Maximum matchings in 3-connected graphs contain contractible edges

by

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Abstract

In this paper it is shown that every maximum matching in a 3-connected graph, other than K_4, contains at least one contractible edge. In the case of a perfect matching, those graphs in which there exists a perfect matching containing precisely one contractible edge are characterized.

Introduction

The existence of contractible edges in 3-connected graphs, as well as in certain types of subgraphs, has proven to be a useful inductive tool [see, 3, 9, 10, 11, 12] with the most notable instances being Tutte's characterization of 3-connected graphs and Thomassen's ingenious proof of Kuratowski's Theorem. These applications have motivated, and sometimes required, deeper studies into the number of, and distribution of contractible edges. In one such study Dean, Hemminger and Toft [5] showed that every longest cycle in a non-K_4, 3-connected graph contains at least two contractible edges of the graph. (Dean, Hemminger and Ota [6] later showed that, except for $K_2 \times K_3$, they in fact contain three; subsequently Aldred and Hemminger [1] characterized the extremal graphs.)

In this context Kaneko and Ota [8] posed the following problem: find other types of subgraphs such that every one in a non-K_4, 3-connected graph G contains at least one contractible edge of G. They put forward maximum matchings as a likely, and useful, candidate and showed they were correct if the matching wasn't a near-perfect one.

In this paper we will prove that all maximum matchings have the desired property; namely, except in K_4 they always contain at least one contractible edge of G (Theorem 6 to follow).

In [8] they also show that the ladders (definition to follow) are the only graphs that have a perfect matching that contains *only* one contractible edge of G. That result is an easy corollary (Theorem 4 here) of our proof of Theorem 6. However, we will only sketch that idea as we can, and do achieve the same end by a simplification of the proof of Theorem 6.

Finally, in a separate paper [2], the authors give a constructive characterization of all 3-connected graphs G that admit a maximum matching that contains only one contractible edge of G. Since all such matchings are either perfect (for ladders, as noted above) or near-perfect, we immediately get the main result of [8]; namely, if a maximum matching M of G leaves more than one vertex of G unsaturated, then M contains at least two contractible edges of G.

Definitions

We consider only finite undirected graphs without loops and multiple edges and use Bondy and Murty [4] as our reference for undefined terms and notation; in particular $\nu(G) = |V(G)|$, $\epsilon(G) = |E(G)|$ and for $A \subseteq V(G)$ or $A \subseteq E(G)$, $G[A]$

denotes the *subgraph* of G *induced by* A and $G - A$ is the subgraph of G obtained *by deleting* A. In fact, if the context is clear, we will often use A in place of $G[A]$; for example, both the edge set and the vertex set of a cycle will be referred to as the cycle. We also let $V(A)$ denote the vertex set of $G[A]$. For $v \in V(G)$, $N_G(v) = \{w \in V(G) : vw \in E(G)\}$, $dg_G(v) = |N_G(v)|$ and $E_G(v) = \{e \in E(G) : v \in V(e)\}$. Likewise, for $e = uv$, $N_G(e) = N_G(u) \cup N_G(v) - \{u, v\}$. In this and similar notation, we will commonly suppress the G if no confusion will result. For a connected graph G, $S \subset V(G)$ is called a *cutset* of G if $G - S$ is disconnected. For an edge e in G, $G \circ e$ will denote the simple graph that results from contracting e, and in a 3-connected graph G, e is called a *contractible edge* of G if $G \circ e$ is also 3-connected; otherwise, e is called a *noncontractible edge* of G.

If $F \subseteq E(G)$, then $F_c(G) = \{e \in F : e \text{ is contractible in } G\}$. For us, this will only be used for $E_c(G)$ and $M_c(G)$ where M is a matching in G. And for $v \in V(G)$ we let $E_c(v) = E(v) \cap E_c(G)$. Similarly $F_n(G) = E(F) - F_c(G)$ and $E_n(v) = E(v) - E_c(v)$. Note that in a non-K_4, 3-connected graph G, an edge e is noncontractible if and only if there is a 3-cut of G that contains $V(e)$. If $e = xy \in E_n(G)$, $S = \{x, y, s\}$ is an associated 3-cut of G, and C is a component of $G - S$, we let $C^+ = G[V(C) \cup \{s\}]$.

For $n \geq 1$, an n-ladder L_n is a graph in which $V(L_n)$ can be labelled with the set $\{x_i, y_i : 0 \leq i \leq n\}$ such that

$$E(L_n) = \{x_iy_i,\ x_ix_{i+1}, y_iy_{i+1} : 1 \leq i \leq n-1\}$$
$$\cup\,(x_0y_0,\ x_0x_1,\ x_0y_1,\ x_ny_n,\ y_0x_n,\ y_0y_n\} \cup D$$

where $D \subseteq D_n = \{x_iy_{i+1},\ y_ix_{i+1} : 1 \leq i \leq n-1\}$. For an example, see Figure 1 with the edge xz contracted. The edges in D are referred to as *optional edges* and the edges in $R = \{x_iy_i : 0 \leq i \leq n\}$ are called the *rungs of* L_n. It should be obvious that R is a perfect matching in L_n and, for $n \geq 2$, that $R_c = \{x_0y_0\}$; moreover, it is easily seen that R is the only matching in L_n that contains only one contractible edge of L_n. It is this property of the n-ladders ($n \geq 2$) that enables us to refer to *the* rungs of L_n.

We let $\mathcal{L}$ denote the class of all n-ladders, $n \geq 1$. Note that $L_1 \simeq K_4$ and so $R_c = \emptyset$ since $E_c(K_4) = \emptyset$.

Our proofs here, and in [2], turn on the concept of a "turncoat edge". To facilitate their definition, we introduce the following notational convention. Let H be a 3-connected graph, let $e = uv \in E_c(H)$ and let $f = xy \in E(H)$. If we denote $H \circ e$ by $\hat{H}$, then we will let $\hat{f} \in E(\hat{H})$ denote the "image" of f, that is, $\hat{f} = xy$ if $x, y \notin V(e)$ and $\hat{f} = \hat{e}y$ if $u = x$ and $\hat{e}$ denotes the contraction of e. We will refer to $\hat{f}$ as the edge of $\hat{H}$ *induced by* f and, as long as no confusion can arise, we will continue to denote it by f. In these terms, an edge $f \in E_n(H)$ is called a *turncoat edge of* H via e if f *becomes* contractible in $\hat{H}$, that is, if $f \in E_n(H)$ and $\hat{f} \in E_c(\hat{H})$.

Thus an edge subtended by a vertex u of degree three is a turncoat via each $e \in E_c(u)$.

Our viewpoint is of some interest in that it enables us to prove something *about* contractible edges (in matchings in this paper) by *using* contractible edges as an inductive tool. And that is why turncoat edges are troublesome; because of them we can have $E_c(G \circ e) \neq E_c(G) - e$. Yu [13] introduced the concept (in a different context) and used the properties given in Lemma 1 as did Hemminger and Yu [7].

The Theorems

Throughout the paper G will be a 3-connected graph other than K_4 and M will be a maximum matching in G. We let $U(M, G) = V(G) - V(M)$.

Lemma 1: If f is a turncoat edge via $e \in E_c(G)$, then e has an endvertex u of degree three with $V(f) \subset N(u)$. Moreover, $N(u)$ is the only 3-cut containing $V(f)$ and $G - (N(u) \cup \{u\})$ is connected.

Proof: Let $f = xy$ and let $S = \{x, y, s\}$ be any 3-cut associated with f. Then $V(e) \not\subset S$ since $e \in E_c(G)$, so let C be the component of $G - S$ with $V(e) \cap V(C) \neq \emptyset$. Then $V(e) \not\subseteq V(C)$ or $f \in E_n(G \circ e)$. For the same reason $|V(C)| = 1$, that is, $e = uv$ with $V(C) = \{u\}$ and $S = N(u)$. Thus $G - S$ only has one other component since $f \notin E_n(G \circ e)$. Since $\{x, y, u\}$ is not a 3-cut, that proves the lemma.

But for matching edges, turncoats do not beget turncoats!

Lemma 2: If $f \in M_n$ is a turncoat edge of G via $e \in M_c(G)$, then no edge of $M_n(G) - \{e, f\}$ is contractible in $(G \circ e) \circ f$.

Proof: Suppose that $e = uv \in M_c(G)$ and $f = xy \in M_n(G) \cap E_c(G \circ e)$. By Lemma 1 we can further assume that $N(u) = \{v, x, y\}$.

If f is the only turncoat in G via e, then the claim is true since neither x nor y can be of degree three in $G \circ e$ and still subtend an edge of M_n: for example, $\{\hat{e}, y\} \subset N_{G \circ e}(x)$ and $\hat{e} \in U(\hat{M}, \hat{G})$.

If $f = xy$ and $h = wz$ are both in $M_n(G) \cap E_c(G \circ e)$, then by Lemma 1 we can assume that $N(u) = \{v, x, y\}$ and $N(v) = \{u, w, z\}$. But then $\{w, z, \hat{f}\}$ is a 3-cut in $(G \circ e) \circ f$, so $wz \in E_n((G \circ e) \circ f)$. Thus the claim holds as before.

Lemma 3: If $M_c = \emptyset$, if $f = xy \in M = M_n$, and if $S = \{x, y, s\}$ is any associated 3-cut of G, then $|U(M, G)|$ is at least as large as the number of components of $G - S$. In particular, $|U(M, G)| \leq 1$ implies that $M_c \neq \emptyset$.

Proof: Let $C = C_1$ be a component of $G - S$ and set $x_1 = x, y_1 = y,\ s_1 = s$ and $S_1 = S$. Now pick $x_2y_2 \in M$ with $x_2 \in V(C_1)$ and let $S_2 = \{x_2, y_2, s_2\}$ be an associated 3-cut where we choose $s_2 = s$ if possible.

Now since $G[V(G) - V(C_1) - \{s_1\}]$ is easily seen to be 2-connected, $G - S_2$ will have a component C_2 such that $V(C_2) \subseteq V(C_1) - \{x_2\}$ if $y_2 = s$ and $V(C_2) \subseteq (V(C_1) \cup \{s\}) - \{x_2, y_2\}$ if $y_2 \neq s$. In either case we see that $|V(C_2)| < |V(C_1)|$. Moreover, it is clear, since we are taking $s_2 = s$ if possible, that we have $V(C_2) \subset V(C_1)$ *unless* $G - S$ only has two components (let D be the other) and $N(s) = \{t, x_2, y_2\}$ with $t \in V(D)$. In this case $ts \in E_c(G)$; for if not and $T = \{s, t, w\}$ is an associated 3-cut of G, then $G - (V(D) \cup \{s\})$ is 2-connected and so $G - T$ has a component wholly contained in $D - \{t\}$ to which s must be adjacent. Thus $ts \in E_c(G)$ and $x_2y_2 \in M$, so $s \in U(M, G)$.

So assume that C is not a component of this latter type. Of course there is at most one of that type and if it exists it will be paired with $s \in U(M, G)$. We will now show that each other component, such as C, contains an M-unsaturated vertex and hence complete the proof of the lemma.

For suppose not and let $x_3y_3 \in M$ with $x_3 \in V(C_2)$. Then just as we got $|V(C_2)| < |V(C_1)|$ we get $|V(C_3)| < |V(C_2)|$. And by the restriction on C we also get $V(C_3) \subset V(C_1)$. Continuing in this manner we get a contradiction since G is finite.

Theorem 4 [8]: Let G be a non-K_4, 3-connected graph and let M be a perfect matching in G. Then $|M_c(G)| = 1$ if and only if G is a ladder.

Proof: As noted, ladders have the requisite property so assume that M is a perfect matching in G with $M_c = \{e = uv\}$. Then $\hat{M} = M - \{e\}$ is a maximum matching in $\hat{G} = G \circ e$ with $\{\hat{e}\} = U(\hat{M}, \hat{G})$. Thus, by Lemma 3, $\hat{M}_c \neq \emptyset$. Of course $\hat{M}_c$ can only contain turncoat edges of G via e, so let $f = xy$ be one such edge (of a possible two), say with $N(u) = \{x, y, v\}$ (by Lemma 1).

Let $\overline{G} = \hat{G} \circ f$ and $\overline{M} = (\hat{M} - \{f\}) \cup \{\hat{e}\hat{f}\}$, so that $\overline{M}$ is a perfect matching in $\overline{G}$. Hence, by Lemma 2, $M_n(G) \cap \overline{M}_c = \emptyset$ and so $\overline{M}_c \subseteq \{\hat{e}\hat{f}\}$. If $\overline{M}_c = \emptyset$, then $\overline{G} \simeq K_4$. In that case, $x, y \notin N(v)$, or else $|M_c(G)| \geq 2$, and so G is a ladder on six vertices. For the same reason, G is a ladder with $M_c = \{e\}$ if $\overline{M}_c = \{\hat{e}\hat{f}\}$.

That completes the proof of Theorem 4. The graph in Figure 1 illustrates the difficulties encountered if we try to use the approach contained therein to characterize the 3-connected graphs G that admit a maximum matching M with $|M_c| = 1$. For, if we let G be that graph and let M be the set of fattened edges, then $\hat{M}$ is not a maximum matching (nor can it be extended to one) and $\hat{M}_c = \emptyset$. To overcome this type of difficulty, we move to a contractible edge at a vertex in $U(M, G)$, such as zw or zx in the graph in Figure 1. But this problem requires a much deeper analysis and is the topic of [2].

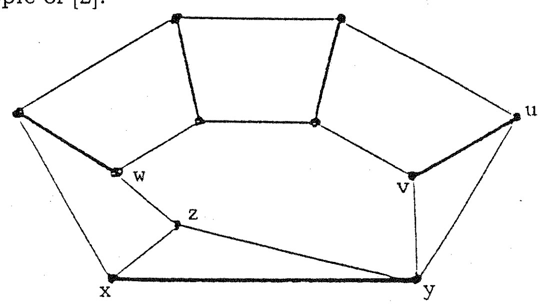

Figure 1

The above proofs of Lemma 3 and of Theorem 4 were obtained by Hemminger and Yu in the summer of 1990. Lemma 3 is a special case of Theorem 6 and its proof is a shortcut version of the following proof of Theorem 6 that was obtained by Aldred and Hemminger in November 1989.

For reference within the proof of Theorem 6 we list the following lemma, whose proof is contained within the proof of Lemma 3.

Lemma 5: Let $S = \{x, y, s\}$ be a 3-cut of G associated with $xy \in E_n(G)$, let C be a component of $G - S$, and let $x'y' \in E_n(G)$ with $x' \in V(C)$ and $y' \neq x, y$. Then either $N(s) = \{x', y', t\}$ with $t \notin C^+$ or $x'y'$ has an associated 3-cut T such that $G - T$ has a component C' wholly contained in C. In the former case, $G - S$ has only two components and $st \in E_c(G)$.

Theorem 6: If G is a non-K_4, 3-connected graph and if M is a maximum matching, then $M_c \neq \emptyset$.

Proof: We assume that $M_c = \emptyset$ and use the notation in the proof of Lemma 3. Only now our goal is to produce an M-alternating path

$$P : x_1, y_1, x_2, y_2, \cdots, x_k, y_k, z$$

in $C^+ = G[V(C) \cup \{s\}]$ that ends with an M-unsaturated vertex z.

So we can assume that $N(y_1) \cap V(C)$ is M-saturated. We now pick $x_2y_2 \in M$ as in Lemma 3, but with the added stipulation that $y_1x_2 \in E(G)$. With $S_2 = \{x_2, y_2, s_2\}$ an associated 3-cut we are done (with $k = 2$ and $z = s_1$) if $N(s_1) = S_2$; for by Lemma 5, $s_1s_2 \in E_c$ and so $s_1 \in U(M, G)$. So we assume that $N(s_1) \neq S_2$ and hence by Lemma 5, that $G - S_2$ contains a component $C_2 \subset C_1$. Thus $s_2 \in V(C_1) \cup S_1$ since s_2 is adjacent to C_2.

If y_2 has an M-unsaturated neighbour x_3 in C_2 we have the desired path P with $k = 2$ and $z = x_3$. So, as before, we can assume that we have edges y_2x_3 and x_3y_3 with $x_3 \in V(C_2)$, with $y_3 \in V(C_2) \cup \{s_2\}$, and with $x_3y_3 \in M = M_n$. Let $S_3 = \{x_3, y_3, s_3\}$ be a 3-cut of G associated with the edge x_3y_3. If $N(s_2) = S_3$, then $s_2 \neq x_1$ or y_1 (e.g. $s_2 = x_1$ means that $N(s_2) \supseteq \{y_1, x_3, y_3, w\}$ for some $w \in N(x_1) - (V(C_1) \cup S_1)$). So we have the desired path P with $k = 3$ and $z = s_2$ since $s_2 s_3 \in E_c(G)$ by Lemma 5. Otherwise, we continue this process as long as possible. If y_k has an M-unsaturated vertex x_{k+1} in C_k for $k \geq 3$, then we have the desired path with $z = x_{k+1}$. So as before, we can assume that we have edges $y_k x_{k+1}$ and $x_{k+1} y_{k+1} \in M = M_n$. Let $S_{k+1} = \{x_{k+1}, y_{k+1}, s_{k+1}\}$ be a 3-cut of G associated with the edge $x_{k+1} y_{k+1}$. If $N(s_k) = S_{k+1}$, then $s_k \neq x_j$ or y_j for $j < k$. For suppose so, say $s_k = x_j$ for $2 \leq j < k$. But then $N(s_k) \supseteq \{x_{k+1}, y_{k+1}, y_j, y_{j-1}\}$, contradicting that $dg(s_k) = 3$. Likewise $s_k \neq y_j$ for $j \neq 1$. And that $s_k \neq x_1$ or y_1 follows as in the case with $k = 2$.

Reversing the roles of x_1 and y_1 and using a component $D \neq C$ of $G - S_1$, we get an M-alternating path

$$P : y_1, x_1, y_2', x_2', \cdots, y_q', x_q', z'$$

that is contained in D^+ and that ends at an M-unsaturated vertex z'. Now $z \neq z'$ for that would require $z = z' = s_1$, that is, $dg(s_1) = 3$ and $k, q \geq 2$ so that $N(s_1) \supseteq \{x_k, y_k, y_q', x_q'\}$. Thus $P \cup P'$ gives an M-augmenting path, which contradicts that M was a maximum matching in G.

That completes the proof of the theorem. And we easily get some extra mileage out of the iterative procedure in its proof. For suppose that G and M are as in the theorem and that $|M_c| = 1$, say $M_c = \{e\}$. So we have $x_1 y_1 \in M_n$, S and C as in the proof of the theorem. The new wrinkle is that the procedure can now end with $x_{k+1} y_{k+1} = e \in E(C_1^+)$ or with $s_k s_{k+1} = e$, $x_{k+1} y_{k+1} \in M_n$ and $N(s_k) = \{x_{k+1}, y_{k+1}, s_{k+1}\}$ (the two options in Lemma 5).

From this we easily get Theorem 4.

Alternate proof of Theorem 4: As noted before, ladders have such matchings so we turn to the converse. And since G has no M-unsaturated vertices the iterative procedure can only end at the contractible edge $e \in M_c$. Thus, for S as above $G - S$ can have only two components, say C and D, and by relabelling, we obtain a cycle

$$Z : v_0, u_1, v_1, \cdots, u_k, v_k, u_0, v_0$$

where $\{u_0v_0 = e\} = M_c$, $u_iv_i \in M_n$ for $1 \le i \le k$, $N(u_0) = \{v_0, u_k, v_k\}$ and $N(v_0) = \{u_0, u_1, v_1\}$.

Moreover, we conclude that Z is a Hamilton cycle in G. For suppose not. Since M is a perfect matching and $M_c = \{e\}$, there is an edge $uv \in M_n$ with $u, v \notin V(Z)$. If $S' = \{u, v, s'\}$ is an associated 3-cut of G, then $Z - \{s'\}$ is connected and so $G - S'$ has a component F disjoint from Z. Working into F, as before with C, we are led to a contradiction via Lemma 5 since we have already accounted for all neighbors of the only contractible edge.

For $1 \le i \le k$, let $S_i = \{u_i, v_i, s_i\}$ be a 3-cut of G associated with u_iv_i where we take $s_i = u_0$ if possible. Thus, by the above, $s_1 = u_0$, $s_k = v_0$, and $s_i \ne v_0$ if $i \ne k$. Since $V(G) = V(Z)$, $G - S_i$ has only two components and, traversing Z as listed, we let C_i be the component on the vertices from u_{i+1} to s_i (including u_{i+1} but not s_i). So $V(C_1) = \{u_2, v_2, \cdots, u_k, v_k\}$.

Now apply Lemma 5 with $x = u_2$ and $y = v_2$. Since G is a ladder if $k = 2$, we can assume that $k \ge 3$. Thus, since $S_2 \ne N(u_0)$, we conclude that $C_2 \subset C_1$. Hence $s_2 \in V(C_1) \cup \{s_1\}$.

Continuing in this way, we get a sequence $C_1 \supset \cdots \supset C_j$ with $s_{i+1} \in V(C_i) \cup \{s_i\}$, $1 \le i < j$, which only ceases when we get $S_j = \{u_j, v_j, v_{j+1}\}$: that is $s_j = v_{j+1}$ where $v_{j+1} = v_0$ if $j = k$. Hence $j = k$; otherwise, the hypotheses of Lemma 5 hold, but not the conclusion. But for the sequence to have gotten to that point we must have had $s_1 = s_2 = \cdots = s_{k-1} = u_0$. Consequently, $[\{u_i, v_i\}, \{u_j, v_j\}] = \emptyset$ for $1 \leq i \leq j-2 \leq k-2$; for otherwise S_{i+1} is not a cutset of G. And because of this, there must be a 2-matching between $\{u_i, v_i\}$ and $\{u_{i+1}, v_{i+1}\}$, $1 \leq i \leq k-1$. Thus G is a ladder as claimed.

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