ldots, r_i\} = \{t_1, t_2, \ldots, t_i\}$ for some $i < \beta$.

If $t_j \notin \{r_1, r_2, \ldots, r_k\} \cup \{s_1, s_2, \ldots, s_k\}$, then, since t_j and v_2 are not adjacent, there are β edges from t_j to $\{s_1, s_2, \ldots, s_k\}$. But $\beta \geq \frac{k}{3}$, and $k \geq 4$, so, since β is an integer, $\beta \geq 2$. Hence G[S] has at least $(k-\beta)\beta + \beta \geq (k-1) + 2 = k + 1$ edges. By the k-regularity, at most $k|S| - 2(k+1) = k(|S|2) - 2$ edges can enter S. However, by Lemma 4, there are at least $k(|S|-2)$ edges from $C_1, C_2, \ldots, C_{|S|-2}$ to S. This gives another contradiction.

Hence we may suppose that $\{t_1, t_2, \ldots, t_k\} \subseteq \{r_1, r_2, \ldots, r_k\} \cup \{s_1, s_2, \ldots, s_k\}$. Since none of the vertices $r_{\beta+1}, r_{\beta+2}, \ldots, r_k$ is adjacent to v_2 and none of the vertices $s_{\beta+1}, s_{\beta+2}, \ldots, s_k$ is adjacent to v_1, there are $(k-\beta)\beta \geq k-1$ edges from $\{r_{\beta+1}, r_{\beta+2}, \ldots, r_k\}$ to $\{s_1, s_2, \ldots, s_k\}$ and at least $k-1$ edges from $\{s_{\beta+1}, s_{\beta+2}, \ldots, s_k\}$ to $\{r_1, r_2, \ldots, r_k\}$.

(1.2.1) Suppose there are at most $(k-\beta)\beta - 2$ edges between $\{r_{\beta+1}, r_{\beta+2}, \ldots, r_k\}$ and $\{s_{\beta+1}, s_{\beta+2}, \ldots, s_k\}$. Then there are at least two edges incident with $\{s_{\beta+1}, s_{\beta+2}, \ldots, s_k\}$ which are not incident with $\{r_{\beta+1}, r_{\beta+2}, \ldots, r_k\}$. Hence G[S] has at least $(k-1) + 2 = k + 1$ edges. Counting the edges into S by the two methods used above, again gives a contradiction.

(1.2.2) Suppose there are $(k-\beta)\beta$ edges from $\{r_{\beta+1}, r_{\beta+2}, \ldots, r_k\}$ to $\{s_{\beta+1}, s_{\beta+2}, \ldots, s_k\}$. Since r_β is not adjacent to v_3, there are $\beta(\geq 2)$ edges from r_β to $\{t_1, t_2, \ldots, t_k\}$. The fact that G[S] contains at least $(k-1) + 2 = k + 1$ edges again leads to a contradiction.

(1.2.3) Hence we may suppose that there are $(k-\beta)\beta-1$ edges between $\{r_{\beta+1}, r_{\beta+2}, \ldots, r_k\}$ and $\{s_{\beta+1}, s_{\beta+2}, \ldots, s_k\}$. Let $u \in \{r_{i+1}, r_{i+2}, \ldots, r_\beta\}$ not be adjacent to $\{r_{\beta+1}, r_{\beta+2}, \ldots, r_k\} \cup \{s_{\beta+1}, s_{\beta+2}, \ldots, s_k\}$. Since u and

v_3 are not adjacent, there are β edges from u to $\{t_1, t_2, \ldots, t_k\}$. The usual counting argument now produces the contradiction. Hence any vertex in $\{r_{i+1}, r_{i+2}, \ldots, r_\beta\}$ must be adjacent to a vertex in $\{r_{\beta+1}, r_{\beta+2}, \ldots, r_k\} \cup \{s_{\beta+1}, s_{\beta+2}, \ldots, s_k\}$.

Suppose two vertices u, v exist in $\{r_{i+1}, r_{i+2}, \ldots, r_\beta\}$. As neither u nor v is adjacent to v_3, there are $\beta \geq 2$ edges from each of u, v to $\{t_1, t_2, \ldots, t_k\}$. Hence again G[S] contains at least k + 1 edges and we again obtain a contradiction.

Hence there is only one vertex in $\{r_{i+1}, r_{i+2}, \ldots, r_\beta\}$. Thus $i = \beta - 1$.

Now consider v_3. First v_3 is adjacent to $r_1, r_2, \ldots, r_{\beta-1}$. Since, by the early part of (1.2), $\{t_1, t_2, \ldots, t_k\} \subseteq \{r_1, r_2, \ldots, r_k\} \cup \{s_1, s_2, \ldots, s_k\}$, and the fact that v_1 and v_3, and v_2 and v_3 have β vertices in common, we see that v_3 is adjacent to precisely one vertex each in $\{r_{\beta+1}, r_{\beta+2}, \ldots, r_k\}$ and $\{s_{\beta+1}, s_{\beta+2}, \ldots, s_k\}$. Hence $k = \beta + 1$.

So there are $(k-\beta)\beta - 1 + 1 = \beta$ edges between $r_k = r_{\beta+1}$ and $\{s_\beta, s_{\beta+1}\}$. So $\beta \leq 2$ and $k = \beta + 1 \leq 3$. This contradicts the fact that $k \geq 4$.

(2) Suppose there are exactly two singleton components of G - S.

Let v_1 and v_2 be the vertices of the singleton components. Let $r_1, r_2, \ldots, r_k$ be the neighbours of v_1, let $s_1, s_2, \ldots, s_k$ be the neighbours in S of v_2. As v_1 and v_2 are not adjacent, $|\{r_1, \ldots, r_k\} \cap \{s_1, \ldots, s_k\}| = \beta$. As each vertex of $\{r_1, \ldots, r_k\} \setminus \{s_1, s_2, \ldots, s_k\}$ is not adjacent to v_2, there are at least $(k-\beta)\beta \geq k - 1$ edges from $\{r_1, r_2, \ldots, r_k\} \setminus \{s_1, s_2, \ldots, s_k\}$ to $\{s_1, s_2, \ldots, s_k\}$. So G[S] contains at least k - 1 edges. S can accept at most $k|S| - 2(k-1) = k|S| - 2k + 2$ edges.

(2.1) Suppose there are at least two odd components of order at least three. Let C_3 and C_4 be two odd components of order at least three.

By Claim 1, there are at least $\frac{3}{2}k$ edges from each of C_3 and C_4 to

S. By Lemma 4, there are at least $(|S|-4)k + 2 \times \frac{3}{2}k = k|S| - k$ edges from $C_1, C_2, \ldots, C_{|S|-2}$ to S, contradicting the fact that S can accept at most $k|S| - 2k + 2$ edges.

(2.2) Suppose there is only one odd component of order at least three.

Now there are only three odd components of G - S, so $|S| = 5$. Let C_3 be the odd component of order at least three.

(2.2.1) Suppose there is an even component C of G - S. If C has order at least four, by Lemma 3, there are at least $\frac{4}{3}k$ edges from C to S. If $|V(C_3)| \geq 5$, by Lemma 3, there are at least $\frac{5}{3}k$ edges from C_3 to S. So the number of edges from S to G-S is $N \geq \frac{4}{3}k + \frac{5}{3}k + 2k = 5k > 5k - 4$, contradicting $N \leq k|S| - 4$.
If $|V(C_3)| = 3$, there are at least $3k - 6$ edges from C_3 to S. $N \geq \frac{4}{3}k + 3k - 6 + 2k = 5k + \frac{4}{3}k - 6 > 5k - 4$ for $k \geq 4$, contradicting $N \leq k|S| - 4$. If C has order two, there are $2k - 2$ edges from C to S. By Lemma 4, $N \geq 3k + 2k - 2 = 5k - 2 > 5k - 4$, contradicting $N \leq k|S| - 4$.

(2.2.2) No even component exists.

(2.2.2.1) Suppose $|V(C_3)| \geq 5$.

By Lemma 3, there are at least $\frac{5}{3}k$ edges from C_3 to S.

$N \geq \frac{5}{3}k + 2k > 3k + 2$, when $k \geq 4$, contradicting the fact that S can accept at most $k|S| - 2k + 2 = 5k - 2k + 2 = 3k + 2$ edges.

(2.2.2.2) Suppose $|V(C_3)| = 3$.

By Claim 1, there are at least $\frac{3}{2}k$ edges from C_3 to S.

$N \geq \frac{3}{2}k + 2k = 3k + \frac{k}{2}$. If $k \geq 5$, $N > 3k + 2$, contradicting the fact that S can accept at most $3k + 2$ edges. So $k = 4$. We see by Lemma 1 that there is no$(10,4,\alpha,\beta)$-graph.

(2.3) No odd component of order at least three exists. Now there are only two odd components of $G - S$. Both are singletons. $|S| = 4$. As C_1 is only adjacent to vertices of S, $k = 4$.

Suppose there is an even component C. By Lemma 4, there are at least k edges from C to S. Hence $N \geq 3k > 2k + 2$. But since $|S| = 4$, this contradicts the fact that $N \leq k|S| - 2k + 2 = 2k + 2$.

So no even component exists. However, by Lemma 1 there is no

$(6, 4, \alpha, \beta)$-graph with $\frac{k}{3} \leq \beta \leq k - 1$. We conclude that there are not precisely two singletons.

(3) There is at most one singleton component of $G - S$.

(3.1) Suppose there are at least four odd components with order at least three. Let C_1, C_2, C_3 and C_4 be four odd components with order at
least three. There are $|S|-6$ other odd components. By Claim 1, there

are $\frac{3}{2}k$ edges from each of C_1, C_2, C_3, C_4 to S. By Lemma 4,

$N \geq 4 \times \frac{3}{2}k + k(|S|-6) = k|S|$, contradicting $N \leq k|S| - 4$.

(3.2) Suppose there are exactly three odd components C_1, C_2 and C_3 with order at least three.

(3.2.1) Suppose there is an even component C of $G - S$.

Since C has order at least two, by Lemma 3, there are at least $2\beta \geq \frac{2}{3}k$ edges from C to S. By Claim 1 there are at least $\frac{3}{2}k$ edges from C_1, C_2, C_3 to S. Hence by Lemma 4, $N \geq \frac{2}{3}k + 3 \times \frac{3}{2}k + k(|S| - 5) = k|S| + \frac{1}{6}k > k|S| - 4$. This contradicts the fact that $N \leq k|S| - 4$.

(3.2.2) No even component exists.

Suppose one of C_1, C_2 and C_3 has order at least seven. There are at least $\frac{7}{3}k$ edges from that component to S. By Claim 1 and Lemma 3, $N \geq \frac{7}{3}k + 2 \times \frac{3}{2}k + k(|S|-5) = k|S| + \frac{k}{3} > k|S| - 4$, contradicting $N \leq k|S| - 4$.

So none of C_1, C_2, C_3 has order larger than five.

(3.2.2.1) Suppose $|V(C_1)| = |V(C_2)| = |V(C_3)| = 5$.

By Lemma 3, there are at least $\frac{5}{3}k$ edges from each of C_i $(i = 1,2,3)$ to S. $N \geq 3 \times \frac{5}{3}k + k(|S|-5) > k|S| - 4$, contradicting $N \leq k|S| - 4$.

(3.2.2.2) Suppose $|V(C_1)| = |V(C_2)| = 5$ and $|V(C_3)| = 3$.

By Lemma 3, there are at least $\frac{5}{3}k$ edges from each of C_1 and C_2 to S and there are at least $3k - 6$ edges from C_3 to S.

$N \geq 2 \times \frac{5}{3}k + 3k - 6 + k(|S|-5) = k|S| + \frac{4}{3}k - 6 > k|S| - 4$ for $k \geq 4$, a contradiction.

(3.2.2.3) Suppose $|V(C_1)| = 5$ and $|V(C_2)| = |V(C_3)| = 3$.

There are at least $\frac{5}{3}k$ edges from C_1 to S and there are at least $3k - 6$ edges from each of C_2 and C_3 to S.

$N \geq \frac{5}{3}k + 2(3k-6) + k(|S|-5) = k|S| + \frac{8}{3}k - 12 > k|S| - 4$ for $k \geq 4$, a contradiction.

(3.2.2.4) Suppose $|V(C_1)| = |VC_2| = |V(C_3)| = 3$.

There are at least $3k-6$ edges from each of C_1, C_2 and C_3 to S. $N \geq 3(3k-6) + k(|S|-5) = k|S| + 4k - 18 > k|S| - 4$ for $k \geq 4$, a contradiction.

(3.3) Suppose there are exactly two odd components C_1 and C_2 with order at least three.

(3.3.1) Suppose that there is an even component C of $G - S$.

If C has order at least four, there are at least $\frac{4}{3}k$ edges from C to S.

$N \geq \frac{4}{3}k + 2 \times \frac{3}{2}k + k(|S|-4) = k|S| + \frac{k}{3} > k|S| - 4$, contradicting $N \leq k|S| - 4$. If C has order two, there are $2k - 2$ edges from C to S. $N \geq 2k - 2 + 2 \times \frac{3}{2}k + k(|S|-4) = k|S| + k - 2 > k|S| - 4$, a contradiction.

(3.3.2) No even component exists.

Suppose one of C_1 and C_2 has order at least seven.

Assume $|V(C_1)| \geq 7$, there are at least $\frac{7}{3}k$ edges from C_1 to S.

If $|V(C_2)| \geq 5$, there are at least $\frac{5}{3}k$ edges from C_2 to S.

$N \geq \frac{7}{3}k + \frac{5}{3}k + k(|S|-4) = k|S| > k|S| - 4$, contradicting $N \leq k|S| - 4$.

If $|V(C_2)| = 3$, there are at least $3k - 6$ edges from C_2 to S.

$N \geq \frac{7}{3}k + 3k - 6 + k(|S| - 4) = k|S| + \frac{4}{3}k - 6 > k|S| - 4$ for $k \geq 4$, a contradiction.

None of C_1 and C_2 has order larger than five.

(3.3.2.1) Suppose $|V(C_1)| = |V(C_2)| = 5$.

There are at least $5k-20$ edges from each of C_1 and C_2 to S. $N \geq 2(5k-20) + k(|S| - 4) = k|S| + 6k - 40$. When $k \geq 7$, $N > k|S| - 4$, a contradiction.

Suppose there is a singleton component C_3. $|S| = 5$. As C_3 is only adjacent to vertices of S, $k \leq 5$. By Lemma 1, the only two possible $(16,k,\alpha,\beta)$-graphs for $4 \leq k \leq 5$ are $(16,5,0,2)$ and $(16,5,2,1)$.

Suppose the graph is a $(16,5,0,2)$-graph. As C_3 is adjacent to all vertices of S and S contains two independent edges, there is a triangle containing C_3, contradicting $\alpha = 0$.

Suppose the graph is a $(16,5,2,1)$-graph. As $\beta \geq \frac{k}{3}$ and $k \geq 4$, $\beta \geq 2$. A $(16,5,2,1)$-graph doesn't satisfy the assumption of the theorem.

Now no singleton exists. $|S| = 4$. We can verify by Lemma 2 that there are no $(14,k,\alpha,\beta)$-graphs for $4 \leq k \leq 6$.

(3.3.2.2) Suppose $|V(C_1)| = 5$ and $|V(C_2)| = 3$.

There are at least $\frac{5}{3}k$ edges from C_1 to S and there are at least $3k - 6$ edges from C_2 to S. $N \geq \frac{5}{3}k + 3k - 6 + k(|S| - 4) = k|S| + \frac{2}{3}k - 6 > k|S| - 4$ for $k \geq 4$, a contradiction.

(3.3.2.3) Suppose $|V(C_1)| = |V(C_2)| = 3$.

There are at least $3k - 6$ edges from each of C_1 and C_2 to S. $N \geq 2(3k-6) + k(|S| - 4) = k|S| + 2k - 12$. $N > k|S| - 4$ for $k \geq 5$, a contradiction.

If $k = 4$, suppose there is a singleton component C_3. $|S| = 5$. We can verify by Lemma 1 that there is no $(12,4,\alpha,\beta)$-graph.

Now no singleton exists and $|S| = 4$. We can verify by Lemma 1 that there is no $(10,4,\alpha,\beta)$-graph.

(3.4) Suppose there is exactly one odd component C_1 with order at least three.

Now $|S| = 4$ and there is a singleton component C_2.

(3.4.1) Suppose there is an even component C of $G - S$.

If C has order at least four, there are at least $\frac{4}{3}k$ edges from C to S.

If $|V(C_1)| \geq 5$, there are at least $\frac{5}{3}k$ edges from C_1 to S.

$N \geq \frac{4}{3}k + \frac{5}{3}k + k = 4k > 4k - 4$, contradicting $N \leq k|S| - 4$. If $|V(C_1)| = 3$, there are at least $3k - 6$ edges from C_1 to S.

$N \geq \frac{4}{3}k + 3k - 6 + k > 4k - 4$ for $k \geq 4$, contradicting $N \leq k|S| - 4$.

If C has order two, there are $2k - 2$ edges from C to S. By Claim 1, there are at least $\frac{3}{2}k$ edges from C_1 to S. $N \geq 2k - 2 + \frac{3}{2}k + k = 4k + \frac{k}{2} - 2 > 4k - 4$, a contradiction.

(3.4.2) No even component exists.

(3.4.2.1) Suppose $|V(C_1)| \geq 9$.

There are at least $\frac{9}{3}k$ edges from C_1 to S. $N \geq \frac{9}{3}k + k = 4k > 4k - 4$, contradicting $N \leq k|S| - 4$.

(3.4.2.2) Suppose $3 \leq |V(C_1)| \leq 7$.

Now $|S| = 4$. C_2 is only adjacent to vertices of S. So $k = 4$. We can verify by Lemma 1 that there are no $(\upsilon,4,\alpha,\beta)$-graphs with

$\frac{k}{3} \leq \beta \leq k - 1$ for $\upsilon = 8$, 10 or 12. ❑

Theorem 5: Every strongly regular graph of even order with $\beta = k$ and $k \geq 4$ is 2-extendable, except the (6,4,2,4)-graph.

Proof: Let G be a strongly regular graph $(\upsilon,k,\alpha,\beta)$ with even order, $\beta = k$ and $k \geq 4$.

If $\alpha = 0$, G is $K_{k,k}$. Hence G is 2-extendable. So assume $\alpha \neq 0$. Let w be a vertex of G and $w_1, w_2, \ldots, w_k$ be the vertices adjacent to w. As $\beta = k$, every vertex of $V(G) - \{w, w_1, w_2, \ldots, w_k\}$ is adjacent to $w_1, w_2, \ldots, w_k$. As $\alpha > 0$, there is an edge $e = w_iw_j$. All the vertices of $V(G) - \{w_1, w_2, \ldots, w_k\}$ are common neighbours of w_i and w_j. So $\alpha \geq \upsilon - k$. Since w and w_i have α common neighbours, there are α edges from w_i to $\{w_1, \ldots, w_k\}$. So $k \geq 2(\upsilon - k)$. Therefore $k \geq \frac{2}{3}\upsilon$.

When $\upsilon \geq 12$, $k \geq \frac{\upsilon}{2} + 2$, so by Lemma 2, G is 2-extendable.
For $\upsilon \leq 10$, $\beta = k \geq 4$ and $\alpha > 0$, the only parameters which satisfy Lemma 1 are given below

(10,9,8,9)	this graph is K_{10} and is 2-extendable
(10,8,6,8)	$k \geq \frac{\upsilon}{2} + 2$, so these graphs are 2-extendable
(10,7,4,7)	see below

(10,6,2,6)	see below
(8,7,6,7)	this graph is K_8 and is 2-extendable
(8,6,4,6)	$k \geq \frac{v}{2} + 2$, so these graphs are 2-extendable
(8,5,2,5)	see below
(6,5,4,5)	this graph is K_6 and is 2-extendable
(6,4,2,4)	see below

In the (10,7,4,7)-graphs, let w be adjacent to the set $W = \{w_1, w_2, \ldots, w_7\}$. Then $G[W]$ is a (7,4,1,4)-graph. However, the neighbours N of w_1 in $G[W]$ are adjacent to three vertices. Since not all members of N are adjacent, the value of α in $G[W]$ is at least 3, a contradiction.

Consider the graphs (10,6,2,6). By previous arguments all the vertices of $V(G) - \{w, w_1, w_2, \ldots, w_6\}$ are adjacent to all the vertices $\{w_1, w_2, \ldots, w_6\}$. Since there must be an edge between w_i and w_j for some i, j, this means that w_i and w_j have at least 4 common neighbours, so $\alpha \geq 4$, a contradiction.

A similar argument with (8,5,2,5) shows that $\alpha \geq 3$, a contradiction.

In Figure 1 we show the (6,4,2,4)-graph. The edges u_1v_1, u_2v_2 cannot be extended to a perfect matching. ❑

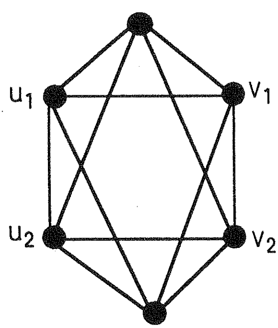

Figure 1

<u>Corollary 2:</u> A strongly regular graph with even order and $k \geq 3$ is 2-extendable when $\beta \geq \frac{k}{3}$, except the Petersen graph and the (6,4,2,4)-graph.

<u>Corollary 3:</u> A strongly regular graph with $k = 3, 4, 5, 6$ is 2-extendable unless it is the Petersen graph or the (6,4,2,4)-graph.

<u>Proof</u>. For $k = 4,5,6$ if $\beta \geq \frac{1}{3}k$ implies $\beta \geq 2$. So we only need test the 2-extendability of $(\upsilon, k, \alpha, 1)$-graphs with $k = 4,5,6$. There are no such graphs. □

We conjecture that all but a few strongly regular graphs are 2-extendable.

5. A family of strongly regular graphs and their n-extendability

Given any n, we now construct a family of strongly regular graphs, each of which is n-extendable.

Let G be a graph and S be a vertex set and $S \cap V(G) = \varnothing$. $G + S$ is defined by $V(G + S) = V(G) \cup S$ and each vertex of S is joined to all vertices of G.

We define a family of graphs S_i $(i = 0,1, \ldots)$ by

(1) $S_0 = C_4$, a 4-cycle.

(2) Assume S_k is defined. $S_{k+1} = S_k + \{u_{k+1}, v_{k+1}\}$, where u_{k+1}, $v_{k+1} \notin V(S_k)$.

Theorem 6 The family S_i $(i = 0,1, \ldots)$ is a family of strongly regular graphs. Each S_i is a $(4+2i, 2+2i, 2i, 2+2i)$-graph $(i = 0,1, \ldots)$.

Proof It is easy to verify that S_0 is a (4,2,0,2)-graph. Assume S_i is a (4+2i, 2+2i, 2i, 2+2i)-graph. By definition, $S_{i+1} = S_i + \{u_{i+1}, v_{i+1}\}$. Hence $V(S_{i+1}) = 4+2i+2 = 4 + 2(i+1)$. As u_{i+1} and v_{i+1} are joined to all vertices of S_i, $d(u_{i+1}) = d(v_{i+1}) = 4 + 2i = 2 + 2(i+1)$. For each vertex u in $V(S_i)$, as u is joined to u_{i+1} and v_{i+1}, $d(u) = 2 + 2i + 2 = 2 + 2(i+1)$. So S_{i+1} is $[2 + 2(i+1)]$-regular.

Let u and v be a pair of non-adjacent vertices. If $u = u_{i+1}$ and $v = v_{i+1}$, all the vertices of S_i are common neighbours of u and v.

So u and v have $4 + 2i = 2 + 2(i+1)$ common neighbours. If $u,v \in V(S_i)$ by the induction hypothesis, u and v have $2 + 2i$ common neighbours in S_i. u_{i+1} and v_{i+1} are also common neighbours of u and v. So u and v have $2 + 2i + 2 = 2 + 2(i+1)$ common neighbours. Hence $\beta(S_{i+1}) = 2 + 2(i+1)$.

Let u and v be a pair of adjacent vertices. If $u = u_{i+1}$ or v_{i+1} and v is in $V(S_i)$, as S_i is $2 + 2i$ regular, u and v have exactly $2 + 2i = 2(i+1)$ common neighbours. If u and v are in $V(S_i)$, by the induction hypothesis, u and v have $2i$ common neighbours in S_i. But u_{i+1} and v_{i+1} are also common neighbours of u and v. So u and v have $2i + 2 = 2(i+1)$ common neighbours. Hence $\alpha(S_{i+1}) = 2(i+1)$. S_{i+1} is therefore a $(4 + 2(i+1),\ 2 + 2(i+1),\ 2(i+1),\ 2 + 2(i+1))$-graph. ❑

Theorem 7 S_i is i-extendable $(i = 0,1, \ldots)$.

Proof As the degree $k = 2 + 2i = 2 + i + i \geq \frac{\upsilon}{2} + i$, S_i is i-extendable by Lemma 2. ❑

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