AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **60(2)** (2014), Pages 227–254

Decompositions of complete 3-uniform hypergraphs into small 3-uniform hypergraphs

Darryn Bryant

School of Mathematics and Physics
The University of Queensland, Qld 4072
Australia
db@maths.uq.edu.au

Sarada Herke

School of Mathematical Sciences
Monash University
Clayton, Vic 3800
Australia
sarada.herke@monash.edu

Barbara Maenhaut

School of Mathematics and Physics
The University of Queensland, Qld 4072
Australia
bmm@maths.uq.edu.au

Wannasiri Wannasit

Mathematics Department
Faculty of Science, Chiang Mai University
Chiang Mai 50200
Thailand
wannasiri.w@cmu.ac.th

Abstract

In this paper we consider the problem of determining all values of v for which there exists a decomposition of the complete 3-uniform hypergraph on v vertices into edge-disjoint copies of a given 3-uniform hypergraph. We solve the problem for each 3-uniform hypergraph having at most three edges and at most six vertices, and for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration.

1 Introduction

Problems concerning the decompositions of graphs into edge-disjoint subgraphs have been well-studied, see for example the surveys [4] and [7]. Formally, a *decomposition* of a graph K is a set $\{G_1, G_2, \ldots, G_s\}$ of subgraphs of K such that $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_s) = E(K)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $1 \le i < j \le s$. If G is a fixed graph and $\mathcal{D} = \{G_1, G_2, \ldots, G_s\}$ is a decomposition such that $G_1 \cong G_2 \cong \cdots \cong G_s \cong G$, then $\mathcal{D}$ is called a *G-decomposition*.

Of special interest is the problem of determining all values of v for which there is a G-decomposition of the complete graph K_v of order v, see the survey [1]. A G-decomposition of K_v is called a *G-design of order v*. If G is a complete graph of order k, then a G-design of order v is an *$S(2, k, v)$-design* or *Steiner system*. An $S(2, k, v)$-design is also known as a *balanced incomplete block design of index* $\lambda = 1$, or $(v, k, 1)$-*BIBD*. Thus, G-designs generalise $S(2, k, v)$-designs.

The notion of G-decompositions of graphs extends naturally to H-decompositions of t-uniform hypergraphs with $t > 2$, and generalises $S(t, k, v)$-designs (a hypergraph is *t-uniform* if each edge is incident with exactly t vertices). A *decomposition* of a hypergraph K is a set $\{H_1, H_2, \ldots, H_s\}$ of subhypergraphs of K such that $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s) = E(K)$ and $E(H_i) \cap E(H_j) = \emptyset$ for $1 \le i < j \le s$. If a hypergraph H_i in a decomposition is isomorphic to a particular hypergraph H, then H_i is called an *H-block*. If $\mathcal{D}$ is a decomposition of K into H-blocks, then $\mathcal{D}$ is called an *H-decomposition* of K, and an H-decomposition of the complete t-uniform hypergraph of order v is called an *H-design of order v*. The problem of determining all v for which there exists an H-design of order v is called the *existence problem for H-designs*.

Clearly, if H is the complete t-uniform hypergraph of order k, then an H-design of order v is equivalent to an $S(t, k, v)$-design. A summary of results on $S(t, k, v)$-designs appears in [10]. Keevash [20] has recently proved the existence conjecture for $S(t, k, v)$-designs, establishing that for given t and k, the obvious necessary conditions are sufficient for the existence of an $S(t, k, v)$-design for all sufficiently large v.

In this paper we are interested in the existence problem for H-designs in cases where H is a small 3-uniform hypergraph. We solve this problem completely for each of the simple 3-uniform hypergraphs having at most three edges and at most six vertices (and no isolated vertices), see Theorems 4.1 and 4.3. Similar results exist in the case of 2-uniform hypergraphs (graphs). The existence of G-designs for each graph G that has at most four vertices was solved in [6]. The existence problem for G-designs where G has five vertices was subsequently studied in [5], and eventually completely solved in [13]. We also solve the existence problem for the 3-uniform hypergraph of order 6 whose four edges form the lines of the Pasch configuration, see Theorem 3.2

Since we shall be concerned only with simple hypergraphs, there will be no confusion if we identify any edge with the set containing its endpoints. Further, we will often describe our hypergraphs by giving their edge set only. Since the hypergraphs we consider will never contain isolated vertices, this is enough to uniquely define them. The complete t-uniform hypergraph with vertex set V has the set of all t-

element subsets of V as its edge set and is denoted by K_V^t. If $v = |V|$, then K_v^t is used to denote any hypergraph isomorphic to K_V^t.

We now give a brief overview of what is already known about the existence problem for H-designs where H is a t-uniform hypergraph ($t \geq 3$), focusing particularly on the case $t = 3$. An $S(3, 4, v)$-design, also known as a *Steiner quadruple system of order* v, is a K_4^3-design of order v. Such designs are known to exist if and only if $v \equiv 2, 4 \pmod 6$ or $v = 1$. This was first proved by Hanani in [18] and several alternative proofs have been given since, see for example [17] and [32]. See [10] for results on the existence of $S(3, k, v)$-designs for $k \geq 5$, which correspond to K_k^3-designs of order v.

In [16], Hanani considered H-designs where H is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron, namely a tetrahedron, an octahedron or an icosahedron. Denote these hypergraphs by T, O and I respectively. The hypergraph T is K_4^3, so T-designs are Steiner quadruple systems and have been discussed above. Hanani [16] completely settled the existence problem for O-designs, and gave necessary conditions for the existence of an I-design. Oriented octahedron designs are considered in [22].

In [11] and [12] the existence problems for $(K_4^3 - e)$-designs and $(K_4^3 + e)$-designs are solved. Here $K_4^3 - e$ denotes the 3-uniform hypergraph obtained by deleting an edge from K_4^3, and $K_4^3 + e$ denotes the hypergraph obtained by adding the edge $\{3, 4, 5\}$ to $K_{\{1,2,3,4\}}^3$. Decompositions of hypergraphs into Hamilton cycles [2, 26] and into hypertrees [31] have also been studied. Baranyai's Theorem [3] yields results on decompositions into hypergraphs, each of which consists of a set of independent edges (see Theorem 2.3).

A *large set* of $S(t, k, v)$-designs is a collection of $S(t, k, v)$-designs that partitions the set of all k-element subsets of the underlying point set of cardinality v. Large sets of $S(t, k, v)$-designs are thus decompositions of K_v^k into hypergraphs, where each hypergraph in the decomposition has the blocks of an $S(t, k, v)$-design as its edge set (however there is, in general, no requirement that these hypergraphs be isomorphic). Large sets of designs have been widely studied, especially for the case $k = 3$. For example, it is known that there is a large set of Steiner triple systems ($S(2, 3, v)$-designs) if and only if $v \equiv 1, 3 \pmod 6$ and $v \neq 7$, see [23, 24, 25, 30]. For other results on large sets of designs see Section 4.4 of [21]. Also see [8], for example, and the references therein.

2 Notation and Preliminaries

Our methods for constructing H-designs involve 3-uniform hypergraph analogues of group divisible designs. To this end we need to define some notation for certain types of "multipartite" hypergraphs. Let $U_1, U_2, \ldots, U_m$ be pairwise disjoint sets. The hypergraph with vertex set $V = U_1 \cup U_2 \cup \cdots \cup U_m$ and edge set consisting of all t-element subsets of V having at most one vertex in each of $U_1, U_2, \ldots, U_m$ is denoted by $K_{U_1,U_2,\ldots,U_m}^t$. If $u_i = |U_i|$ for $i = 1, 2, \ldots, m$, then the notation $K_{u_1,u_2,\ldots,u_m}^t$ denotes any hypergraph that is isomorphic to $K_{U_1,U_2,\ldots,U_m}^t$, and if $u_1 = u_2 = \cdots = u_m = u$, then the notation $K^t_{m[u]}$ may be used instead of $K^t_{u_1,u_2,\dots,u_m}$.

For pairwise disjoint sets $U_1, U_2, \dots, U_m$, the hypergraph with vertex set $V = U_1 \cup U_2 \cup \dots \cup U_m$ and edge set consisting of all t-element subsets of V having at most $t-1$ vertices in each of $U_1, U_2, \dots, U_m$ is denoted by $L^t_{U_1,U_2,\dots,U_m}$. If $u_i = |U_i|$ for $i = 1, 2, \dots, m$, then the notation $L^t_{u_1,u_2,\dots,u_m}$ denotes any hypergraph that is isomorphic to $L^t_{U_1,U_2,\dots,U_m}$, and if $u_1 = u_2 = \dots = u_m = u$, then the notation $L^t_{m[u]}$ may be used instead of $L^t_{u_1,u_2,\dots,u_m}$.

Decompositions of $K^t_{m[u]}$ into K^t_k have been called transverse systems and H designs, see for example [19, 27, 28], and decompositions of $L^t_{m[u]}$ into K^t_k have been called group divisible systems and G designs, see for example [17, 27]. In this paper, when we use the notation H-design (or G-design) we will mean as defined in the introduction. Decompositions of $L^t_{u_1,u_2,\dots,u_m}$ are *candelabra systems* having no stem. Candelabra systems have been used to construct $S(3, k, v)$-designs and other hypergraph decompositions, see for example [12, 19, 29].

If H' is a subhypergraph of H, then $H \setminus H'$ denotes the hypergraph obtained from H by deleting the edges of H'. If H is a hypergraph with vertex set V and $S \subseteq V$, then we define the *degree* of S in H, denoted $\deg_H(S)$, by $\deg_H(S) = |\{e \in E(H) : S \subseteq e\}|$. Note that if $S = \{x\}$, then $\deg_H(\{x\})$ is the degree of the vertex x, and the notation $\deg_H(x)$ may be used rather than $\deg_H(\{x\})$. Note also that $\deg_H(\emptyset) = |E(H)|$.

Lemma 2.1 *Let H and K be hypergraphs. If there exists an H-decomposition of K, then for each subset R of $V(K)$,* $\gcd(\{\deg_H(S) : S \subseteq V(H), |S| = |R|\})$ *divides* $\deg_K(R)$.

Proof: Let $\mathcal{D}$ be an H-decomposition of K. For each $R \subseteq V(K)$, the H-blocks in $\mathcal{D}$ partition the edges of K that contain R. Thus, $\deg_K(R) = \sum_{G\in\mathcal{D}} \deg_G(R)$. Since $\deg_G(R) \in \{\deg_H(S) : S \subseteq V(H), |S| = |R|\}$ for each $G \in \mathcal{D}$, it follows that $\gcd(\{\deg_H(S) : S \subseteq V(H), |S| = |R|\})$ divides $\deg_K(R)$. □

We are specifically interested in H-designs where H is 3-uniform. Applying Lemma 2.1 with $K = K^3_v$ we obtain the following necessary conditions for the existence of an H-design of order v in the case H is 3-uniform. Conditions (1), (2) and (3) are obtained by taking $R = \emptyset$, taking $R = \{x\}$ for some $x \in V(K^3_v)$, and taking $R = \{x, y\}$ for distinct $x, y \in V(K^3_v)$ respectively.

Lemma 2.2 *Let H be a 3-uniform hypergraph. If there exists an H-design of order v, then*

(1) $|E(H)|$ *divides* $\binom{v}{3}$*;*

(2) $\gcd(\{\deg_H(x) : x \in V(H)\})$ *divides* $\binom{v-1}{2}$*; and*

(3) $\gcd(\{\deg_H(\{x, y\}) : x, y \in V(H), x \neq y\})$ *divides* $v - 2$.

The well-known theorem of Baranyai [3] has the following result as an immediate corollary.

Theorem 2.3 ([3]) *Let H be the t-uniform hypergraph consisting of m independent edges. There is an H-design of order v if and only if m divides $\binom{v}{t}$ and $mt \le v$.*

The constructions outlined in the following sections depend on many small examples. These examples are given in the Appendix.

3 P-designs

The hypergraph with vertex set $\{0,1,2,3,4,5\}$ and edge set $\{\{0,1,2\},\{0,4,5\},\{1,3,5\},\{2,3,4\}\}$ is denoted by $[0,1,2,3,4,5]_P$ and the notation P will be used for any hypergraph that is isomorphic to $[0,1,2,3,4,5]_P$. Thus, the edges of P form the lines of the Pasch configuration. In this section we settle the existence problem for P-designs. The existence problem for the 2-uniform hypergraph of order 6 whose edge set consists of all two element subsets of the edges of P, equivalently the edges of the octahedron, was settled in [14]. It is well known that there is no large set of Steiner triple systems of order 7, see Theorem 2.63 in [9], and the following result shows that this implies there is no P-design of order 6.

Lemma 3.1 *There is no P-design of order* 6.

Proof: Suppose $\mathcal{D}$ is a P-decomposition of K_V^3 where $|V| = 6$. It is easy to see that $|\mathcal{D}| = 5$. We say that a pair $\{x,y\}$ of vertices is covered by a P-block $H \in \mathcal{D}$ when $\{x,y\} \subseteq e$ for some $e \in E(H)$. Since $|\mathcal{D}| = 5$, since no block covers any pair more than once, and since each pair $\{x,y\}$ is covered by four P-blocks (one for each vertex other than x and y), there is exactly one P-block in $\mathcal{D}$ which does not cover $\{x,y\}$.

The three uncovered pairs from each P-block of $\mathcal{D}$ form a 1-factor of K_6^2, and it follows from the preceding paragraph that the five 1-factors formed from the five P-blocks of $\mathcal{D}$ constitute a 1-factorisation of K_6^2. This means that if we add a new vertex ∞, and add three new edges to each P-block $H \in \mathcal{D}$, namely $\{\infty, x, y\}$ for each uncovered pair $\{x,y\}$ from H, then the result is a large set of Steiner triple systems of order 7. Since no such set exists (see Theorem 2.63 in [9]), we conclude that there is no P-design of order 6. □

Theorem 3.2 *There is a P-design of order v if and only if $v \equiv 1, 2$ or $6 \pmod 8$; except that there is no P-design of order* 6.

Proof: Conditions (1) and (2) of Lemma 2.2 imply that if there exists a P-design of order v, then 4 divides $\binom{v}{3}$ and 2 divides $\binom{v-1}{2}$. It follows immediately that $v \equiv 1, 2$ or $6 \pmod 8$. For $v = 1$ and $v = 2$, P-designs of order v exist trivially, and Lemma 3.1 rules out the existence of a P-design of order 6.

Let O denote any hypergraph that is isomorphic to the hypergraph with vertex set $\{0,1,2,3,4,5\}$ and edge set $\{\{0,1,2\},\{0,1,5\},\{0,2,4\},\{0,4,5\},\{1,2,3\},\{1,3,5\},\{2,3,4\},\{3,4,5\}\}$. The eight edges of the hypergraph O correspond to the eight faces of an octahedron, where the three pairs of opposite vertices are $\{0,3\}$, $\{1,4\}$ and $\{2,5\}$. It is easy to see that there is a P-decomposition of O;

$$\{[0,1,2,3,4,5]_P, [0,1,5,3,4,2]_P\}$$

is a P-decomposition of the above copy of O. It follows that any hypergraph having an O-decomposition, also has a P-decomposition. In [16] it was shown that there is an O-design of order v if and only if $v \equiv 2 \pmod 8$, and this proves the existence of a P-design of order v for all $v \equiv 2 \pmod 8$.

A P-design of order 9 is given in Example A.1 and a P-design of order 14 is given in Example A.2. It remains to construct a P-design of order v for all $v \equiv 1$ or $6 \pmod 8$ with $v \geq 17$. Let $v = 8n + \epsilon$ where $\epsilon \in \{1,6\}$ and $n \geq 2$. Let $V_1, V_2, \ldots, V_n$ be pairwise disjoint sets, each of cardinality 8, and let $\infty_1, \infty_2, \ldots, \infty_6$ be distinct points, none of which is in $V_1 \cup \cdots \cup V_n$.

For $i = 1, 2, \ldots, n$, let $\mathcal{D}_i$ be a P-decomposition of $K^3_{V_i \cup \{\infty_1\}}$ (which exists by Example A.1) and for $i = 1, 2, \ldots, n-1$, let $\mathcal{D}'_i$ be a P-decomposition of $K^3_{V_i \cup \{\infty_1,\ldots,\infty_6\}} \setminus K^3_{\{\infty_1,\ldots,\infty_6\}}$ (which exists by Example A.3). For $1 \leq i < j \leq n$, let $\mathcal{D}_{i,j}$ be a P-decomposition of $L^3_{V_i,V_j} \cup K^3_{V_i,V_j,\{\infty_1\}}$ (which exists by Example A.4) and let $\mathcal{D}'_{i,j}$ be a P-decomposition of $L^3_{V_i,V_j} \cup K^3_{V_i,V_j,\{\infty_1,\ldots,\infty_6\}}$ (which exists by Example A.5). In [16], Hanani gave an O-decomposition of $K^3_{8,8,8}$ and hence there exists a P-decomposition of $K^3_{8,8,8}$. For $1 \leq i < j < k \leq n$ let $\mathcal{D}_{i,j,k}$ be a P-decomposition of $K^3_{V_i,V_j,V_k}$. Then

$$\left(\bigcup_{1\leq i\leq n} \mathcal{D}_i\right) \cup \left(\bigcup_{1\leq i<j\leq n} \mathcal{D}_{i,j}\right) \cup \left(\bigcup_{1\leq i<j<k\leq n} \mathcal{D}_{i,j,k}\right)$$

is a P-decomposition of $K_{V_1\cup\cdots\cup V_n\cup\{\infty_1\}}$ and

$$\mathcal{D}^* \cup \left(\bigcup_{1\leq i\leq n-1} \mathcal{D}'_i\right) \cup \left(\bigcup_{1\leq i<j\leq n} \mathcal{D}'_{i,j}\right) \cup \left(\bigcup_{1\leq i<j<k\leq n} \mathcal{D}_{i,j,k}\right)$$

is a P-decomposition of $K_{V_1\cup\cdots\cup V_n\cup\{\infty_1,\ldots,\infty_6\}}$, where $\mathcal{D}^*$ is a P-decomposition of $K_{V_n\cup\{\infty_1,\ldots,\infty_6\}}$ (which exists by Example A.2). □

4 H-designs for small H

Up to isomorphism, there are eleven (simple) 3-uniform hypergraphs having order at most 6 and size at most 3 (and no isolated vertices). If H is the 3-uniform hypergraph with a single edge $\{1,2,3\}$, then trivially there is an H-decomposition of any 3-uniform hypergraph. The notation we use for the ten 3-uniform hypergraphs having order at most 6 and either 2 or 3 edges is given in the following table. The first row tells us that $[1,2,3,4,5,6]_{H_{2,1}}$ is the hypergraph with vertex set $V(H_{2,1}) = \{1,2,3,4,5,6\}$ and edge set $E(H_{2,1}) = \{\{1,2,3\},\{4,5,6\}\}$, and that $H_{2,1}$ is used to denote any hypergraph that is isomorphic to $[1,2,3,4,5,6]_{H_{2,1}}$. The other nine rows similarly give the notation for the other nine hypergraphs.

$H_{2,1}$	$[1,2,3,4,5,6]_{H_{2,1}}$	$V(H_{2,1}) = \{1,2,3,4,5,6\}$	$E(H_{2,1}) = \{\{1,2,3\},\{4,5,6\}\}$
$H_{2,2}$	$[1,2,3,4,5]_{H_{2,2}}$	$V(H_{2,2}) = \{1,2,3,4,5\}$	$E(H_{2,2}) = \{\{1,2,3\},\{1,4,5\}\}$
$H_{2,3}$	$[1,2,3,4]_{H_{2,3}}$	$V(H_{2,3}) = \{1,2,3,4\}$	$E(H_{2,3}) = \{\{1,2,3\},\{1,2,4\}\}$
$H_{3,1}$	$[1,2,3,4,5,6]_{H_{3,1}}$	$V(H_{3,1}) = \{1,2,3,4,5,6\}$	$E(H_{3,1}) = \{\{1,2,3\},\{1,2,4\},\{4,5,6\}\}$
$H_{3,2}$	$[1,2,3,4,5,6]_{H_{3,2}}$	$V(H_{3,2}) = \{1,2,3,4,5,6\}$	$E(H_{3,2}) = \{\{1,2,3\},\{1,2,4\},\{2,5,6\}\}$
$H_{3,3}$	$[1,2,3,4,5,6]_{H_{3,3}}$	$V(H_{3,3}) = \{1,2,3,4,5,6\}$	$E(H_{3,3}) = \{\{1,2,3\},\{1,4,5\},\{2,4,6\}\}$
$H_{3,4}$	$[1,2,3,4,5]_{H_{3,4}}$	$V(H_{3,4}) = \{1,2,3,4,5\}$	$E(H_{3,4}) = \{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$
$H_{3,5}$	$[1,2,3,4,5]_{H_{3,5}}$	$V(H_{3,5}) = \{1,2,3,4,5\}$	$E(H_{3,5}) = \{\{1,2,3\},\{1,2,4\},\{1,4,5\}\}$
$H_{3,6}$	$[1,2,3,4,5]_{H_{3,6}}$	$V(H_{3,6}) = \{1,2,3,4,5\}$	$E(H_{3,6}) = \{\{1,2,3\},\{1,2,4\},\{3,4,5\}\}$
$H_{3,7}$	$[1,2,3,4]_{H_{3,7}}$	$V(H_{3,7}) = \{1,2,3,4\}$	$E(H_{3,7}) = \{\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$

The following theorem settles the existence problem for H-designs where H is any one of the three simple 3-uniform hypergraphs with two edges.

Theorem 4.1 *Let H be a simple 3-uniform hypergraph with two edges. There exists an H-design of order v if and only if $v \equiv 0, 1$ or $2 \pmod 4$, $v \neq 4, 5$ if $H \cong H_{2,1}$ and $v \neq 4$ if $H \cong H_{2,2}$.*

Proof: Condition (1) of Lemma 2.2 implies that the condition $v \equiv 0, 1$ or $2 \pmod 4$ is necessary for existence and since $H_{2,1}$ has six vertices and $H_{2,2}$ has five vertices the other restrictions on v are obvious. We now show that these conditions are also sufficient. Let H be a simple 3-uniform hypergraph with two edges. For $v = 1$ and $v = 2$, H-designs of order v exist trivially. If $H \cong H_{2,1}$, then the existence of an H-design of order v for all $v \equiv 0, 1$ or $2 \pmod 4$, $v \geq 6$ follows from Theorem 2.3. Thus, we can assume that $H \cong H_{2,j}$ where $j \in \{2, 3\}$. Since there does not exist an $H_{2,2}$-design of order 4, the case of $H \cong H_{2,2}$ and $v \equiv 0 \pmod 4$ will be treated separately.

Suppose $H \cong H_{2,2}$ and $v \equiv 0 \pmod 4$. An H-design of order 8 is given in Example A.8. Thus we can assume $v \geq 12$. Let $v = 4n$ where $n \geq 3$. Let $V_1, \ldots, V_n$ be pairwise disjoint sets, each of cardinality 4. For $1 \leq i < j < k \leq n$, let $\mathcal{D}_{i,j,k}$ be an H-decomposition of $K^3_{V_i,V_j,V_k}$ (which exists by Example A.14). For $1 \leq i < j \leq n-1$, let $\mathcal{D}_{i,j}$ be an H-decomposition of $L^3_{V_i,V_j}$ (which exists by Example A.11). For $1 \leq i \leq n-2$, let $\mathcal{D}_i$ be an H-decomposition of $K^3_{V_i \cup V_n} \setminus K^3_{V_n}$ (which exists by Example A.10). Finally, let $\mathcal{D}_n$ be an H-decomposition of $K^3_{V_{n-1} \cup V_n}$ (which exists by Example A.8). Then

$$\left(\bigcup_{1\leq i<j<k\leq n} \mathcal{D}_{i,j,k}\right) \cup \left(\bigcup_{1\leq i<j\leq n-1} \mathcal{D}_{i,j}\right) \cup \left(\bigcup_{1\leq i\leq n-2} \mathcal{D}_i\right) \cup \mathcal{D}_n$$

is the required H-decomposition of $K_{V_1\cup\cdots\cup V_n}$.

Suppose then that $H \cong H_{2,2}$ with $v \equiv 1, 2 \pmod 4$, or $H \cong H_{2,3}$ with $v \equiv 0, 1$ or $2 \pmod 4$. An H-design of order 5 is given in Example A.6, an H-design of order 6 is given in Example A.7, and if $H \cong H_{2,3}$ then an H-design of order 4 is given in Example A.9. Thus we can assume $v \geq 8$. Let $v = 4n + \epsilon$ where $\epsilon \in \{0, 1, 2\}$

and $n \geq 2$. Let $V_1, \ldots, V_n$ be pairwise vertex-disjoint sets, each of cardinality 4. For $1 \leq i < j < k \leq n$, let $\mathcal{D}_{i,j,k}$ be an H-decomposition of $K^3_{V_i,V_j,V_k}$ (which exists by Example A.14). For $\epsilon = 0, 1, 2$ respectively and for $1 \leq i < j \leq n$, let $\mathcal{D}^{\epsilon}_{i,j}$ be an H-decomposition of $L^3_{V_i,V_j}$, $L^3_{V_i,V_j} \cup K^3_{V_i,V_j,\{\infty_1\}}$, $L^3_{V_i,V_j} \cup K^3_{V_i,V_j,\{\infty_1,\infty_2\}}$ respectively (these decompositions exist by Examples A.11, A.12 and A.13). For $\epsilon = 0, 1, 2$ respectively and for $1 \leq i \leq n$, let $\mathcal{D}^{\epsilon}_i$ be an H-decomposition of $K^3_{V_i}$, $K^3_{V_i \cup \{\infty_1\}}$, $K^3_{V_i \cup \{\infty_1,\infty_2\}}$ respectively (these decompositions exist by Examples A.9, A.6 and A.7). Then

$$\left(\bigcup_{1 \leq i<j<k \leq n} \mathcal{D}_{i,j,k} \right) \cup \left(\bigcup_{1 \leq i<j \leq n} \mathcal{D}^{\epsilon}_{i,j} \right) \cup \left(\bigcup_{1 \leq i \leq n} \mathcal{D}^{\epsilon}_{i} \right)$$

is the required H-decomposition of $K_{V_1 \cup \cdots \cup V_n \cup S_\infty}$ where $S_\infty = \emptyset, \{\infty_1\}, \{\infty_1, \infty_2\}$ respectively when $\epsilon = 0, 1, 2$ respectively. □

The next lemma is needed in the proof of Theorem 4.3 which follows.

Lemma 4.2 *Let $K^{2,3}_n$ denote any hypergraph that is isomorphic to the hypergraph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{S : S \subseteq \{1, 2, \ldots, n\}, |S| \in \{2, 3\}\}$, let $[1, 2, 3]_A$ denote the hypergraph with vertex set $V(A) = \{1, 2, 3\}$ and edge set $E(A) = \{\{1, 2\}, \{1, 2, 3\}\}$, and let A denote any hypergraph that is isomorphic to $[1, 2, 3]_A$. For each positive integer n, there is a decomposition of $K^{2,3}_n$ in which each block is isomorphic to K^3_4, K^2_2 or A.*

Proof: Let $N = \{1, 2, \ldots, n\}$ and recall that there exists a K^3_4-decomposition of K^3_n (Steiner quadruple system of order n) for all $n \equiv 2, 4 \pmod 6$. It is sufficient to show that for each n, there is a decomposition $\mathcal{D}$ of some K, where $K^3_N \subseteq K \subseteq K^{2,3}_N$ and each block of $\mathcal{D}$ is isomorphic to either K^3_4 or A. To see this, observe that the required decomposition can be obtained from any such $\mathcal{D}$ simply by adding the blocks of a K^2_2-decomposition of $K^{2,3}_N \setminus K$ (which clearly exists because $K^3_N \subseteq K$ implies $K^{2,3}_N \setminus K$ is 2-uniform).

If $n \equiv 2, 4 \pmod 6$, then a K^3_4-decomposition of K^3_N suffices for $\mathcal{D}$. If $n \equiv 3, 5 \pmod 6$, then let $\mathcal{D}_1$ be a K^3_4-decomposition of $K^3_{\{1,2,\ldots,n-1\}}$ and let

$$\mathcal{D}_2 = \{[x, y, n]_A : 1 \leq x < y \leq n - 1\}.$$

Then $\mathcal{D}_1 \cup \mathcal{D}_2$ is the required decomposition $\mathcal{D}$.

If $n \equiv 1 \pmod 6$, then let $\mathcal{D}_0$ be a K^3_4-decomposition of $K^3_{\{1,2,\ldots,n+1\}}$ and let $\mathcal{D}_1 = \{H : H \in \mathcal{D}_0, n + 1 \notin V(H)\}$. Note that

$$\mathcal{D}_2 = \{K^2_{\{x,y,z\}} : K^3_{\{n+1,x,y,z\}} \in \mathcal{D}_0\}$$

is a K^2_3-decomposition of K^2_n (a Steiner triple system of order n). Thus, we can let

$$\mathcal{D}_3 = \{[x, y, z]_A : K^2_{\{x,y,z\}} \in \mathcal{D}_2, x < y < z\}$$

and $\mathcal{D}_1 \cup \mathcal{D}_3$ is the required decomposition $\mathcal{D}$.

If $n \equiv 0 \pmod 6$, then let $\mathcal{D}_0$ be a K_4^3-decomposition of $K^3_{\{1,2,\dots,n+2\}}$, let

$$\mathcal{D}_1 = \{K^3_{\{w,x,y,z\}} \in \mathcal{D}_0 : \{w,x,y,z\} \subseteq \{1,2,\dots,n\}\}$$

and let

$$\mathcal{D}_2 = \{K^3_{\{x,y,z\}} : K^3_{\{w,x,y,z\}} \in \mathcal{D}_0, w \in \{n+1,n+2\}, \{x,y,z\} \subseteq \{1,2,\dots,n\}\}.$$

Thus, $\mathcal{D}_1 \cup \mathcal{D}_2$ is a decomposition of K_N^3 in which each block is either K_4^3 (if the block is in $\mathcal{D}_1$) or K_3^3 (if the block is in $\mathcal{D}_2$). To each block $G \in \mathcal{D}_2$ we add an edge $\{x,y\}$, where $x,y \in V(G)$, to obtain a set $\mathcal{D}_3$ of A-blocks. If we can do this so that the added edges are pairwise distinct, then the union $\mathcal{D}_1 \cup \mathcal{D}_3$ is the required decomposition $\mathcal{D}$. We now show that this is indeed possible.

For each block $K^3_{\{x,y,z\}} \in \mathcal{D}_2$, any one of the three edges $\{x,y\}$, $\{x,z\}$ or $\{y,z\}$ can be added. Also, if $x,y \in \{1,2,\dots,n\}$ are distinct, then there are at most two blocks of $\mathcal{D}_2$ to which the edge $\{x,y\}$ can be added (otherwise either $\{x,y,n+1\}$ or $\{x,y,n+2\}$ would occur in more than one block of $\mathcal{D}_0$). It thus follows from Hall's Marriage Theorem [15] that the required selection of pairwise distinct edges exists. □

We are now ready to give Theorem 4.3 which settles the existence problem for H-designs when H is any one of the seven simple 3-uniform hypergraphs that have three edges and at most six vertices.

Theorem 4.3 *Let H be a simple 3-uniform hypergraph with three edges and at most six vertices. There exists an H-design of order v if and only if $v \equiv 0, 1$ or $2 \pmod 9$.*

Proof: Condition (1) of Lemma 2.2 implies that the condition $v \equiv 0, 1$ or $2 \pmod 9$ is necessary for existence. We now show that it is also sufficient. Let H be a simple 3-uniform hypergraph with three edges and at most six vertices. If $H \cong H_{3,7}$, then the existence of an H-design of order v for all $v \equiv 0, 1$ or $2 \pmod 9$ is proved in [11]. Thus, we can assume that $H \cong H_{3,j}$ where $j \in \{1,2,3,4,5,6\}$.

For $v = 1$ and $v = 2$, H-designs of order v exist trivially, and H-designs of orders 9, 10 and 11 are given in Examples A.15, A.16 and A.17. Thus we can assume $v \geq 18$. Let $v = 9n + \epsilon$ where $\epsilon \in \{0,1,2\}$ and $n \geq 2$. Let $V_1,\dots,V_n$ be pairwise disjoint sets, each of cardinality 9. The hypergraph $H_{3,6}$ differs from $H_{3,1},\dots,H_{3,5}$ in that $H_{3,6}$ is not 3-colourable (there is no function $\gamma : V(H_{3,6}) \mapsto \{c_1,c_2,c_3\}$ such that the three endpoints of each edge of $H_{3,6}$ are assigned c_1, c_2 and c_3). This means that there is no $H_{3,6}$-decomposition of $K^3_{3[9]}$, and hence it will be treated separately.

First let $H \in \{H_{3,1},\dots,H_{3,5}\}$. For $1 \leq i < j < k \leq n$, let $\mathcal{D}_{i,j,k}$ be an H-decomposition of $K^3_{V_i,V_j,V_k}$ (which exists by Example A.21). For $\epsilon = 0,1,2$ respectively and for $1 \leq i < j \leq n$, let $\mathcal{D}^{\epsilon}_{i,j}$ be an H-decomposition of $L^3_{V_i,V_j}$, $L^3_{V_i,V_j} \cup K^3_{V_i,V_j,\{\infty_1\}}$, $L^3_{V_i,V_j} \cup K^3_{V_i,V_j,\{\infty_1,\infty_2\}}$ respectively (these decompositions exist by Examples A.18, A.19 and A.20). For $\epsilon = 0,1,2$ respectively and for $1 \leq i \leq n$, let $\mathcal{D}_i^\epsilon$ be an H-decomposition of $K^3_{V_i}$, $K^3_{V_i\cup\{\infty_1\}}$, $K^3_{V_i\cup\{\infty_1,\infty_2\}}$ respectively (these decompositions exist by Examples A.15, A.16 and A.17). Then

$$\left(\bigcup_{1\le i<j<k\le n}\mathcal{D}_{i,j,k}\right)\cup\left(\bigcup_{1\le i<j\le n}\mathcal{D}^\epsilon_{i,j}\right)\cup\left(\bigcup_{1\le i\le n}\mathcal{D}^\epsilon_{i}\right)$$

is the required H-decomposition of $K_{V_1\cup\cdots\cup V_n\cup S_\infty}$, where $S_\infty=\emptyset,\{\infty_1\},\{\infty_1,\infty_2\}$ respectively when $\epsilon=0,1,2$ respectively.

Now let $H\cong H_{3,6}$, define A as in Lemma 4.2, and let $\mathcal{D}=\mathcal{D}_{K_4^3}\cup\mathcal{D}_{K_2^2}\cup\mathcal{D}_A$ be a decomposition of $K^{2,3}_{\{1,\ldots,n\}}$ in which each block of $\mathcal{D}_{K_4^3}$ is isomorphic to K_4^3, each block of $\mathcal{D}_{K_2^2}$ is isomorphic to K_2^2 and each block of $\mathcal{D}_A$ is isomorphic to A. The decomposition $\mathcal{D}$ exists by Lemma 4.2.

For each $K^3_{\{i,j,k,l\}}\in\mathcal{D}_{K_4^3}$, let $\mathcal{D}_{i,j,k,l}$ be an H-decomposition of $K^3_{V_i,V_j,V_k,V_l}$ (which exists by Example A.25). For $\epsilon=0,1,2$ respectively and for each $[i,j,k]_A\in\mathcal{D}_A$, let $\mathcal{D}^\epsilon_{i,j,k}$ be an H-decomposition of $K^3_{V_i,V_j,V_k}\cup L^3_{V_i,V_j}$, $K^3_{V_i,V_j,V_k}\cup L^3_{V_i,V_j}\cup K^3_{V_i,V_j,\{\infty_1\}}$, $K^3_{V_i,V_j,V_k}\cup L^3_{V_i,V_j}\cup K^3_{V_i,V_j,\{\infty_1,\infty_2\}}$ respectively (these decompositions exist by Examples A.22, A.23 and A.24). For $\epsilon=0,1,2$ respectively and for each $K^2_{\{i,j\}}\in\mathcal{D}_{K_2^2}$, let $\mathcal{D}^\epsilon_{i,j}$ be an H-decomposition of $L^3_{V_i,V_j}$, $L^3_{V_i,V_j}\cup K^3_{V_i,V_j,\{\infty_1\}}$, $L^3_{V_i,V_j}\cup K^3_{V_i,V_j,\{\infty_1,\infty_2\}}$ respectively (these decompositions exist by Examples A.18, A.19 and A.20). For $\epsilon=0,1,2$ respectively and for $1\le i\le n$, let $\mathcal{D}^\epsilon_i$ be an H-decomposition of $K^3_{V_i}$, $K^3_{V_i\cup\{\infty_1\}}$, $K^3_{V_i\cup\{\infty_1,\infty_2\}}$ respectively (which exist by Examples A.15, A.16 and A.17). Then

$$\left(\bigcup_{K^3_{\{i,j,k,l\}}\in\mathcal{D}_{K_4^3}}\mathcal{D}_{i,j,k,l}\right)\cup\left(\bigcup_{[i,j,k]_A\in\mathcal{D}_A}\mathcal{D}^\epsilon_{i,j,k}\right)\cup\left(\bigcup_{K^2_{\{i,j\}}\in\mathcal{D}_{K_2^2}}\mathcal{D}^\epsilon_{i,j}\right)\cup\left(\bigcup_{1\le i\le n}\mathcal{D}^\epsilon_{i}\right)$$

is the required H-decomposition of $K_{V_1\cup\cdots\cup V_n\cup S_\infty}$ where $S_\infty=\emptyset,\{\infty_1\},\{\infty_1,\infty_2\}$ respectively when $\epsilon=0,1,2$ respectively. □

Acknowledgements

The first and third authors acknowledge the support of the Australian Research Council. The second author was a PhD student in the School of Mathematics and Physics at The University of Queensland when this research was carried out. The fourth author was visiting the School of Mathematics and Physics at The University of Queensland when this research was carried out, supported by Chiang Mai University.

Appendix

A Example Decompositions

Example A.1 *A P-design of order 9:* The union of the orbits of the following P-blocks under the permutation $(0\ 1\ 2\ 3\ 4\ 5\ 6)(\infty_1)(\infty_2)$ is a P-design of order 9 with point set $\{0, 1, 2, 3, 4, 5, 6, \infty_1, \infty_2\}$.

$$[0, 1, 3, \infty_1, 4, 6]_P \quad [0, 1, 6, \infty_1, 3, \infty_2]_P \quad [0, 2, 5, 3, \infty_2, 6]_P$$

Example A.2 *A P-design of order* 14*:* The union of the orbits of the following P-blocks under the permutation $(0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12)(\infty)$ is a P-design of order 14 with point set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \infty\}$.

$$[0, 1, 2, 4, 6, 9]_P \quad [0, 1, 11, 6, 9, 3]_P \quad [0, 1, 9, 11, 4, 12]_P \quad [0, 1, 8, 4, 5, 11]_P$$

$$[0, 1, 7, 5, \infty, 12]_P \quad [0, 1, 6, \infty, 9, 5]_P \quad [0, 1, 10, \infty, 4, 6]_P$$

Example A.3 *A P-decomposition of* $K^3_{14} \setminus K^3_6$*:* Let $U = \{u_0, \ldots, u_5\}$ and $V = \{v_0, \ldots, v_7\}$ where $U \cap V = \emptyset$. The union of the following sets of P-blocks form a P-decomposition of $K^3_{V\cup U} \setminus K^3_U$. The orbits of

$$[v_0, v_1, v_2, v_4, v_7, v_5]_P \ [v_0, v_3, v_7, v_4, u_0, u_3]_P \ [v_0, v_1, v_3, v_4, u_4, u_1]_P \ [v_0, v_2, v_6, v_5, u_5, u_2]_P$$

under the permutation $(v_0\ v_1\ \cdots\ v_7)(u_0)(u_1)(u_2)(u_3)(u_4)(u_5)$, the orbits of

$$\begin{array}{lll} [v_0, u_0, u_1, v_1, u_3, u_5]_P & [v_2, u_0, u_1, v_3, u_3, u_5]_P & [v_4, u_0, u_1, v_5, u_3, u_5]_P \\ [v_6, u_0, u_1, v_7, u_3, u_5]_P & & \end{array}$$

under the permutation $(v_0)(v_1)(v_2)(v_3)(v_4)(v_5)(v_6)(v_7)(u_0\ u_1\ \cdots\ u_5)$, and the orbits of

$$\begin{array}{lll} [u_0, v_0, v_1, u_3, v_4, v_5]_P & [u_0, v_1, v_2, u_3, v_5, v_6]_P & [u_0, v_2, v_3, u_3, v_6, v_7]_P \\ [u_0, v_3, v_4, u_3, v_7, v_0]_P & [u_0, v_0, v_2, u_3, v_6, v_4]_P & [u_0, v_1, v_3, u_3, v_7, v_5]_P \\ [u_0, v_0, v_4, u_3, v_2, v_6]_P & [u_0, v_1, v_5, u_3, v_3, v_7]_P & [u_0, v_0, v_6, u_3, v_4, v_2]_P \\ [u_0, v_1, v_7, u_3, v_5, v_3]_P & & \end{array}$$

under the permutation $(v_0)(v_1)(v_2)(v_3)(v_4)(v_5)(v_6)(v_7)(u_0\ u_1\ u_2)(u_3\ u_4\ u_5)$.

Example A.4 *A P-decomposition of* $K^3_{\{0,2,\cdots,14\},\{1,3,\cdots,15\},\{\infty\}} \cup L^3_{\{0,2,\cdots,14\},\{1,3,\cdots,15\}}$*:* The union of the orbits of the following P-blocks under the permutation $(0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15)(\infty)$ form a P-decomposition of $K^3_{\{0,2,\cdots,14\},\{1,3,\cdots,15\},\{\infty\}} \cup L^3_{\{0,2,\cdots,14\},\{1,3,\cdots,15\}}$.

$$[1, 2, 7, \infty, 6, 5]_P \quad [0, 2, 3, 12, \infty, 11]_P \quad [0, 1, 2, 11, 3, 14]_P \quad [0, 1, 3, 8, 13, 6]_P$$

$$[1, 4, 13, 2, 10, 9]_P \quad [0, 3, 7, 8, 4, 15]_P \quad [0, 1, 4, 7, 15, 10]_P \quad [0, 7, 12, 5, 11, 14]_P$$

Example A.5 *A P-decomposition of* $K^3_{\mathbb{Z}_8\times\{0\},\ \mathbb{Z}_8\times\{1\},\ \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\infty_6\}} \cup L^3_{\mathbb{Z}_8\times\{0\},\ \mathbb{Z}_8\times\{1\}}$*:* The union of the orbits of the following P-blocks under the permutation $(i, j) \to (i + 1, j)$ for $i \in \mathbb{Z}_8,\ j \in \mathbb{Z}_2$ and $(\infty_k \to \infty_k)$ for $k = \{1, 2, 3, 4, 5, 6\}$ form a P-decomposition of $K^3_{\mathbb{Z}_8\times\{0\},\ \mathbb{Z}_8\times\{1\},\ \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\infty_6\}} \cup L^3_{\mathbb{Z}_8\times\{0\},\ \mathbb{Z}_8\times\{1\}}$.

$$[(0, 0), (0, 1), \infty_1, (1, 0), (2, 1), \infty_2]_P \quad [(0, 0), (2, 1), \infty_1, (1, 0), (4, 1), \infty_2]_P$$

$$[(0,0),(4,1),\infty_1,(1,0),(6,1),\infty_2]_P \quad [(0,0),(6,1),\infty_1,(1,0),(0,1),\infty_2]_P$$
$$[(0,0),(0,1),\infty_3,(1,0),(2,1),\infty_4]_P \quad [(0,0),(2,1),\infty_3,(1,0),(4,1),\infty_4]_P$$
$$[(0,0),(4,1),\infty_3,(1,0),(6,1),\infty_4]_P \quad [(0,0),(6,1),\infty_3,(1,0),(0,1),\infty_4]_P$$
$$[(0,0),(0,1),\infty_5,(1,0),(2,1),\infty_6]_P \quad [(0,0),(2,1),\infty_5,(1,0),(4,1),\infty_6]_P$$
$$[(0,0),(4,1),\infty_5,(1,0),(6,1),\infty_6]_P \quad [(0,0),(6,1),\infty_5,(1,0),(0,1),\infty_6]_P$$
$$[(0,0),(1,0),(0,1),(3,0),(1,1),(2,1)]_P \quad [(0,1),(1,1),(0,0),(3,1),(1,0),(2,0)]_P$$
$$[(0,0),(1,0),(1,1),(3,0),(3,1),(4,1)]_P \quad [(0,1),(1,1),(1,0),(3,1),(3,0),(4,0)]_P$$
$$[(0,0),(3,0),(1,1),(5,1),(4,0),(0,1)]_P \quad [(0,1),(3,1),(1,0),(5,0),(4,1),(0,0)]_P$$
$$[(0,0),(4,0),(2,1),(7,0),(5,1),(3,1)]_P \quad [(0,1),(4,1),(2,0),(7,1),(5,0),(3,0)]_P$$
$$[(0,0),(3,0),(3,1),(5,0),(6,1),(7,1)]_P \quad [(0,1),(3,1),(3,0),(5,1),(6,0),(7,0)]_P$$
$$[(0,0),(3,0),(0,1),(7,0),(2,1),(4,1)]_P \quad [(0,1),(3,1),(0,0),(7,1),(2,0),(4,0)]_P$$
$$[(0,0),(1,0),(6,1),(4,0),(4,1),(5,1)]_P \quad [(0,1),(1,1),(6,0),(4,1),(4,0),(5,0)]_P$$

Example A.6 *An $H_{2,i}$-design of order 5 for $i \in \{2,3\}$:* In each case the orbit of the given $H_{2,i}$-block under the action of $\mathbb{Z}_5$ is an $H_{2,i}$-design of order 5 with point set $\mathbb{Z}_5$.

$H_{2,2}$	$[0,1,4,2,3]_{H_{2,2}}$
$H_{2,3}$	$[0,1,2,3]_{H_{2,3}}$

Example A.7 *An $H_{2,i}$-design of order 6 for $i \in \{2,3\}$:* In each case the union of the orbits of the given $H_{2,i}$-blocks under the permutation $(0\ 1\ 2\ 3\ 4)(\infty)$ is an $H_{2,i}$-design of order 6 with point set $\{0,1,2,3,4,\infty\}$.

$H_{2,2}$	$[0,1,4,2,3]_{H_{2,2}}$ $[\infty,0,1,2,4]_{H_{2,2}}$
$H_{2,3}$	$[0,1,2,3]_{H_{2,3}}$ $[\infty,0,1,2]_{H_{2,3}}$

Example A.8 *An $H_{2,2}$-design of order 8:* The union of the orbits of the following $H_{2,2}$-blocks under the permutation $(0\ 1\ 2\ 3\ 4\ 5\ 6)(\infty)$ is an $H_{2,2}$-design of order 8 with point set $\{0,1,2,3,4,5,6,\infty\}$.

$$[2,0,1,4,6]_{H_{2,2}} \quad [1,0,3,2,6]_{H_{2,2}} \quad [4,0,1,\infty,5]_{H_{2,2}} \quad [\infty,0,2,3,6]_{H_{2,2}}$$

Example A.9 *An $H_{2,3}$-design of order 4:*
The union of $[0,1,2,3]_{H_{2,3}}$ and $[2,3,0,1]_{H_{2,3}}$ is an $H_{2,3}$-design of order 4 with point set $\{0,1,2,3\}$.

Example A.10 *An $H_{2,2}$-decomposition of $K_8^3 \setminus K_4^3$:* Let $U = \{u_0, \ldots, u_3\}$ and $V = \{v_0, \ldots, v_3\}$ where $U \cap V = \emptyset$. The union of the following sets of $H_{2,2}$-blocks form an $H_{2,2}$-decomposition of $K_{U\cup V}^3 \setminus K_V^3$. The orbits of

$$[v_0, v_1, u_2, u_0, u_1]_{H_{2,2}} \quad [v_1, v_2, u_2, u_0, u_1]_{H_{2,2}} \quad [v_2, v_0, u_2, u_0, u_1]_{H_{2,2}}$$
$$[v_3, v_0, u_2, u_0, u_1]_{H_{2,2}} \quad [v_3, v_2, u_2, v_1, u_0]_{H_{2,2}}$$

under the permutation $(u_0, u_1, u_2, u_3)(v_0)(v_1)(v_2)(v_3)$, together with the following set of $H_{2,2}$-blocks

$$\{[u_1, u_0, u_2, u_3, v_0]_{H_{2,2}} \quad [u_2, u_1, u_3, u_0, v_0]_{H_{2,2}} \quad [u_3, u_0, u_2, u_1, v_1]_{H_{2,2}}$$
$$[u_0, u_1, u_3, u_2, v_1]_{H_{2,2}} \quad [v_2, u_0, u_2, u_1, u_3]_{H_{2,2}} \quad [v_3, u_1, u_3, u_0, u_2]_{H_{2,2}}\}.$$

Example A.11 *An $H_{2,i}$-decomposition of $L^3_{\mathbb{Z}_4\times\{0\},\ \mathbb{Z}_4\times\{1\}}$ for $i \in \{2,3\}$:* In each case the union of the orbits of the given $H_{2,i}$-blocks under the permutation $(j,k) \to (j+1,k)$ for $j \in \mathbb{Z}_4, k \in \mathbb{Z}_2$ form an $H_{2,i}$-decomposition of $L^3_{\mathbb{Z}_4\times\{0\},\ \mathbb{Z}_4\times\{1\}}$ for $i \in \{2,3\}$.

$H_{2,2}$	$[(3,1),(0,0),(1,0),(2,0),(3,0)]_{H_{2,2}}$	$[(3,0),(0,1),(1,1),(2,1),(3,1)]_{H_{2,2}}$
	$[(0,1),(0,0),(1,0),(2,0),(3,0)]_{H_{2,2}}$	$[(0,0),(0,1),(1,1),(2,1),(3,1)]_{H_{2,2}}$
	$[(0,1),(0,0),(2,0),(1,0),(3,0)]_{H_{2,2}}$	$[(0,0),(0,1),(2,1),(1,1),(3,1)]_{H_{2,2}}$
$H_{2,3}$	$[(0,0),(1,0),(0,1),(1,1)]_{H_{2,3}}$	$[(0,1),(1,1),(0,0),(1,0)]_{H_{2,3}}$
	$[(0,0),(1,0),(2,1),(3,1)]_{H_{2,3}}$	$[(0,1),(1,1),(2,0),(3,0)]_{H_{2,3}}$
	$[(0,0),(2,0),(0,1),(1,1)]_{H_{2,3}}$	$[(0,1),(2,1),(0,0),(1,0)]_{H_{2,3}}$

Example A.12 *An $H_{2,i}$-decomposition of $K^3_{\mathbb{Z}_4\times\{0\},\ \mathbb{Z}_4\times\{1\},\{\infty\}}$ for $i \in \{2,3\}$:* In each case the union of the orbits of the given $H_{2,i}$-blocks under the permutation $(j,k) \to (j+1,k)$ for $j \in \mathbb{Z}_4, k \in \mathbb{Z}_2$ and $\infty \to \infty$ form an $H_{2,i}$-decomposition of $K^3_{\mathbb{Z}_4\times\{0\},\ \mathbb{Z}_4\times\{1\},\{\infty\}}$ for $i \in \{2,3\}$.

$H_{2,2}$	$[\infty,(0,0),(0,1),(1,0),(2,1)]_{H_{2,2}}$	$[\infty,(0,0),(2,1),(1,0),(0,1)]_{H_{2,2}}$
$H_{2,3}$	$[\infty,(0,0),(0,1),(1,1)]_{H_{2,3}}$	$[\infty,(0,0),(2,1),(3,1)]_{H_{2,3}}$

Example A.13 *An $H_{2,i}$-decomposition of $K^3_{\mathbb{Z}_4\times\{0\},\ \mathbb{Z}_4\times\{1\},\{\infty_1,\infty_2\}}$ for $i \in \{2,3\}$:* In each case the union of the orbits of the given $H_{2,i}$-blocks under the permutation $(j,k) \to (j+1,k)$ for $j \in \mathbb{Z}_4, k \in \mathbb{Z}_2$ and $\infty_l \to \infty_l$ for $l = \{1,2\}$ form an $H_{2,i}$-decomposition of $K^3_{\mathbb{Z}_4\times\{0\},\ \mathbb{Z}_4\times\{1\},\{\infty_1,\infty_2\}}$ for $i \in \{2,3\}$.

$H_{2,2}$	$[\infty_1,(0,0),(0,1),(1,0),(2,1)]_{H_{2,2}}$	$[\infty_1,(0,0),(2,1),(1,0),(0,1)]_{H_{2,2}}$
	$[\infty_2,(0,0),(0,1),(1,0),(2,1)]_{H_{2,2}}$	$[\infty_2,(0,0),(2,1),(1,0),(0,1)]_{H_{2,2}}$
$H_{2,3}$	$[\infty_1,(0,0),(0,1),(1,1)]_{H_{2,3}}$	$[\infty_1,(0,0),(2,1),(3,1)]_{H_{2,3}}$
	$[\infty_2,(0,0),(0,1),(1,1)]_{H_{2,3}}$	$[\infty_2,(0,0),(2,1),(3,1)]_{H_{2,3}}$

Example A.14 *An $H_{2,i}$-decomposition of $K_{\mathbb{Z}_4\times\{0\},\ \mathbb{Z}_4\times\{1\},\ \mathbb{Z}_4\times\{2\}}$ for $i \in \{2,3\}$:* In each case the union of the orbits of the given $H_{2,i}$-blocks under the permutation $(j,k) \to (j+1,k)$ for $j \in \mathbb{Z}_4, k \in \mathbb{Z}_3$ form an $H_{2,i}$-decomposition of $K^3_{\mathbb{Z}_4\times\{0\},\ \mathbb{Z}_4\times\{1\},\ \mathbb{Z}_4\times\{2\}}$ for $i \in \{2,3\}$.

$H_{2,2}$	$[(0,0),(0,1),(0,2),(1,1),(1,2)]_{H_{2,2}}$	$[(0,0),(0,1),(1,2),(1,1),(0,2)]_{H_{2,2}}$
	$[(0,0),(0,1),(2,2),(1,1),(3,2)]_{H_{2,2}}$	$[(0,0),(0,1),(3,2),(1,1),(2,2)]_{H_{2,2}}$
	$[(0,0),(2,1),(0,2),(3,1),(1,2)]_{H_{2,2}}$	$[(0,0),(2,1),(1,2),(3,1),(0,2)]_{H_{2,2}}$
	$[(0,0),(2,1),(2,2),(3,1),(3,2)]_{H_{2,2}}$	$[(0,0),(2,1),(3,2),(3,1),(2,2)]_{H_{2,2}}$
$H_{2,3}$	$[(0,0),(0,1),(0,2),(1,2)]_{H_{2,3}}$	$[(0,0),(0,1),(2,2),(3,2)]_{H_{2,3}}$
	$[(0,0),(1,1),(0,2),(1,2)]_{H_{2,3}}$	$[(0,0),(1,1),(2,2),(3,2)]_{H_{2,3}}$
	$[(0,0),(2,1),(0,2),(1,2)]_{H_{2,3}}$	$[(0,0),(2,1),(2,2),(3,2)]_{H_{2,3}}$
	$[(0,0),(3,1),(0,2),(1,2)]_{H_{2,3}}$	$[(0,0),(3,1),(2,2),(3,2)]_{H_{2,3}}$

Example A.15 *An $H_{3,i}$-design of order* 9 *for* $i \in \{1,2,3,4,5,6\}$*:* In each case the union of the orbits of the given $H_{3,i}$-blocks under the permutation $(0\ 1\ 2\ 3\ 4\ 5\ 6)(\infty_1)(\infty_2)$ is an $H_{3,i}$-design of order 9 with point set $\{0,1,2,3,4,5,6,\infty_1,\infty_2\}$.

$H_{3,1}$	$[0,1,\infty_2,\infty_1,3,5]_{H_{3,1}}$	$[0,3,\infty_1,\infty_2,4,6]_{H_{3,1}}$	$[0,1,3,2,4,6]_{H_{3,1}}$	$[0,1,5,4,\infty_1,\infty_2]_{H_{3,1}}$
$H_{3,2}$	$[0,1,\infty_1,\infty_2,2,3]_{H_{3,2}}$	$[2,0,\infty_1,\infty_2,4,5]_{H_{3,2}}$	$[3,0,\infty_1,\infty_2,2,4]_{H_{3,2}}$	$[0,1,4,5,\infty_1,\infty_2]_{H_{3,2}}$
$H_{3,3}$	$[\infty_1,1,0,2,4,3]_{H_{3,3}}$	$[0,1,\infty_2,4,5,6]_{H_{3,3}}$	$[0,\infty_2,3,\infty_1,4,1]_{H_{3,3}}$	$[1,4,0,6,2,\infty_2]_{H_{3,3}}$
$H_{3,4}$	$[0,2,1,4,\infty_1]_{H_{3,4}}$	$[0,1,3,4,5]_{H_{3,4}}$	$[0,\infty_1,1,3,\infty_2]_{H_{3,4}}$	$[\infty_2,0,1,2,3]_{H_{3,4}}$
$H_{3,5}$	$[1,0,3,2,\infty_1]_{H_{3,5}}$	$[0,1,5,4,2]_{H_{3,5}}$	$[\infty_2,0,1,2,5]_{H_{3,5}}$	$[\infty_1,0,2,3,\infty_2]_{H_{3,5}}$
$H_{3,6}$	$[0,1,2,4,6]_{H_{3,6}}$	$[0,1,3,5,\infty_1]_{H_{3,6}}$	$[0,3,\infty_1,\infty_2,4]_{H_{3,6}}$	$[\infty_2,1,0,6,\infty_1]_{H_{3,6}}$

Example A.16 *An $H_{3,i}$-design of order* 10 *for* $i \in \{1,2,3,4,5,6\}$*:* In each case the union of the orbits of the given $H_{3,i}$-blocks under the action of $\mathbb{Z}_{10}$ is an $H_{3,i}$-design of order 10 with point set $\mathbb{Z}_{10}$.

$H_{3,1}$	$[0,1,2,3,6,9]_{H_{3,1}}$	$[0,1,4,5,2,7]_{H_{3,1}}$	$[0,1,6,7,8,5]_{H_{3,1}}$	$[0,2,4,5,1,7]_{H_{3,1}}$
$H_{3,2}$	$[0,1,2,3,4,7]_{H_{3,2}}$	$[1,0,4,5,2,7]_{H_{3,2}}$	$[1,0,6,7,4,8]_{H_{3,2}}$	$[2,0,4,5,1,8]_{H_{3,2}}$
$H_{3,3}$	$[2,0,1,5,3,7]_{H_{3,3}}$	$[1,0,4,6,2,8]_{H_{3,3}}$	$[1,6,0,8,2,3]_{H_{3,3}}$	$[1,0,8,3,7,6]_{H_{3,3}}$
$H_{3,4}$	$[0,1,2,4,5]_{H_{3,4}}$	$[0,1,6,7,8]_{H_{3,4}}$	$[0,2,4,6,7]_{H_{3,4}}$	$[0,3,1,6,8]_{H_{3,4}}$
$H_{3,5}$	$[0,3,1,6,8]_{H_{3,5}}$	$[0,2,5,6,7]_{H_{3,5}}$	$[0,1,8,7,2]_{H_{3,5}}$	$[1,0,5,2,7]_{H_{3,5}}$
$H_{3,6}$	$[0,1,2,3,6]_{H_{3,6}}$	$[0,1,5,6,2]_{H_{3,6}}$	$[2,5,0,8,6]_{H_{3,6}}$	$[0,2,6,7,4]_{H_{3,6}}$

Example A.17 *An $H_{3,i}$-design of order* 11 *for* $i \in \{1,2,3,4,5,6\}$*:* In each case the union of the orbits of the given $H_{3,i}$-blocks under the action of $\mathbb{Z}_{11}$ is an $H_{3,i}$-design of order 11 with point set $\mathbb{Z}_{11}$.

$H_{3,1}$	$[0,1,2,3,4,7]_{H_{3,1}}$	$[0,1,7,6,5,10]_{H_{3,1}}$	$[0,1,9,8,6,10]_{H_{3,1}}$	$[0,2,6,5,1,7]_{H_{3,1}}$	$[0,3,6,7,4,9]_{H_{3,1}}$
$H_{3,2}$	$[1,0,2,3,4,10]_{H_{3,2}}$	$[0,1,6,7,2,5]_{H_{3,2}}$	$[1,0,8,9,2,4]_{H_{3,2}}$	$[0,2,5,6,4,10]_{H_{3,2}}$	$[0,3,6,7,5,10]_{H_{3,2}}$
$H_{3,3}$	$[2,0,1,5,3,6]_{H_{3,3}}$	$[1,4,0,6,2,8]_{H_{3,3}}$	$[1,0,7,2,9,3]_{H_{3,3}}$	$[2,0,5,8,4,3]_{H_{3,3}}$	$[2,7,0,10,4,3]_{H_{3,3}}$
$H_{3,4}$	$[0,1,3,5,6]_{H_{3,4}}$	$[0,1,7,8,9]_{H_{3,4}}$	$[0,2,4,5,6]_{H_{3,4}}$	$[0,2,1,7,8]_{H_{3,4}}$	$[0,3,6,7,10]_{H_{3,4}}$
$H_{3,5}$	$[1,0,6,2,9]_{H_{3,5}}$	$[0,1,9,7,8]_{H_{3,5}}$	$[2,0,8,7,3]_{H_{3,5}}$	$[2,0,5,4,8]_{H_{3,5}}$	$[3,0,7,6,4]_{H_{3,5}}$
$H_{3,6}$	$[0,1,4,5,10]_{H_{3,6}}$	$[0,1,7,8,5]_{H_{3,6}}$	$[0,2,6,4,9]_{H_{3,6}}$	$[0,2,7,8,9]_{H_{3,6}}$	$[0,3,6,7,9]_{H_{3,6}}$

Example A.18 *An $H_{3,i}$-decomposition of $L^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\}}$ for* $i \in \{1,2,3,4,5,6\}$*:* In each case the union of the orbits of the given $H_{3,i}$-blocks under the permutation $(j,k) \to (j+1,k)$ for $j \in \mathbb{Z}_9, k \in \mathbb{Z}_2$ form an $H_{3,i}$-decomposition of $L^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\}}$ for $i \in \{1,2,3,4,5,6\}$.

$H_{3,1}$	$[(0,0),(1,0),(0,1),(1,1),(2,1),(3,0)]_{H_{3,1}}$	$[(0,0),(2,0),(0,1),(1,1),(3,1),(3,0)]_{H_{3,1}}$
	$[(0,0),(3,0),(1,1),(0,1),(3,1),(2,0)]_{H_{3,1}}$	$[(0,0),(4,0),(0,1),(1,1),(5,1),(3,0)]_{H_{3,1}}$
	$[(0,0),(1,0),(4,1),(3,1),(2,1),(7,0)]_{H_{3,1}}$	$[(0,0),(2,0),(4,1),(3,1),(5,1),(8,0)]_{H_{3,1}}$
	$[(0,0),(3,0),(4,1),(3,1),(6,1),(8,0)]_{H_{3,1}}$	$[(0,0),(4,0),(4,1),(3,1),(7,1),(8,0)]_{H_{3,1}}$
	$[(0,0),(1,0),(6,1),(7,1),(8,1),(6,0)]_{H_{3,1}}$	$[(0,0),(2,0),(7,1),(6,1),(8,1),(5,0)]_{H_{3,1}}$
	$[(0,0),(3,0),(7,1),(6,1),(0,1),(5,0)]_{H_{3,1}}$	$[(0,0),(4,0),(7,1),(6,1),(1,1),(5,0)]_{H_{3,1}}$
	$[(0,1),(1,1),(0,0),(1,0),(2,0),(3,1)]_{H_{3,1}}$	$[(0,1),(2,1),(0,0),(1,0),(3,0),(3,1)]_{H_{3,1}}$
	$[(0,1),(3,1),(1,0),(0,0),(3,0),(2,1)]_{H_{3,1}}$	$[(0,1),(4,1),(0,0),(1,0),(5,0),(3,1)]_{H_{3,1}}$
	$[(0,1),(1,1),(4,0),(3,0),(2,0),(7,1)]_{H_{3,1}}$	$[(0,1),(2,1),(4,0),(3,0),(5,0),(8,1)]_{H_{3,1}}$
	$[(0,1),(3,1),(4,0),(3,0),(6,0),(8,1)]_{H_{3,1}}$	$[(0,1),(4,1),(4,0),(3,0),(7,0),(8,1)]_{H_{3,1}}$
	$[(0,1),(1,1),(6,0),(7,0),(8,0),(6,1)]_{H_{3,1}}$	$[(0,1),(2,1),(7,0),(6,0),(8,0),(5,1)]_{H_{3,1}}$
	$[(0,1),(3,1),(7,0),(6,0),(0,0),(5,1)]_{H_{3,1}}$	$[(0,1),(4,1),(7,0),(6,0),(1,0),(5,1)]_{H_{3,1}}$
$H_{3,2}$	$[(0,0),(1,0),(0,1),(1,1),(2,0),(3,1)]_{H_{3,2}}$	$[(0,0),(1,0),(3,1),(4,1),(2,0),(6,1)]_{H_{3,2}}$
	$[(0,0),(1,0),(6,1),(7,1),(2,0),(0,1)]_{H_{3,2}}$	$[(0,0),(2,0),(0,1),(1,1),(4,0),(4,1)]_{H_{3,2}}$
	$[(0,0),(2,0),(3,1),(4,1),(4,0),(7,1)]_{H_{3,2}}$	$[(0,0),(2,0),(6,1),(7,1),(4,0),(1,1)]_{H_{3,2}}$
	$[(0,0),(3,0),(0,1),(1,1),(6,0),(5,1)]_{H_{3,2}}$	$[(0,0),(3,0),(3,1),(4,1),(6,0),(8,1)]_{H_{3,2}}$
	$[(0,0),(3,0),(6,1),(7,1),(6,0),(2,1)]_{H_{3,2}}$	$[(0,0),(4,0),(0,1),(1,1),(8,0),(6,1)]_{H_{3,2}}$
	$[(0,0),(4,0),(3,1),(4,1),(8,0),(0,1)]_{H_{3,2}}$	$[(0,0),(4,0),(6,1),(7,1),(8,0),(3,1)]_{H_{3,2}}$
	$[(0,1),(1,1),(0,0),(1,0),(2,1),(3,0)]_{H_{3,2}}$	$[(0,1),(1,1),(3,0),(4,0),(2,1),(6,0)]_{H_{3,2}}$
	$[(0,1),(1,1),(6,0),(7,0),(2,1),(0,0)]_{H_{3,2}}$	$[(0,1),(2,1),(0,0),(1,0),(4,1),(4,0)]_{H_{3,2}}$
	$[(0,1),(2,1),(3,0),(4,0),(4,1),(7,0)]_{H_{3,2}}$	$[(0,1),(2,1),(6,0),(7,0),(4,1),(1,0)]_{H_{3,2}}$
	$[(0,1),(3,1),(0,0),(1,0),(6,1),(5,0)]_{H_{3,2}}$	$[(0,1),(3,1),(3,0),(4,0),(6,1),(8,0)]_{H_{3,2}}$
	$[(0,1),(3,1),(6,0),(7,0),(6,1),(2,0)]_{H_{3,2}}$	$[(0,1),(4,1),(0,0),(1,0),(8,1),(6,0)]_{H_{3,2}}$
	$[(0,1),(4,1),(3,0),(4,0),(8,1),(0,0)]_{H_{3,2}}$	$[(0,1),(4,1),(6,0),(7,0),(8,1),(3,0)]_{H_{3,2}}$
$H_{3,3}$	$[(1,0),(0,0),(0,1),(2,0),(2,1),(6,1)]_{H_{3,3}}$	$[(1,0),(0,0),(2,1),(2,0),(4,1),(7,1)]_{H_{3,3}}$
	$[(1,0),(0,0),(4,1),(2,0),(6,1),(8,1)]_{H_{3,3}}$	$[(4,0),(0,0),(3,1),(8,0),(8,1),(5,1)]_{H_{3,3}}$
	$[(4,0),(0,0),(5,1),(8,0),(1,1),(7,1)]_{H_{3,3}}$	$[(4,0),(0,0),(7,1),(8,0),(3,1),(6,1)]_{H_{3,3}}$
	$[(2,0),(0,0),(0,1),(4,0),(3,1),(2,1)]_{H_{3,3}}$	$[(2,0),(0,0),(2,1),(4,0),(5,1),(1,1)]_{H_{3,3}}$
	$[(2,0),(0,0),(4,1),(4,0),(7,1),(0,1)]_{H_{3,3}}$	$[(3,0),(0,0),(0,1),(6,0),(4,1),(8,1)]_{H_{3,3}}$
	$[(3,0),(0,0),(3,1),(6,0),(7,1),(2,1)]_{H_{3,3}}$	$[(3,0),(0,0),(6,1),(6,0),(1,1),(5,1)]_{H_{3,3}}$
	$[(1,1),(0,1),(0,0),(2,1),(2,0),(6,0)]_{H_{3,3}}$	$[(1,1),(0,1),(2,0),(2,1),(4,0),(7,0)]_{H_{3,3}}$
	$[(1,1),(0,1),(4,0),(2,1),(6,0),(8,0)]_{H_{3,3}}$	$[(4,1),(0,1),(3,0),(8,1),(8,0),(5,0)]_{H_{3,3}}$
	$[(4,1),(0,1),(5,0),(8,1),(1,0),(7,0)]_{H_{3,3}}$	$[(4,1),(0,1),(7,0),(8,1),(3,0),(6,0)]_{H_{3,3}}$
	$[(2,1),(0,1),(0,0),(4,1),(3,0),(2,0)]_{H_{3,3}}$	$[(2,1),(0,1),(2,0),(4,1),(5,0),(1,0)]_{H_{3,3}}$
	$[(2,1),(0,1),(4,0),(4,1),(7,0),(0,0)]_{H_{3,3}}$	$[(3,1),(0,1),(0,0),(6,1),(4,0),(8,0)]_{H_{3,3}}$
	$[(3,1),(0,1),(3,0),(6,1),(7,0),(2,0)]_{H_{3,3}}$	$[(3,1),(0,1),(6,0),(6,1),(1,0),(5,0)]_{H_{3,3}}$
$H_{3,4}$	$[(0,0),(1,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$	$[(0,0),(2,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$
	$[(0,0),(1,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$	$[(0,0),(2,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$
	$[(0,0),(1,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$	$[(0,0),(2,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$
	$[(0,0),(3,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$	$[(0,0),(4,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$
	$[(0,0),(3,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$	$[(0,0),(4,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$
	$[(0,0),(3,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$	$[(0,0),(4,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$
	$[(0,1),(1,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$	$[(0,1),(2,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$
	$[(0,1),(1,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$	$[(0,1),(2,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$
	$[(0,1),(1,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$	$[(0,1),(2,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$
	$[(0,1),(3,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$	$[(0,1),(4,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$
	$[(0,1),(3,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$	$[(0,1),(4,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$
	$[(0,1),(3,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$	$[(0,1),(4,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$

$H_{3,5}$	$[(0,0),(1,0),(1,1),(0,1),(2,0)]_{H_{3,5}}$	$[(0,0),(2,0),(1,1),(2,1),(1,0)]_{H_{3,5}}$
	$[(0,0),(1,0),(4,1),(3,1),(2,0)]_{H_{3,5}}$	$[(0,0),(2,0),(4,1),(5,1),(1,0)]_{H_{3,5}}$
	$[(0,0),(1,0),(7,1),(6,1),(2,0)]_{H_{3,5}}$	$[(0,0),(2,0),(7,1),(8,1),(1,0)]_{H_{3,5}}$
	$[(0,0),(3,0),(1,1),(0,1),(4,0)]_{H_{3,5}}$	$[(0,0),(4,0),(1,1),(2,1),(3,0)]_{H_{3,5}}$
	$[(0,0),(3,0),(4,1),(3,1),(4,0)]_{H_{3,5}}$	$[(0,0),(4,0),(4,1),(5,1),(3,0)]_{H_{3,5}}$
	$[(0,0),(3,0),(7,1),(6,1),(4,0)]_{H_{3,5}}$	$[(0,0),(4,0),(7,1),(8,1),(3,0)]_{H_{3,5}}$
	$[(0,1),(1,1),(1,0),(0,0),(2,1)]_{H_{3,5}}$	$[(0,1),(2,1),(1,0),(2,0),(1,1)]_{H_{3,5}}$
	$[(0,1),(1,1),(4,0),(3,0),(2,1)]_{H_{3,5}}$	$[(0,1),(2,1),(4,0),(5,0),(1,1)]_{H_{3,5}}$
	$[(0,1),(1,1),(7,0),(6,0),(2,1)]_{H_{3,5}}$	$[(0,1),(2,1),(7,0),(8,0),(1,1)]_{H_{3,5}}$
	$[(0,1),(3,1),(1,0),(0,0),(4,1)]_{H_{3,5}}$	$[(0,1),(4,1),(1,0),(2,0),(3,1)]_{H_{3,5}}$
	$[(0,1),(3,1),(4,0),(3,0),(4,1)]_{H_{3,5}}$	$[(0,1),(4,1),(4,0),(5,0),(3,1)]_{H_{3,5}}$
	$[(0,1),(3,1),(7,0),(6,0),(4,1)]_{H_{3,5}}$	$[(0,1),(4,1),(7,0),(8,0),(3,1)]_{H_{3,5}}$
$H_{3,6}$	$[(0,0),(2,0),(0,1),(1,1),(8,0)]_{H_{3,6}}$	$[(0,0),(2,0),(2,1),(3,1),(8,0)]_{H_{3,6}}$
	$[(0,0),(2,0),(4,1),(5,1),(8,0)]_{H_{3,6}}$	$[(0,0),(2,0),(7,1),(8,1),(8,0)]_{H_{3,6}}$
	$[(0,0),(3,0),(0,1),(1,1),(7,0)]_{H_{3,6}}$	$[(0,0),(3,0),(2,1),(3,1),(7,0)]_{H_{3,6}}$
	$[(0,0),(3,0),(4,1),(5,1),(7,0)]_{H_{3,6}}$	$[(0,0),(3,0),(6,1),(7,1),(6,0)]_{H_{3,6}}$
	$[(0,0),(4,0),(0,1),(1,1),(2,0)]_{H_{3,6}}$	$[(0,0),(4,0),(3,1),(6,1),(2,0)]_{H_{3,6}}$
	$[(0,0),(4,0),(4,1),(8,1),(6,0)]_{H_{3,6}}$	$[(0,0),(4,0),(5,1),(7,1),(2,0)]_{H_{3,6}}$
	$[(0,1),(2,1),(0,0),(1,0),(8,1)]_{H_{3,6}}$	$[(0,1),(2,1),(2,0),(3,0),(8,1)]_{H_{3,6}}$
	$[(0,1),(2,1),(4,0),(5,0),(8,1)]_{H_{3,6}}$	$[(0,1),(2,1),(7,0),(8,0),(8,1)]_{H_{3,6}}$
	$[(0,1),(3,1),(0,0),(1,0),(7,1)]_{H_{3,6}}$	$[(0,1),(3,1),(2,0),(3,0),(7,1)]_{H_{3,6}}$
	$[(0,1),(3,1),(4,0),(5,0),(7,1)]_{H_{3,6}}$	$[(0,1),(3,1),(6,0),(7,0),(6,1)]_{H_{3,6}}$
	$[(0,1),(4,1),(0,0),(1,0),(2,1)]_{H_{3,6}}$	$[(0,1),(4,1),(3,0),(6,0),(2,1)]_{H_{3,6}}$
	$[(0,1),(4,1),(4,0),(8,0),(6,1)]_{H_{3,6}}$	$[(0,1),(4,1),(5,0),(7,0),(2,1)]_{H_{3,6}}$

Example A.19 *An $H_{3,i}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \infty} \cup L^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\}}$ for $i \in \{1,2,3,4,5,6\}$:* In each case the union of the orbits of the given $H_{3,i}$-blocks under the permutation $(j,k) \to (j+1,k)$ for $j \in \mathbb{Z}_9, k \in \mathbb{Z}_2$ and $\infty \to \infty$ form an $H_{3,i}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \infty} \cup L^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\}}$ for $i \in \{1,2,3,4,5,6\}$.

$H_{3,1}$	$[(0,1),(2,1),(0,0),(1,0),(3,0),(3,1)]_{H_{3,1}}$	$[(0,0),(2,0),(0,1),(1,1),(3,1),(3,0)]_{H_{3,1}}$
	$[(0,0),(3,0),(1,1),(0,1),(3,1),(2,0)]_{H_{3,1}}$	$[(0,0),(4,0),(0,1),(1,1),(5,1),(3,0)]_{H_{3,1}}$
	$[(0,0),(1,0),(4,1),(3,1),(2,1),(7,0)]_{H_{3,1}}$	$[(0,0),(2,0),(4,1),(3,1),(5,1),(8,0)]_{H_{3,1}}$
	$[(0,0),(3,0),(4,1),(3,1),(6,1),(8,0)]_{H_{3,1}}$	$[(0,0),(4,0),(4,1),(3,1),(7,1),(8,0)]_{H_{3,1}}$
	$[(0,0),(1,0),(6,1),(7,1),(8,1),(6,0)]_{H_{3,1}}$	$[(0,0),(2,0),(7,1),(6,1),(8,1),(5,0)]_{H_{3,1}}$
	$[(0,0),(3,0),(7,1),(6,1),(0,1),(5,0)]_{H_{3,1}}$	$[(0,0),(4,0),(7,1),(6,1),(1,1),(5,0)]_{H_{3,1}}$
	$[(0,1),(3,1),(1,0),(0,0),(3,0),(2,1)]_{H_{3,1}}$	$[(0,1),(4,1),(0,0),(1,0),(5,0),(3,1)]_{H_{3,1}}$
	$[(0,1),(1,1),(4,0),(3,0),(2,0),(7,1)]_{H_{3,1}}$	$[(0,1),(2,1),(4,0),(3,0),(5,0),(8,1)]_{H_{3,1}}$
	$[(0,1),(3,1),(4,0),(3,0),(6,0),(8,1)]_{H_{3,1}}$	$[(0,1),(4,1),(4,0),(3,0),(7,0),(8,1)]_{H_{3,1}}$
	$[(0,1),(1,1),(6,0),(7,0),(8,0),(6,1)]_{H_{3,1}}$	$[(0,1),(2,1),(7,0),(6,0),(8,0),(5,1)]_{H_{3,1}}$
	$[(0,1),(3,1),(7,0),(6,0),(0,0),(5,1)]_{H_{3,1}}$	$[(0,1),(4,1),(7,0),(6,0),(1,0),(5,1)]_{H_{3,1}}$
	$[(0,0),(1,0),(0,1),(1,1),(2,0),\infty]_{H_{3,1}}$	$[\infty,(0,0),(0,1),(1,1),(2,1),(3,0)]_{H_{3,1}}$
	$[\infty,(0,0),(2,1),(3,1),(1,0),(2,0)]_{H_{3,1}}$	$[\infty,(0,0),(4,1),(5,1),(6,1),(5,0)]_{H_{3,1}}$
	$[\infty,(0,0),(6,1),(7,1),(8,1),(8,0)]_{H_{3,1}}$	

$H_{3,2}$	$[(0,0),(1,0),(0,1),(1,1),(2,0),(3,1)]_{H_{3,2}}$ $[(0,0),(1,0),(3,1),(4,1),(2,0),(6,1)]_{H_{3,2}}$
	$[(0,0),(1,0),(6,1),(7,1),(2,0),(0,1)]_{H_{3,2}}$ $[(0,0),(2,0),(0,1),(1,1),(4,0),(4,1)]_{H_{3,2}}$
	$[(0,0),(2,0),(3,1),(4,1),(4,0),(7,1)]_{H_{3,2}}$ $[(0,0),(2,0),(6,1),(7,1),(4,0),(1,1)]_{H_{3,2}}$
	$[(0,0),(3,0),(0,1),(1,1),(6,0),(5,1)]_{H_{3,2}}$ $[(0,0),(3,0),(3,1),(4,1),(6,0),(8,1)]_{H_{3,2}}$
	$[(0,0),(3,0),(6,1),(7,1),(6,0),(2,1)]_{H_{3,2}}$ $[(0,0),(4,0),(0,1),(1,1),(8,0),(6,1)]_{H_{3,2}}$
	$[(0,0),(4,0),(3,1),(4,1),(8,0),(0,1)]_{H_{3,2}}$ $[(0,0),(4,0),(6,1),(7,1),(8,0),(3,1)]_{H_{3,2}}$
	$[(0,1),(1,1),(0,0),(1,0),(2,1),(3,0)]_{H_{3,2}}$ $[(0,1),(1,1),(3,0),(4,0),(2,1),(6,0)]_{H_{3,2}}$
	$[(0,1),(1,1),(6,0),(7,0),(2,1),(0,0)]_{H_{3,2}}$ $[(0,1),(2,1),(0,0),(1,0),(4,1),(4,0)]_{H_{3,2}}$
	$[(0,1),(2,1),(3,0),(4,0),(4,1),(7,0)]_{H_{3,2}}$ $[(0,1),(2,1),(6,0),(7,0),(4,1),(1,0)]_{H_{3,2}}$
	$[(0,1),(3,1),(0,0),(1,0),(6,1),(5,0)]_{H_{3,2}}$ $[(0,1),(3,1),(3,0),(4,0),(6,1),(8,0)]_{H_{3,2}}$
	$[(0,1),(3,1),(6,0),(7,0),(6,1),(2,0)]_{H_{3,2}}$ $[(0,1),(4,1),(0,0),(1,0),(8,1),(6,0)]_{H_{3,2}}$
	$[(0,1),(4,1),(3,0),(4,0),(8,1),(0,0)]_{H_{3,2}}$ $[(0,1),(4,1),(6,0),(7,0),(8,1),(3,0)]_{H_{3,2}}$
	$[(0,0),\infty,(0,1),(1,1),(3,1),(1,0)]_{H_{3,2}}$ $[(0,0),\infty,(3,1),(4,1),(6,1),(1,0)]_{H_{3,2}}$
	$[(0,0),\infty,(6,1),(7,1),(0,1),(1,0)]_{H_{3,2}}$
$H_{3,3}$	$[(1,1),(0,1),(2,0),(2,1),(4,0),(7,0)]_{H_{3,3}}$ $[(1,0),(0,0),(2,1),(2,0),(4,1),(7,1)]_{H_{3,3}}$
	$[(1,0),(0,0),(4,1),(2,0),(6,1),(8,1)]_{H_{3,3}}$ $[(4,0),(0,0),(3,1),(8,0),(8,1),(5,1)]_{H_{3,3}}$
	$[(4,0),(0,0),(5,1),(8,0),(1,1),(7,1)]_{H_{3,3}}$ $[(4,0),(0,0),(7,1),(8,0),(3,1),(6,1)]_{H_{3,3}}$
	$[(2,0),(0,0),(0,1),(4,0),(3,1),(2,1)]_{H_{3,3}}$ $[(2,0),(0,0),(2,1),(4,0),(5,1),(1,1)]_{H_{3,3}}$
	$[(2,0),(0,0),(4,1),(4,0),(7,1),(0,1)]_{H_{3,3}}$ $[(3,0),(0,0),(0,1),(6,0),(4,1),(8,1)]_{H_{3,3}}$
	$[(3,0),(0,0),(3,1),(6,0),(7,1),(2,1)]_{H_{3,3}}$ $[(3,0),(0,0),(6,1),(6,0),(1,1),(5,1)]_{H_{3,3}}$
	$[(1,1),(0,1),(4,0),(2,1),(6,0),(8,0)]_{H_{3,3}}$ $[(4,1),(0,1),(3,0),(8,1),(8,0),(5,0)]_{H_{3,3}}$
	$[(4,1),(0,1),(5,0),(8,1),(1,0),(7,0)]_{H_{3,3}}$ $[(4,1),(0,1),(7,0),(8,1),(3,0),(6,0)]_{H_{3,3}}$
	$[(2,1),(0,1),(0,0),(4,1),(3,0),(2,0)]_{H_{3,3}}$ $[(2,1),(0,1),(2,0),(4,1),(5,0),(1,0)]_{H_{3,3}}$
	$[(2,1),(0,1),(4,0),(4,1),(7,0),(0,0)]_{H_{3,3}}$ $[(3,1),(0,1),(0,0),(6,1),(4,0),(8,0)]_{H_{3,3}}$
	$[(3,1),(0,1),(3,0),(6,1),(7,0),(2,0)]_{H_{3,3}}$ $[(3,1),(0,1),(6,0),(6,1),(1,0),(5,0)]_{H_{3,3}}$
	$[(0,0),\infty,(0,1),(1,0),(1,1),(2,1)]_{H_{3,3}}$ $[(1,1),\infty,(4,0),(3,1),(7,0),(0,0)]_{H_{3,3}}$
	$[\infty,(0,0),(4,1),(1,0),(6,1),(0,1)]_{H_{3,3}}$ $[\infty,(0,0),(7,1),(2,0),(1,1),(6,1)]_{H_{3,3}}$
	$[(1,1),(0,0),(0,1),(2,1),(2,0),\infty]_{H_{3,3}}$
$H_{3,4}$	$[(0,0),(1,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$ $[(0,0),(2,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$
	$[(0,0),(1,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$ $[(0,0),(2,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$
	$[(0,0),(1,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$ $[(0,0),(2,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$
	$[(0,0),(3,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$ $[(0,0),(4,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$
	$[(0,0),(3,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$ $[(0,0),(4,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$
	$[(0,0),(3,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$ $[(0,0),(4,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$
	$[(0,1),(1,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$ $[(0,1),(2,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$
	$[(0,1),(1,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$ $[(0,1),(2,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$
	$[(0,1),(1,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$ $[(0,1),(2,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$
	$[(0,1),(3,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$ $[(0,1),(4,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$
	$[(0,1),(3,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$ $[(0,1),(4,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$
	$[(0,1),(3,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$ $[(0,1),(4,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$
	$[\infty,(0,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$ $[\infty,(0,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$
	$[\infty,(0,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$

$H_{3,5}$	$[(0,0),(1,0),(1,1),(0,1),(2,0)]_{H_{3,5}}$	$[(0,0),(2,0),(1,1),(2,1),(1,0)]_{H_{3,5}}$
	$[(0,0),(1,0),(4,1),(3,1),(2,0)]_{H_{3,5}}$	$[(0,0),(2,0),(4,1),(5,1),(1,0)]_{H_{3,5}}$
	$[(0,0),(1,0),(7,1),(6,1),(2,0)]_{H_{3,5}}$	$[(0,0),(2,0),(7,1),(8,1),(1,0)]_{H_{3,5}}$
	$[(0,0),(3,0),(1,1),(0,1),(4,0)]_{H_{3,5}}$	$[(0,0),(4,0),(1,1),(2,1),(3,0)]_{H_{3,5}}$
	$[(0,0),(3,0),(4,1),(3,1),(4,0)]_{H_{3,5}}$	$[(0,0),(4,0),(4,1),(5,1),(3,0)]_{H_{3,5}}$
	$[(0,0),(3,0),(7,1),(6,1),(4,0)]_{H_{3,5}}$	$[(0,0),(4,0),(7,1),(8,1),(3,0)]_{H_{3,5}}$
	$[(0,1),(1,1),(1,0),(0,0),(2,1)]_{H_{3,5}}$	$[(0,1),(2,1),(1,0),(2,0),(1,1)]_{H_{3,5}}$
	$[(0,1),(1,1),(4,0),(3,0),(2,1)]_{H_{3,5}}$	$[(0,1),(2,1),(4,0),(5,0),(1,1)]_{H_{3,5}}$
	$[(0,1),(1,1),(7,0),(6,0),(2,1)]_{H_{3,5}}$	$[(0,1),(2,1),(7,0),(8,0),(1,1)]_{H_{3,5}}$
	$[(0,1),(3,1),(1,0),(0,0),(4,1)]_{H_{3,5}}$	$[(0,1),(4,1),(1,0),(2,0),(3,1)]_{H_{3,5}}$
	$[(0,1),(3,1),(4,0),(3,0),(4,1)]_{H_{3,5}}$	$[(0,1),(4,1),(4,0),(5,0),(3,1)]_{H_{3,5}}$
	$[(0,1),(3,1),(7,0),(6,0),(4,1)]_{H_{3,5}}$	$[(0,1),(4,1),(7,0),(8,0),(3,1)]_{H_{3,5}}$
	$[\infty,(0,0),(1,1),(0,1),(7,0)]_{H_{3,5}}$	$[\infty,(0,0),(4,1),(3,1),(7,0)]_{H_{3,5}}$
	$[\infty,(0,0),(7,1),(6,1),(7,0)]_{H_{3,5}}$	
$H_{3,6}$	$[(0,1),(2,1),(2,0),(3,0),(8,1)]_{H_{3,6}}$	$[(0,0),(2,0),(2,1),(3,1),(8,0)]_{H_{3,6}}$
	$[(0,0),(2,0),(4,1),(5,1),(8,0)]_{H_{3,6}}$	$[(0,0),(2,0),(7,1),(8,1),(8,0)]_{H_{3,6}}$
	$[(0,0),(3,0),(0,1),(1,1),(7,0)]_{H_{3,6}}$	$[(0,0),(3,0),(2,1),(3,1),(7,0)]_{H_{3,6}}$
	$[(0,0),(3,0),(4,1),(5,1),(7,0)]_{H_{3,6}}$	$[(0,0),(3,0),(6,1),(7,1),(6,0)]_{H_{3,6}}$
	$[(0,0),(4,0),(0,1),(1,1),(2,0)]_{H_{3,6}}$	$[(0,0),(4,0),(3,1),(6,1),(2,0)]_{H_{3,6}}$
	$[(0,0),(4,0),(4,1),(8,1),(6,0)]_{H_{3,6}}$	$[(0,0),(4,0),(5,1),(7,1),(2,0)]_{H_{3,6}}$
	$[(0,1),(2,1),(4,0),(5,0),(8,1)]_{H_{3,6}}$	$[(0,1),(2,1),(7,0),(8,0),(8,1)]_{H_{3,6}}$
	$[(0,1),(3,1),(0,0),(1,0),(7,1)]_{H_{3,6}}$	$[(0,1),(3,1),(2,0),(3,0),(7,1)]_{H_{3,6}}$
	$[(0,1),(3,1),(4,0),(5,0),(7,1)]_{H_{3,6}}$	$[(0,1),(3,1),(6,0),(7,0),(6,1)]_{H_{3,6}}$
	$[(0,1),(4,1),(0,0),(1,0),(2,1)]_{H_{3,6}}$	$[(0,1),(4,1),(3,0),(6,0),(2,1)]_{H_{3,6}}$
	$[(0,1),(4,1),(4,0),(8,0),(6,1)]_{H_{3,6}}$	$[(0,1),(4,1),(5,0),(7,0),(2,1)]_{H_{3,6}}$
	$[\infty,(0,0),(1,1),(3,1),(2,0)]_{H_{3,6}}$	$[\infty,(2,1),(0,0),(7,0),(8,1)]_{H_{3,6}}$
	$[\infty,(0,0),(7,1),(8,1),(6,0)]_{H_{3,6}}$	$[\infty,(6,1),(0,0),(1,0),(8,1)]_{H_{3,6}}$
	$[(0,0),(0,1),(2,0),(2,1),\infty]_{H_{3,6}}$	

Example A.20 *An $H_{3,i}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \{\infty_1,\infty_2\}} \cup L^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\}}$ for $i \in \{1,2,3,4,5,6\}$:* In each case the union of the orbits of the given $H_{3,i}$-blocks under the permutation $(j,k) \to (j+1,k)$ for $j \in \mathbb{Z}_9, k \in \mathbb{Z}_2$ and $\infty_l \to \infty_l$ for $l \in \{1,2,\}$ form an $H_{3,i}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \{\infty_1,\infty_2\}} \cup L^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\}}$ for $i \in \{1,2,3,4,5,6\}$.

$H_{3,1}$	$[(0,0),(1,0),(0,1),(1,1),(2,1),(3,0)]_{H_{3,1}}$	$[(0,0),(2,0),(0,1),(1,1),(3,1),(3,0)]_{H_{3,1}}$
	$[(0,0),(3,0),(1,1),(0,1),(3,1),(2,0)]_{H_{3,1}}$	$[(0,0),(4,0),(0,1),(1,1),(5,1),(3,0)]_{H_{3,1}}$
	$[(0,0),(1,0),(4,1),(3,1),(2,1),(7,0)]_{H_{3,1}}$	$[(0,0),(2,0),(4,1),(3,1),(5,1),(8,0)]_{H_{3,1}}$
	$[(0,0),(3,0),(4,1),(3,1),(6,1),(8,0)]_{H_{3,1}}$	$[(0,0),(4,0),(4,1),(3,1),(7,1),(8,0)]_{H_{3,1}}$
	$[(0,0),(1,0),(6,1),(7,1),(8,1),(6,0)]_{H_{3,1}}$	$[(0,0),(2,0),(7,1),(6,1),(8,1),(5,0)]_{H_{3,1}}$
	$[(0,0),(3,0),(7,1),(6,1),(0,1),(5,0)]_{H_{3,1}}$	$[(0,0),(4,0),(7,1),(6,1),(1,1),(5,0)]_{H_{3,1}}$
	$[(0,1),(1,1),(0,0),(1,0),(2,0),(3,1)]_{H_{3,1}}$	$[(0,1),(2,1),(0,0),(1,0),(3,0),(3,1)]_{H_{3,1}}$
	$[(0,1),(3,1),(1,0),(0,0),(3,0),(2,1)]_{H_{3,1}}$	$[(0,1),(4,1),(0,0),(1,0),(5,0),(3,1)]_{H_{3,1}}$
	$[(0,1),(1,1),(4,0),(3,0),(2,0),(7,1)]_{H_{3,1}}$	$[(0,1),(2,1),(4,0),(3,0),(5,0),(8,1)]_{H_{3,1}}$
	$[(0,1),(3,1),(4,0),(3,0),(6,0),(8,1)]_{H_{3,1}}$	$[(0,1),(4,1),(4,0),(3,0),(7,0),(8,1)]_{H_{3,1}}$
	$[(0,1),(1,1),(6,0),(7,0),(8,0),(6,1)]_{H_{3,1}}$	$[(0,1),(2,1),(7,0),(6,0),(8,0),(5,1)]_{H_{3,1}}$
	$[(0,1),(3,1),(7,0),(6,0),(0,0),(5,1)]_{H_{3,1}}$	$[(0,1),(4,1),(7,0),(6,0),(1,0),(5,1)]_{H_{3,1}}$
	$[(0,0),(0,1),\infty_2,\infty_1,(1,0),(2,1)]_{H_{3,1}}$	$[(0,0),(2,1),\infty_1,\infty_2,(2,0),(3,1)]_{H_{3,1}}$
	$[(0,0),(3,1),\infty_2,\infty_1,(1,0),(5,1)]_{H_{3,1}}$	$[(0,0),(5,1),\infty_1,\infty_2,(2,0),(6,1)]_{H_{3,1}}$
	$[(0,0),(6,1),\infty_2,\infty_1,(1,0),(8,1)]_{H_{3,1}}$	$[(0,0),(8,1),\infty_1,\infty_2,(2,0),(0,1)]_{H_{3,1}}$

$H_{3,2}$	$[(0,0),(1,0),(0,1),(1,1),(2,0),(3,1)]_{H_{3,2}}$	$[(0,0),(1,0),(3,1),(4,1),(2,0),(6,1)]_{H_{3,2}}$
	$[(0,0),(1,0),(6,1),(7,1),(2,0),(0,1)]_{H_{3,2}}$	$[(0,0),(2,0),(0,1),(1,1),(4,0),(4,1)]_{H_{3,2}}$
	$[(0,0),(2,0),(3,1),(4,1),(4,0),(7,1)]_{H_{3,2}}$	$[(0,0),(2,0),(6,1),(7,1),(4,0),(1,1)]_{H_{3,2}}$
	$[(0,0),(3,0),(0,1),(1,1),(6,0),(5,1)]_{H_{3,2}}$	$[(0,0),(3,0),(3,1),(4,1),(6,0),(8,1)]_{H_{3,2}}$
	$[(0,0),(3,0),(6,1),(7,1),(6,0),(2,1)]_{H_{3,2}}$	$[(0,0),(4,0),(0,1),(1,1),(8,0),(6,1)]_{H_{3,2}}$
	$[(0,0),(4,0),(3,1),(4,1),(8,0),(0,1)]_{H_{3,2}}$	$[(0,0),(4,0),(6,1),(7,1),(8,0),(3,1)]_{H_{3,2}}$
	$[(0,1),(1,1),(0,0),(1,0),(2,1),(3,0)]_{H_{3,2}}$	$[(0,1),(1,1),(3,0),(4,0),(2,1),(6,0)]_{H_{3,2}}$
	$[(0,1),(1,1),(6,0),(7,0),(2,1),(0,0)]_{H_{3,2}}$	$[(0,1),(2,1),(0,0),(1,0),(4,1),(4,0)]_{H_{3,2}}$
	$[(0,1),(2,1),(3,0),(4,0),(4,1),(7,0)]_{H_{3,2}}$	$[(0,1),(2,1),(6,0),(7,0),(4,1),(1,0)]_{H_{3,2}}$
	$[(0,1),(3,1),(0,0),(1,0),(6,1),(5,0)]_{H_{3,2}}$	$[(0,1),(3,1),(3,0),(4,0),(6,1),(8,0)]_{H_{3,2}}$
	$[(0,1),(3,1),(6,0),(7,0),(6,1),(2,0)]_{H_{3,2}}$	$[(0,1),(4,1),(0,0),(1,0),(8,1),(6,0)]_{H_{3,2}}$
	$[(0,1),(4,1),(3,0),(4,0),(8,1),(0,0)]_{H_{3,2}}$	$[(0,1),(4,1),(6,0),(7,0),(8,1),(3,0)]_{H_{3,2}}$
	$[(0,0),\infty_1,(0,1),(1,1),(3,1),(1,0)]_{H_{3,2}}$	$[(0,0),\infty_2,(0,1),(1,1),(3,1),(1,0)]_{H_{3,2}}$
	$[(0,0),\infty_1,(3,1),(4,1),(6,1),(1,0)]_{H_{3,2}}$	$[(0,0),\infty_2,(3,1),(4,1),(6,1),(1,0)]_{H_{3,2}}$
	$[(0,0),\infty_1,(6,1),(7,1),(0,1),(1,0)]_{H_{3,2}}$	$[(0,0),\infty_2,(6,1),(7,1),(0,1),(1,0)]_{H_{3,2}}$
$H_{3,3}$	$[(1,0),(0,0),(0,1),(2,0),(2,1),(6,1)]_{H_{3,3}}$	$[(1,0),(0,0),(2,1),(2,0),(4,1),(7,1)]_{H_{3,3}}$
	$[(1,0),(0,0),(4,1),(2,0),(6,1),(8,1)]_{H_{3,3}}$	$[(4,0),(0,0),(3,1),(8,0),(8,1),(5,1)]_{H_{3,3}}$
	$[(4,0),(0,0),(5,1),(8,0),(1,1),(7,1)]_{H_{3,3}}$	$[(4,0),(0,0),(7,1),(8,0),(3,1),(6,1)]_{H_{3,3}}$
	$[(2,0),(0,0),(0,1),(4,0),(3,1),(2,1)]_{H_{3,3}}$	$[(2,0),(0,0),(2,1),(4,0),(5,1),(1,1)]_{H_{3,3}}$
	$[(2,0),(0,0),(4,1),(4,0),(7,1),(0,1)]_{H_{3,3}}$	$[(3,0),(0,0),(0,1),(6,0),(4,1),(8,1)]_{H_{3,3}}$
	$[(3,0),(0,0),(3,1),(6,0),(7,1),(2,1)]_{H_{3,3}}$	$[(3,0),(0,0),(6,1),(6,0),(1,1),(5,1)]_{H_{3,3}}$
	$[(1,1),(0,1),(0,0),(2,1),(2,0),(6,0)]_{H_{3,3}}$	$[(1,1),(0,1),(2,0),(2,1),(4,0),(7,0)]_{H_{3,3}}$
	$[(1,1),(0,1),(4,0),(2,1),(6,0),(8,0)]_{H_{3,3}}$	$[(4,1),(0,1),(3,0),(8,1),(8,0),(5,0)]_{H_{3,3}}$
	$[(4,1),(0,1),(5,0),(8,1),(1,0),(7,0)]_{H_{3,3}}$	$[(4,1),(0,1),(7,0),(8,1),(3,0),(6,0)]_{H_{3,3}}$
	$[(2,1),(0,1),(0,0),(4,1),(3,0),(2,0)]_{H_{3,3}}$	$[(2,1),(0,1),(2,0),(4,1),(5,0),(1,0)]_{H_{3,3}}$
	$[(2,1),(0,1),(4,0),(4,1),(7,0),(0,0)]_{H_{3,3}}$	$[(3,1),(0,1),(0,0),(6,1),(4,0),(8,0)]_{H_{3,3}}$
	$[(3,1),(0,1),(3,0),(6,1),(7,0),(2,0)]_{H_{3,3}}$	$[(3,1),(0,1),(6,0),(6,1),(1,0),(5,0)]_{H_{3,3}}$
	$[\infty_1,(6,1),(5,0),(0,0),(0,1),\infty_2]_{H_{3,3}}$	$[\infty_2,(6,1),(5,0),(0,0),(0,1),\infty_1]_{H_{3,3}}$
	$[\infty_1,(7,1),(4,0),(0,0),(2,1),\infty_2]_{H_{3,3}}$	$[\infty_2,(7,1),(4,0),(0,0),(2,1),\infty_1]_{H_{3,3}}$
	$[\infty_1,(8,1),(3,0),(0,0),(4,1),\infty_2]_{H_{3,3}}$	$[\infty_2,(8,1),(3,0),(0,0),(4,1),\infty_1]_{H_{3,3}}$
$H_{3,4}$	$[(0,0),(1,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$	$[(0,0),(2,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$
	$[(0,0),(1,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$	$[(0,0),(2,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$
	$[(0,0),(1,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$	$[(0,0),(2,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$
	$[(0,0),(3,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$	$[(0,0),(4,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$
	$[(0,0),(3,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$	$[(0,0),(4,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$
	$[(0,0),(3,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$	$[(0,0),(4,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$
	$[(0,1),(1,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$	$[(0,1),(2,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$
	$[(0,1),(1,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$	$[(0,1),(2,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$
	$[(0,1),(1,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$	$[(0,1),(2,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$
	$[(0,1),(3,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$	$[(0,1),(4,1),(0,0),(1,0),(2,0)]_{H_{3,4}}$
	$[(0,1),(3,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$	$[(0,1),(4,1),(3,0),(4,0),(5,0)]_{H_{3,4}}$
	$[(0,1),(3,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$	$[(0,1),(4,1),(6,0),(7,0),(8,0)]_{H_{3,4}}$
	$[\infty_1,(0,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$	$[\infty_2,(0,0),(0,1),(1,1),(2,1)]_{H_{3,4}}$
	$[\infty_1,(0,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$	$[\infty_2,(0,0),(3,1),(4,1),(5,1)]_{H_{3,4}}$
	$[\infty_1,(0,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$	$[\infty_2,(0,0),(6,1),(7,1),(8,1)]_{H_{3,4}}$

$H_{3,5}$	$[(0,0),(1,0),(1,1),(0,1),(2,0)]_{H_{3,5}}$	$[(0,0),(2,0),(1,1),(2,1),(1,0)]_{H_{3,5}}$
	$[(0,0),(1,0),(4,1),(3,1),(2,0)]_{H_{3,5}}$	$[(0,0),(2,0),(4,1),(5,1),(1,0)]_{H_{3,5}}$
	$[(0,0),(1,0),(7,1),(6,1),(2,0)]_{H_{3,5}}$	$[(0,0),(2,0),(7,1),(8,1),(1,0)]_{H_{3,5}}$
	$[(0,0),(3,0),(1,1),(0,1),(4,0)]_{H_{3,5}}$	$[(0,0),(4,0),(1,1),(2,1),(3,0)]_{H_{3,5}}$
	$[(0,0),(3,0),(4,1),(3,1),(4,0)]_{H_{3,5}}$	$[(0,0),(4,0),(4,1),(5,1),(3,0)]_{H_{3,5}}$
	$[(0,0),(3,0),(7,1),(6,1),(4,0)]_{H_{3,5}}$	$[(0,0),(4,0),(7,1),(8,1),(3,0)]_{H_{3,5}}$
	$[(0,1),(1,1),(1,0),(0,0),(2,1)]_{H_{3,5}}$	$[(0,1),(2,1),(1,0),(2,0),(1,1)]_{H_{3,5}}$
	$[(0,1),(1,1),(4,0),(3,0),(2,1)]_{H_{3,5}}$	$[(0,1),(2,1),(4,0),(5,0),(1,1)]_{H_{3,5}}$
	$[(0,1),(1,1),(7,0),(6,0),(2,1)]_{H_{3,5}}$	$[(0,1),(2,1),(7,0),(8,0),(1,1)]_{H_{3,5}}$
	$[(0,1),(3,1),(1,0),(0,0),(4,1)]_{H_{3,5}}$	$[(0,1),(4,1),(1,0),(2,0),(3,1)]_{H_{3,5}}$
	$[(0,1),(3,1),(4,0),(3,0),(4,1)]_{H_{3,5}}$	$[(0,1),(4,1),(4,0),(5,0),(3,1)]_{H_{3,5}}$
	$[(0,1),(3,1),(7,0),(6,0),(4,1)]_{H_{3,5}}$	$[(0,1),(4,1),(7,0),(8,0),(3,1)]_{H_{3,5}}$
	$[\infty_1,(0,0),(1,1),(0,1),(7,0)]_{H_{3,5}}$	$[\infty_2,(0,0),(1,1),(0,1),(7,0)]_{H_{3,5}}$
	$[\infty_1,(0,0),(4,1),(3,1),(7,0)]_{H_{3,5}}$	$[\infty_2,(0,0),(4,1),(3,1),(7,0)]_{H_{3,5}}$
	$[\infty_1,(0,0),(7,1),(6,1),(7,0)]_{H_{3,5}}$	$[\infty_2,(0,0),(7,1),(6,1),(7,0)]_{H_{3,5}}$
$H_{3,6}$	$[(0,0),(2,0),(4,1),(5,1),(8,0)]_{H_{3,6}}$	$[(0,0),(2,0),(7,1),(8,1),(8,0)]_{H_{3,6}}$
	$[(0,0),(3,0),(0,1),(1,1),(7,0)]_{H_{3,6}}$	$[(0,0),(3,0),(2,1),(3,1),(7,0)]_{H_{3,6}}$
	$[(0,0),(3,0),(4,1),(5,1),(7,0)]_{H_{3,6}}$	$[(0,0),(3,0),(6,1),(7,1),(6,0)]_{H_{3,6}}$
	$[(0,0),(4,0),(0,1),(1,1),(2,0)]_{H_{3,6}}$	$[(0,0),(4,0),(3,1),(6,1),(2,0)]_{H_{3,6}}$
	$[(0,0),(4,0),(4,1),(8,1),(6,0)]_{H_{3,6}}$	$[(0,0),(4,0),(5,1),(7,1),(2,0)]_{H_{3,6}}$
	$[(0,1),(2,1),(4,0),(5,0),(8,1)]_{H_{3,6}}$	$[(0,1),(2,1),(7,0),(8,0),(8,1)]_{H_{3,6}}$
	$[(0,1),(3,1),(0,0),(1,0),(7,1)]_{H_{3,6}}$	$[(0,1),(3,1),(2,0),(3,0),(7,1)]_{H_{3,6}}$
	$[(0,1),(3,1),(4,0),(5,0),(7,1)]_{H_{3,6}}$	$[(0,1),(3,1),(6,0),(7,0),(6,1)]_{H_{3,6}}$
	$[(0,1),(4,1),(0,0),(1,0),(2,1)]_{H_{3,6}}$	$[(0,1),(4,1),(3,0),(6,0),(2,1)]_{H_{3,6}}$
	$[(0,1),(4,1),(4,0),(8,0),(6,1)]_{H_{3,6}}$	$[(0,1),(4,1),(5,0),(7,0),(2,1)]_{H_{3,6}}$
	$[\infty_1,(0,0),(1,1),(3,1),(2,0)]_{H_{3,6}}$	$[\infty_1,(2,1),(0,0),(7,0),(8,1)]_{H_{3,6}}$
	$[\infty_1,(0,0),(7,1),(8,1),(6,0)]_{H_{3,6}}$	$[\infty_1,(6,1),(0,0),(1,0),(8,1)]_{H_{3,6}}$
	$[(0,0),(0,1),(2,0),(2,1),\infty_1]_{H_{3,6}}$	$[(2,0),(2,1),(0,0),(0,1),\infty_2]_{H_{3,6}}$
	$[\infty_2,(0,0),(1,1),(3,1),(4,0)]_{H_{3,6}}$	$[\infty_2,(2,1),(0,0),(7,0),(1,1)]_{H_{3,6}}$
	$[\infty_2,(0,0),(5,1),(6,1),(2,0)]_{H_{3,6}}$	$[\infty_2,(8,1),(0,0),(1,0),(6,1)]_{H_{3,6}}$

Example A.21 *An $H_{3,i}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\mathbb{Z}_9\times\{1\},\mathbb{Z}_9\times\{2\}}$ for $i \in \{1,2,3,4,5\}$:* In each case the union of the orbits of the given $H_{3,i}$-blocks under the permutation $(j,k) \to (j+1,k)$ for $j \in \mathbb{Z}_9, k \in \mathbb{Z}_3$ form an $H_{3,i}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}}$ for $i \in \{1,2,3,4,5\}$.

$H_{3,1}$	$[(0,0),(0,1),(0,2),(1,2),(8,0),(8,1)]_{H_{3,1}}$	$[(0,0),(1,1),(0,2),(1,2),(8,0),(0,1)]_{H_{3,1}}$
	$[(0,0),(0,1),(3,2),(4,2),(8,0),(8,1)]_{H_{3,1}}$	$[(0,0),(1,1),(3,2),(4,2),(8,0),(0,1)]_{H_{3,1}}$
	$[(0,0),(0,1),(6,2),(7,2),(8,0),(8,1)]_{H_{3,1}}$	$[(0,0),(1,1),(6,2),(7,2),(8,0),(0,1)]_{H_{3,1}}$
	$[(0,0),(2,1),(0,2),(1,2),(8,0),(1,1)]_{H_{3,1}}$	$[(0,0),(3,1),(0,2),(1,2),(8,0),(2,1)]_{H_{3,1}}$
	$[(0,0),(2,1),(3,2),(4,2),(8,0),(1,1)]_{H_{3,1}}$	$[(0,0),(3,1),(3,2),(4,2),(8,0),(2,1)]_{H_{3,1}}$
	$[(0,0),(2,1),(6,2),(7,2),(8,0),(1,1)]_{H_{3,1}}$	$[(0,0),(3,1),(6,2),(7,2),(8,0),(2,1)]_{H_{3,1}}$
	$[(0,0),(4,1),(0,2),(1,2),(8,0),(3,1)]_{H_{3,1}}$	$[(0,0),(5,1),(0,2),(1,2),(8,0),(4,1)]_{H_{3,1}}$
	$[(0,0),(4,1),(3,2),(4,2),(8,0),(3,1)]_{H_{3,1}}$	$[(0,0),(5,1),(3,2),(4,2),(8,0),(4,1)]_{H_{3,1}}$
	$[(0,0),(4,1),(6,2),(7,2),(8,0),(3,1)]_{H_{3,1}}$	$[(0,0),(5,1),(6,2),(7,2),(8,0),(4,1)]_{H_{3,1}}$
	$[(0,0),(6,1),(0,2),(1,2),(8,0),(5,1)]_{H_{3,1}}$	$[(0,0),(7,1),(0,2),(1,2),(8,0),(6,1)]_{H_{3,1}}$
	$[(0,0),(6,1),(3,2),(4,2),(8,0),(5,1)]_{H_{3,1}}$	$[(0,0),(7,1),(3,2),(4,2),(8,0),(6,1)]_{H_{3,1}}$
	$[(0,0),(6,1),(6,2),(7,2),(8,0),(5,1)]_{H_{3,1}}$	$[(0,0),(7,1),(6,2),(7,2),(8,0),(6,1)]_{H_{3,1}}$
	$[(0,0),(8,1),(0,2),(1,2),(8,0),(7,1)]_{H_{3,1}}$	$[(0,0),(8,1),(3,2),(4,2),(8,0),(7,1)]_{H_{3,1}}$
	$[(0,0),(8,1),(6,2),(7,2),(8,0),(7,1)]_{H_{3,1}}$	

$H_{3,2}$	$[(0,1),(0,0),(0,2),(1,2),(2,2),(1,1)]_{H_{3,2}}$	$[(1,1),(0,0),(0,2),(1,2),(2,2),(0,1)]_{H_{3,2}}$
	$[(0,1),(0,0),(3,2),(4,2),(5,2),(1,1)]_{H_{3,2}}$	$[(1,1),(0,0),(3,2),(4,2),(5,2),(0,1)]_{H_{3,2}}$
	$[(0,1),(0,0),(6,2),(7,2),(8,2),(1,1)]_{H_{3,2}}$	$[(1,1),(0,0),(6,2),(7,2),(8,2),(0,1)]_{H_{3,2}}$
	$[(2,1),(0,0),(0,2),(1,2),(2,2),(3,1)]_{H_{3,2}}$	$[(3,1),(0,0),(0,2),(1,2),(2,2),(2,1)]_{H_{3,2}}$
	$[(2,1),(0,0),(3,2),(4,2),(5,2),(3,1)]_{H_{3,2}}$	$[(3,1),(0,0),(3,2),(4,2),(5,2),(2,1)]_{H_{3,2}}$
	$[(2,1),(0,0),(6,2),(7,2),(8,2),(3,1)]_{H_{3,2}}$	$[(3,1),(0,0),(6,2),(7,2),(8,2),(2,1)]_{H_{3,2}}$
	$[(4,1),(0,0),(0,2),(1,2),(2,2),(5,1)]_{H_{3,2}}$	$[(5,1),(0,0),(0,2),(1,2),(2,2),(4,1)]_{H_{3,2}}$
	$[(4,1),(0,0),(3,2),(4,2),(5,2),(5,1)]_{H_{3,2}}$	$[(5,1),(0,0),(3,2),(4,2),(5,2),(4,1)]_{H_{3,2}}$
	$[(4,1),(0,0),(6,2),(7,2),(8,2),(5,1)]_{H_{3,2}}$	$[(5,1),(0,0),(6,2),(7,2),(8,2),(4,1)]_{H_{3,2}}$
	$[(6,1),(0,0),(0,2),(1,2),(2,2),(7,1)]_{H_{3,2}}$	$[(7,1),(0,0),(0,2),(1,2),(2,2),(8,1)]_{H_{3,2}}$
	$[(6,1),(0,0),(3,2),(4,2),(5,2),(7,1)]_{H_{3,2}}$	$[(7,1),(0,0),(3,2),(4,2),(5,2),(8,1)]_{H_{3,2}}$
	$[(6,1),(0,0),(6,2),(7,2),(8,2),(7,1)]_{H_{3,2}}$	$[(7,1),(0,0),(6,2),(7,2),(8,2),(8,1)]_{H_{3,2}}$
	$[(8,1),(0,0),(0,2),(1,2),(2,2),(6,1)]_{H_{3,2}}$	$[(8,1),(0,0),(3,2),(4,2),(5,2),(6,1)]_{H_{3,2}}$
	$[(8,1),(0,0),(6,2),(7,2),(8,2),(6,1)]_{H_{3,2}}$	
$H_{3,3}$	$[(0,0),(0,1),(0,2),(1,2),(1,1),(7,0)]_{H_{3,3}}$	$[(0,0),(3,1),(0,2),(1,2),(4,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(0,1),(1,2),(2,2),(1,1),(7,0)]_{H_{3,3}}$	$[(0,0),(3,1),(1,2),(2,2),(4,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(0,1),(2,2),(3,2),(1,1),(7,0)]_{H_{3,3}}$	$[(0,0),(3,1),(2,2),(3,2),(4,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(0,1),(3,2),(4,2),(1,1),(7,0)]_{H_{3,3}}$	$[(0,0),(3,1),(3,2),(4,2),(4,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(0,1),(4,2),(5,2),(1,1),(7,0)]_{H_{3,3}}$	$[(0,0),(3,1),(4,2),(5,2),(4,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(0,1),(5,2),(6,2),(1,1),(7,0)]_{H_{3,3}}$	$[(0,0),(3,1),(5,2),(6,2),(4,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(0,1),(6,2),(7,2),(1,1),(7,0)]_{H_{3,3}}$	$[(0,0),(3,1),(6,2),(7,2),(4,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(0,1),(7,2),(8,2),(1,1),(7,0)]_{H_{3,3}}$	$[(0,0),(3,1),(7,2),(8,2),(4,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(0,1),(8,2),(0,2),(1,1),(7,0)]_{H_{3,3}}$	$[(0,0),(3,1),(8,2),(0,2),(4,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(6,1),(0,2),(1,2),(7,1),(7,0)]_{H_{3,3}}$	$[(0,0),(6,1),(5,2),(6,2),(7,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(6,1),(1,2),(2,2),(7,1),(7,0)]_{H_{3,3}}$	$[(0,0),(6,1),(6,2),(7,2),(7,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(6,1),(2,2),(3,2),(7,1),(7,0)]_{H_{3,3}}$	$[(0,0),(6,1),(7,2),(8,2),(7,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(6,1),(3,2),(4,2),(7,1),(7,0)]_{H_{3,3}}$	$[(0,0),(6,1),(8,2),(0,2),(7,1),(7,0)]_{H_{3,3}}$
	$[(0,0),(6,1),(4,2),(5,2),(7,1),(7,0)]_{H_{3,3}}$	
$H_{3,4}$	$[(0,0),(0,1),(0,2),(1,2),(2,2)]_{H_{3,4}}$	$[(0,0),(1,1),(0,2),(1,2),(2,2)]_{H_{3,4}}$
	$[(0,0),(0,1),(3,2),(4,2),(5,2)]_{H_{3,4}}$	$[(0,0),(1,1),(3,2),(4,2),(5,2)]_{H_{3,4}}$
	$[(0,0),(0,1),(6,2),(7,2),(8,2)]_{H_{3,4}}$	$[(0,0),(1,1),(6,2),(7,2),(8,2)]_{H_{3,4}}$
	$[(0,0),(2,1),(0,2),(1,2),(2,2)]_{H_{3,4}}$	$[(0,0),(3,1),(0,2),(1,2),(2,2)]_{H_{3,4}}$
	$[(0,0),(2,1),(3,2),(4,2),(5,2)]_{H_{3,4}}$	$[(0,0),(3,1),(3,2),(4,2),(5,2)]_{H_{3,4}}$
	$[(0,0),(2,1),(6,2),(7,2),(8,2)]_{H_{3,4}}$	$[(0,0),(3,1),(6,2),(7,2),(8,2)]_{H_{3,4}}$
	$[(0,0),(4,1),(0,2),(1,2),(2,2)]_{H_{3,4}}$	$[(0,0),(5,1),(0,2),(1,2),(2,2)]_{H_{3,4}}$
	$[(0,0),(4,1),(3,2),(4,2),(5,2)]_{H_{3,4}}$	$[(0,0),(5,1),(3,2),(4,2),(5,2)]_{H_{3,4}}$
	$[(0,0),(4,1),(6,2),(7,2),(8,2)]_{H_{3,4}}$	$[(0,0),(5,1),(6,2),(7,2),(8,2)]_{H_{3,4}}$
	$[(0,0),(6,1),(0,2),(1,2),(2,2)]_{H_{3,4}}$	$[(0,0),(7,1),(0,2),(1,2),(2,2)]_{H_{3,4}}$
	$[(0,0),(6,1),(3,2),(4,2),(5,2)]_{H_{3,4}}$	$[(0,0),(7,1),(3,2),(4,2),(5,2)]_{H_{3,4}}$
	$[(0,0),(6,1),(6,2),(7,2),(8,2)]_{H_{3,4}}$	$[(0,0),(7,1),(6,2),(7,2),(8,2)]_{H_{3,4}}$
	$[(0,0),(8,1),(0,2),(1,2),(2,2)]_{H_{3,4}}$	$[(0,0),(8,1),(3,2),(4,2),(5,2)]_{H_{3,4}}$
	$[(0,0),(8,1),(6,2),(7,2),(8,2)]_{H_{3,4}}$	

$H_{3,5}$	$[(0,0),(0,1),(1,2),(0,2),(1,1)]_{H_{3,5}}$	$[(0,0),(1,1),(1,2),(2,2),(0,1)]_{H_{3,5}}$
	$[(0,0),(0,1),(4,2),(3,2),(1,1)]_{H_{3,5}}$	$[(0,0),(1,1),(4,2),(5,2),(0,1)]_{H_{3,5}}$
	$[(0,0),(0,1),(7,2),(6,2),(1,1)]_{H_{3,5}}$	$[(0,0),(1,1),(7,2),(8,2),(0,1)]_{H_{3,5}}$
	$[(0,0),(2,1),(1,2),(0,2),(3,1)]_{H_{3,5}}$	$[(0,0),(3,1),(1,2),(2,2),(2,1)]_{H_{3,5}}$
	$[(0,0),(2,1),(4,2),(3,2),(3,1)]_{H_{3,5}}$	$[(0,0),(3,1),(4,2),(5,2),(2,1)]_{H_{3,5}}$
	$[(0,0),(2,1),(7,2),(6,2),(3,1)]_{H_{3,5}}$	$[(0,0),(3,1),(7,2),(8,2),(2,1)]_{H_{3,5}}$
	$[(0,0),(4,1),(1,2),(0,2),(5,1)]_{H_{3,5}}$	$[(0,0),(5,1),(1,2),(2,2),(4,1)]_{H_{3,5}}$
	$[(0,0),(4,1),(4,2),(3,2),(5,1)]_{H_{3,5}}$	$[(0,0),(5,1),(4,2),(5,2),(4,1)]_{H_{3,5}}$
	$[(0,0),(4,1),(7,2),(6,2),(5,1)]_{H_{3,5}}$	$[(0,0),(5,1),(7,2),(8,2),(4,1)]_{H_{3,5}}$
	$[(0,0),(6,1),(1,2),(0,2),(7,1)]_{H_{3,5}}$	$[(0,0),(8,1),(0,2),(1,2),(7,1)]_{H_{3,5}}$
	$[(0,0),(6,1),(3,2),(2,2),(7,1)]_{H_{3,5}}$	$[(0,0),(8,1),(2,2),(3,2),(7,1)]_{H_{3,5}}$
	$[(0,0),(6,1),(5,2),(4,2),(7,1)]_{H_{3,5}}$	$[(0,0),(8,1),(4,2),(5,2),(7,1)]_{H_{3,5}}$
	$[(0,0),(6,1),(7,2),(6,2),(7,1)]_{H_{3,5}}$	$[(0,0),(8,1),(6,2),(8,2),(6,1)]_{H_{3,5}}$
	$[(0,0),(7,1),(8,2),(7,2),(8,1)]_{H_{3,5}}$	

Example A.22 *An $H_{3,6}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}} \cup L^3_{\mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}}$:* The union of the orbits of the following $H_{3,6}$-blocks under the permutation $(i,j) \to (i+1,j)$ for $i \in \mathbb{Z}_9, j \in \mathbb{Z}_3$ form an $H_{3,6}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}} \cup L^3_{\mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}}$.

$[(0,0),(0,2),(0,1),(3,1),(1,2)]_{H_{3,6}}$	$[(0,0),(2,1),(0,2),(3,2),(8,1)]_{H_{3,6}}$
$[(0,0),(1,2),(0,1),(3,1),(2,2)]_{H_{3,6}}$	$[(0,0),(2,1),(1,2),(4,2),(8,1)]_{H_{3,6}}$
$[(0,0),(2,2),(0,1),(3,1),(3,2)]_{H_{3,6}}$	$[(0,0),(2,1),(2,2),(5,2),(8,1)]_{H_{3,6}}$
$[(0,0),(3,2),(0,1),(3,1),(4,2)]_{H_{3,6}}$	$[(0,0),(2,1),(6,2),(7,2),(6,1)]_{H_{3,6}}$
$[(0,0),(4,2),(0,1),(3,1),(5,2)]_{H_{3,6}}$	$[(0,0),(4,1),(0,2),(5,2),(3,1)]_{H_{3,6}}$
$[(0,0),(5,2),(0,1),(3,1),(6,2)]_{H_{3,6}}$	$[(0,0),(4,1),(1,2),(6,2),(3,1)]_{H_{3,6}}$
$[(0,0),(6,2),(0,1),(3,1),(7,2)]_{H_{3,6}}$	$[(0,0),(4,1),(2,2),(4,2),(8,1)]_{H_{3,6}}$
$[(0,0),(7,2),(0,1),(3,1),(8,2)]_{H_{3,6}}$	$[(0,0),(4,1),(3,2),(7,2),(2,1)]_{H_{3,6}}$
$[(0,0),(8,2),(0,1),(3,1),(0,2)]_{H_{3,6}}$	$[(0,0),(8,1),(0,2),(1,2),(3,1)]_{H_{3,6}}$
$[(0,0),(0,2),(1,1),(6,1),(8,2)]_{H_{3,6}}$	$[(0,0),(8,1),(3,2),(6,2),(3,1)]_{H_{3,6}}$
$[(0,0),(1,2),(1,1),(6,1),(0,2)]_{H_{3,6}}$	$[(0,0),(8,1),(4,2),(7,2),(5,1)]_{H_{3,6}}$
$[(0,0),(3,2),(1,1),(6,1),(2,2)]_{H_{3,6}}$	$[(0,0),(8,1),(5,2),(8,2),(7,1)]_{H_{3,6}}$
$[(0,0),(4,2),(1,1),(6,1),(3,2)]_{H_{3,6}}$	$[(0,0),(7,1),(0,2),(4,2),(2,1)]_{H_{3,6}}$
$[(0,0),(5,2),(1,1),(6,1),(4,2)]_{H_{3,6}}$	$[(0,0),(7,1),(1,2),(5,2),(4,1)]_{H_{3,6}}$
$[(0,0),(6,2),(1,1),(6,1),(5,2)]_{H_{3,6}}$	$[(0,0),(7,1),(2,2),(6,2),(6,1)]_{H_{3,6}}$
$[(0,0),(7,2),(1,1),(6,1),(1,2)]_{H_{3,6}}$	$[(0,0),(7,1),(3,2),(7,2),(8,1)]_{H_{3,6}}$
$[(0,0),(8,2),(1,1),(2,1),(1,2)]_{H_{3,6}}$	$[(0,0),(5,1),(0,2),(3,2),(3,1)]_{H_{3,6}}$
$[(0,0),(8,2),(4,1),(5,1),(2,2)]_{H_{3,6}}$	$[(0,0),(5,1),(1,2),(4,2),(6,1)]_{H_{3,6}}$

$[(0,0),(8,2),(6,1),(7,1),(2,2)]_{H_{3,6}}$ $[(0,0),(5,1),(2,2),(5,2),(6,1)]_{H_{3,6}}$

$[(0,0),(2,2),(6,1),(8,1),(3,2)]_{H_{3,6}}$ $[(0,0),(5,1),(6,2),(7,2),(2,1)]_{H_{3,6}}$

$[(0,1),(2,1),(0,2),(1,2),(8,1)]_{H_{3,6}}$ $[(0,2),(2,2),(0,1),(1,1),(8,2)]_{H_{3,6}}$

$[(0,1),(2,1),(2,2),(3,2),(8,1)]_{H_{3,6}}$ $[(0,2),(2,2),(2,1),(3,1),(8,2)]_{H_{3,6}}$

$[(0,1),(2,1),(4,2),(5,2),(8,1)]_{H_{3,6}}$ $[(0,2),(2,2),(4,1),(5,1),(8,2)]_{H_{3,6}}$

$[(0,1),(2,1),(7,2),(8,2),(8,1)]_{H_{3,6}}$ $[(0,2),(2,2),(7,1),(8,1),(8,2)]_{H_{3,6}}$

$[(0,1),(4,1),(0,2),(1,2),(2,1)]_{H_{3,6}}$ $[(0,2),(4,2),(0,1),(1,1),(2,2)]_{H_{3,6}}$

$[(0,1),(3,2),(2,2),(1,1),(0,0)]_{H_{3,6}}$

Example A.23 *An $H_{3,6}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}} \cup L^3_{\mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}} \cup K^3_{\mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\},\ \{\infty\}}$:* The union of the orbits of the following $H_{3,6}$-blocks form an $H_{3,6}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}} \cup L^3_{\mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}} \cup K^3_{\mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\},\ \{\infty\}}$ under the permutation $(i,j)\to(i+1,j)$ for $i\in\mathbb{Z}_9$, $j\in\mathbb{Z}_3$ and $\infty\to\infty$.

$[(0,0),(0,2),(0,1),(3,1),(1,2)]_{H_{3,6}}$ $[(0,0),(2,1),(0,2),(3,2),(8,1)]_{H_{3,6}}$

$[(0,0),(1,2),(0,1),(3,1),(2,2)]_{H_{3,6}}$ $[(0,0),(2,1),(1,2),(4,2),(8,1)]_{H_{3,6}}$

$[(0,0),(2,2),(0,1),(3,1),(3,2)]_{H_{3,6}}$ $[(0,0),(2,1),(2,2),(5,2),(8,1)]_{H_{3,6}}$

$[(0,0),(3,2),(0,1),(3,1),(4,2)]_{H_{3,6}}$ $[(0,0),(2,1),(6,2),(7,2),(6,1)]_{H_{3,6}}$

$[(0,0),(4,2),(0,1),(3,1),(5,2)]_{H_{3,6}}$ $[(0,0),(4,1),(0,2),(5,2),(3,1)]_{H_{3,6}}$

$[(0,0),(5,2),(0,1),(3,1),(6,2)]_{H_{3,6}}$ $[(0,0),(4,1),(1,2),(6,2),(3,1)]_{H_{3,6}}$

$[(0,0),(6,2),(0,1),(3,1),(7,2)]_{H_{3,6}}$ $[(0,0),(4,1),(2,2),(4,2),(8,1)]_{H_{3,6}}$

$[(0,0),(7,2),(0,1),(3,1),(8,2)]_{H_{3,6}}$ $[(0,0),(4,1),(3,2),(7,2),(2,1)]_{H_{3,6}}$

$[(0,0),(8,2),(0,1),(3,1),(0,2)]_{H_{3,6}}$ $[(0,0),(8,1),(0,2),(1,2),(3,1)]_{H_{3,6}}$

$[(0,0),(0,2),(1,1),(6,1),(8,2)]_{H_{3,6}}$ $[(0,0),(8,1),(3,2),(6,2),(3,1)]_{H_{3,6}}$

$[(0,0),(1,2),(1,1),(6,1),(0,2)]_{H_{3,6}}$ $[(0,0),(8,1),(4,2),(7,2),(5,1)]_{H_{3,6}}$

$[(0,0),(3,2),(1,1),(6,1),(2,2)]_{H_{3,6}}$ $[(0,0),(8,1),(5,2),(8,2),(7,1)]_{H_{3,6}}$

$[(0,0),(4,2),(1,1),(6,1),(3,2)]_{H_{3,6}}$ $[(0,0),(7,1),(0,2),(4,2),(2,1)]_{H_{3,6}}$

$[(0,0),(5,2),(1,1),(6,1),(4,2)]_{H_{3,6}}$ $[(0,0),(7,1),(1,2),(5,2),(4,1)]_{H_{3,6}}$

$[(0,0),(6,2),(1,1),(6,1),(5,2)]_{H_{3,6}}$ $[(0,0),(7,1),(2,2),(6,2),(6,1)]_{H_{3,6}}$

$[(0,0),(7,2),(1,1),(6,1),(1,2)]_{H_{3,6}}$ $[(0,0),(7,1),(3,2),(7,2),(8,1)]_{H_{3,6}}$

$[(0,0),(8,2),(1,1),(2,1),(1,2)]_{H_{3,6}}$ $[(0,0),(5,1),(0,2),(3,2),(3,1)]_{H_{3,6}}$

$[(0,0),(8,2),(4,1),(5,1),(2,2)]_{H_{3,6}}$ $[(0,0),(5,1),(1,2),(4,2),(6,1)]_{H_{3,6}}$

$$[(0,0),(8,2),(6,1),(7,1),(2,2)]_{H_{3,6}} \quad [(0,0),(5,1),(2,2),(5,2),(6,1)]_{H_{3,6}}$$
$$[(0,0),(2,2),(6,1),(8,1),(3,2)]_{H_{3,6}} \quad [(0,0),(5,1),(6,2),(7,2),(2,1)]_{H_{3,6}}$$
$$[(0,1),(2,1),(2,2),(3,2),(8,1)]_{H_{3,6}} \quad [(0,2),(2,2),(2,1),(3,1),(8,2)]_{H_{3,6}}$$
$$[(0,1),(2,1),(4,2),(5,2),(8,1)]_{H_{3,6}} \quad [(0,2),(2,2),(4,1),(5,1),(8,2)]_{H_{3,6}}$$
$$[(0,1),(2,1),(7,2),(8,2),(8,1)]_{H_{3,6}} \quad [(0,2),(2,2),(7,1),(8,1),(8,2)]_{H_{3,6}}$$
$$[(0,1),(4,1),(0,2),(1,2),(2,1)]_{H_{3,6}} \quad [(0,2),(4,2),(0,1),(1,1),(2,2)]_{H_{3,6}}$$
$$[\infty,(0,1),(1,2),(3,2),(2,1)]_{H_{3,6}} \quad [\infty,(4,2),(0,1),(2,1),(1,2)]_{H_{3,6}}$$
$$[\infty,(0,1),(7,2),(8,2),(6,1)]_{H_{3,6}} \quad [\infty,(6,2),(0,1),(1,1),(8,2)]_{H_{3,6}}$$
$$[(0,1),(0,2),(2,2),(2,1),\infty]_{H_{3,6}} \quad [(0,1),(3,2),(2,2),(1,1),(0,0)]_{H_{3,6}}$$

Example A.24 *An $H_{3,6}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}} \cup L^3_{\mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}} \cup K^3_{\mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\},\ \{\infty_1,\infty_2\}}$:* The union of the orbits of the following $H_{3,6}$-blocks under the permutation $(i,j) \to (i+1,j)$ for $i \in \mathbb{Z}_9, j \in \mathbb{Z}_3$ and $\infty_k \to \infty_k$ form an $H_{3,6}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}} \cup L^3_{\mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\}} \cup K^3_{\mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\},\ \{\infty_1,\infty_2\}}$.

$$[(0,0),(0,2),(0,1),(3,1),(1,2)]_{H_{3,6}} \quad [(0,0),(2,1),(0,2),(3,2),(8,1)]_{H_{3,6}}$$
$$[(0,0),(1,2),(0,1),(3,1),(2,2)]_{H_{3,6}} \quad [(0,0),(2,1),(1,2),(4,2),(8,1)]_{H_{3,6}}$$
$$[(0,0),(2,2),(0,1),(3,1),(3,2)]_{H_{3,6}} \quad [(0,0),(2,1),(2,2),(5,2),(8,1)]_{H_{3,6}}$$
$$[(0,0),(3,2),(0,1),(3,1),(4,2)]_{H_{3,6}} \quad [(0,0),(2,1),(6,2),(7,2),(6,1)]_{H_{3,6}}$$
$$[(0,0),(4,2),(0,1),(3,1),(5,2)]_{H_{3,6}} \quad [(0,0),(4,1),(0,2),(5,2),(3,1)]_{H_{3,6}}$$
$$[(0,0),(5,2),(0,1),(3,1),(6,2)]_{H_{3,6}} \quad [(0,0),(4,1),(1,2),(6,2),(3,1)]_{H_{3,6}}$$
$$[(0,0),(6,2),(0,1),(3,1),(7,2)]_{H_{3,6}} \quad [(0,0),(4,1),(2,2),(4,2),(8,1)]_{H_{3,6}}$$
$$[(0,0),(7,2),(0,1),(3,1),(8,2)]_{H_{3,6}} \quad [(0,0),(4,1),(3,2),(7,2),(2,1)]_{H_{3,6}}$$
$$[(0,0),(8,2),(0,1),(3,1),(0,2)]_{H_{3,6}} \quad [(0,0),(8,1),(0,2),(1,2),(3,1)]_{H_{3,6}}$$
$$[(0,0),(0,2),(1,1),(6,1),(8,2)]_{H_{3,6}} \quad [(0,0),(8,1),(3,2),(6,2),(3,1)]_{H_{3,6}}$$
$$[(0,0),(1,2),(1,1),(6,1),(0,2)]_{H_{3,6}} \quad [(0,0),(8,1),(4,2),(7,2),(5,1)]_{H_{3,6}}$$
$$[(0,0),(3,2),(1,1),(6,1),(2,2)]_{H_{3,6}} \quad [(0,0),(8,1),(5,2),(8,2),(7,1)]_{H_{3,6}}$$
$$[(0,0),(4,2),(1,1),(6,1),(3,2)]_{H_{3,6}} \quad [(0,0),(7,1),(0,2),(4,2),(2,1)]_{H_{3,6}}$$
$$[(0,0),(5,2),(1,1),(6,1),(4,2)]_{H_{3,6}} \quad [(0,0),(7,1),(1,2),(5,2),(4,1)]_{H_{3,6}}$$
$$[(0,0),(6,2),(1,1),(6,1),(5,2)]_{H_{3,6}} \quad [(0,0),(7,1),(2,2),(6,2),(6,1)]_{H_{3,6}}$$
$$[(0,0),(7,2),(1,1),(6,1),(1,2)]_{H_{3,6}} \quad [(0,0),(7,1),(3,2),(7,2),(8,1)]_{H_{3,6}}$$
$$[(0,0),(8,2),(1,1),(2,1),(1,2)]_{H_{3,6}} \quad [(0,0),(5,1),(0,2),(3,2),(3,1)]_{H_{3,6}}$$
$$[(0,0),(8,2),(4,1),(5,1),(2,2)]_{H_{3,6}} \quad [(0,0),(5,1),(1,2),(4,2),(6,1)]_{H_{3,6}}$$
$$[(0,0),(8,2),(6,1),(7,1),(2,2)]_{H_{3,6}} \quad [(0,0),(5,1),(2,2),(5,2),(6,1)]_{H_{3,6}}$$

$[(0,0),(2,2),(6,1),(8,1),(3,2)]_{H_{3,6}}$ $[(0,0),(5,1),(6,2),(7,2),(2,1)]_{H_{3,6}}$

$[(0,1),(2,1),(4,2),(5,2),(8,1)]_{H_{3,6}}$ $[(0,2),(2,2),(4,1),(5,1),(8,2)]_{H_{3,6}}$

$[(0,1),(2,1),(7,2),(8,2),(8,1)]_{H_{3,6}}$ $[(0,2),(2,2),(7,1),(8,1),(8,2)]_{H_{3,6}}$

$[(0,1),(4,1),(0,2),(1,2),(2,1)]_{H_{3,6}}$ $[(0,2),(4,2),(0,1),(1,1),(2,2)]_{H_{3,6}}$

$[\infty_1,(0,1),(1,2),(3,2),(2,1)]_{H_{3,6}}$ $[\infty_1,(4,2),(0,1),(2,1),(1,2)]_{H_{3,6}}$

$[\infty_1,(0,1),(7,2),(8,2),(6,1)]_{H_{3,6}}$ $[\infty_1,(6,2),(0,1),(1,1),(8,2)]_{H_{3,6}}$

$[\infty_2,(0,1),(1,2),(3,2),(4,1)]_{H_{3,6}}$ $[\infty_2,(4,2),(0,1),(2,1),(3,2)]_{H_{3,6}}$

$[\infty_2,(0,1),(5,2),(6,2),(2,1)]_{H_{3,6}}$ $[\infty_2,(8,2),(0,1),(1,1),(6,2)]_{H_{3,6}}$

$[(0,1),(0,2),(2,2),(2,1),\infty_1]_{H_{3,6}}$ $[(2,1),(2,2),(0,1),(0,2),\infty_2]_{H_{3,6}}$

$[(0,1),(3,2),(2,2),(1,1),(0,0)]_{H_{3,6}}$

Example A.25 *An $H_{3,6}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\},\ \mathbb{Z}_9\times\{3\}}$:*
The union of the orbits of the following $H_{3,6}$-blocks under the permutation $(i,j)\to(i+1,j)$ and $(i,j)\to(i,j+1)$ for $i\in\mathbb{Z}_9, j\in\mathbb{Z}_4$ form an $H_{3,6}$-decomposition of $K^3_{\mathbb{Z}_9\times\{0\},\ \mathbb{Z}_9\times\{1\},\ \mathbb{Z}_9\times\{2\},\ \mathbb{Z}_9\times\{3\}}$.

$[(0,1),(0,2),(0,0),(8,3),(1,1)]_{H_{3,6}}$ $[(0,1),(1,2),(0,0),(6,3),(6,1)]_{H_{3,6}}$

$[(1,1),(8,2),(0,0),(2,3),(7,1)]_{H_{3,6}}$ $[(1,1),(0,2),(0,0),(0,3),(3,1)]_{H_{3,6}}$

$[(2,1),(7,2),(0,0),(5,3),(5,1)]_{H_{3,6}}$ $[(2,1),(8,2),(0,0),(3,3),(0,1)]_{H_{3,6}}$

$[(3,1),(6,2),(0,0),(8,3),(2,1)]_{H_{3,6}}$ $[(3,1),(7,2),(0,0),(6,3),(7,1)]_{H_{3,6}}$

$[(4,1),(5,2),(0,0),(2,3),(8,1)]_{H_{3,6}}$ $[(4,1),(6,2),(0,0),(0,3),(5,1)]_{H_{3,6}}$

$[(5,1),(4,2),(0,0),(5,3),(6,1)]_{H_{3,6}}$ $[(5,1),(5,2),(0,0),(3,3),(1,1)]_{H_{3,6}}$

$[(6,1),(3,2),(0,0),(8,3),(3,1)]_{H_{3,6}}$ $[(6,1),(4,2),(0,0),(6,3),(8,1)]_{H_{3,6}}$

$[(7,1),(2,2),(0,0),(2,3),(0,1)]_{H_{3,6}}$ $[(7,1),(3,2),(0,0),(0,3),(4,1)]_{H_{3,6}}$

$[(8,1),(1,2),(0,0),(5,3),(4,1)]_{H_{3,6}}$ $[(8,1),(2,2),(0,0),(3,3),(2,1)]_{H_{3,6}}$

$[(0,1),(2,2),(0,0),(4,3),(3,1)]_{H_{3,6}}$ $[(5,1),(6,2),(0,0),(1,3),(6,1)]_{H_{3,6}}$

$[(1,1),(1,2),(0,0),(7,3),(8,1)]_{H_{3,6}}$ $[(6,1),(5,2),(0,0),(4,3),(4,1)]_{H_{3,6}}$

$[(2,1),(0,2),(0,0),(1,3),(5,1)]_{H_{3,6}}$ $[(7,1),(4,2),(0,0),(7,3),(1,1)]_{H_{3,6}}$

$[(3,1),(8,2),(0,0),(4,3),(2,1)]_{H_{3,6}}$ $[(8,1),(3,2),(0,0),(1,3),(7,1)]_{H_{3,6}}$

$[(4,1),(7,2),(0,0),(7,3),(0,1)]_{H_{3,6}}$

References

[1] P. Adams, D. Bryant and M. Buchanan, A survey on the existence of G-designs, *J. Combin. Des.*, 16 (2008), 373–410.

[2] R. F. Bailey and B. Stevens, Hamiltonian decompositions of complete k-uniform hypergraphs, *Discrete Math.*, 310 (2010), 3088–3095.

[3] Zs. Baranyai, On the factorization of the complete uniform hypergraph, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdos on his 60th birthday), Vol. I, pp. 91–108. *Colloq. Math. Soc. Janos Bolyai*, 10 North-Holland, Amsterdam, 1975.

[4] L. W. Beineke, Graph decompositions. Surveys in graph theory (San Francisco, CA, 1995). Congr. Numer., 115 (1996), 213–226.

[5] J.-C. Bermond, C. Huang, A. Rosa and D. Sotteau, Decomposition of complete graphs into isomorphic subgraphs with five vertices, *Ars Combin.*, 10 (1980), 211–254.

[6] J.-C. Bermond and J. Schönheim, G-decomposition of K_n, where G has four vertices or less, *Discrete Math.*, 19 (1977), 113–120.

[7] D. Bryant and S. El-Zanati, Graph Decompositions, in *The CRC Handbook of Combinatorial Designs*, 2nd edition (Eds. C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton (2007), 477–486.

[8] Y. Chang and J. Zhou, Large sets of Kirkman triple systems and related designs. *J. Combin. Theory Ser. A*, 120 (2013), 649–670.

[9] C. Colbourn, Triple Systems, in *The CRC Handbook of Combinatorial Designs*, 2nd edition (Eds. C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton (2007), 58–71.

[10] C. J. Colbourn and R. Mathon, Steiner Systems, in *The CRC Handbook of Combinatorial Designs*, 2nd edition (Eds. C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton (2007), 102–110.

[11] T. Feng and Y. Chang, Decompositions of the 3-uniform hypergraphs $K_v^{(3)}$ into hypergraphs of a certain type, *Sci China Ser A*, 50 (2007) 1035–1044

[12] T. Feng and Y. Chang, Decompositions of 3-uniform hypergraph $K_v^{(3)}$ into hypergraph $K_4^{(3)} + e$, *Util. Math.*, 87 (2012), 49–64.

[13] G. Ge, S. Hu, E. Kolotoglu and H. Wei, A complete solution to spectrum problem for five-vertex graphs with application to traffic grooming in optical networks, *preprint.*

[14] T. S. Griggs, M. J. de Resmini and A. Rosa, Decomposing Steiner triple systems into four-line configurations, *Ann. Discrete Math.*, 52 (1992) 215–226.

[15] P. Hall, On Representatives of Subsets, *J. London Math. Soc.*, 10 (1935) 26–30.

[16] H. Hanani, Decomposition of hypergraphs into octahedra, Second International Conference on Combinatorial Mathematics (New York, 1978), pp. 260–264, *Ann. New York Acad. Sci.*, 319 New York Acad. Sci., New York, 1979.

[17] H. Hanani, On some tactical configurations, *Canad. J. Math.*, 15 (1963) 702–722

[18] H. Hanani, On quadruple systems, *Canad. J. Math.*, 12 (1960) 145–157.

[19] A. Hartman and K. T. Phelps, Steiner Quadruple Systems, in Contemporary Design Theory A Collection of Surveys, eds. J.H. Dinitz and D.R. Stinson, John Wiley & Sons Inc., 1992.

[20] P. Keevash, The existence of designs, `arXiv:1401.3665`, (2014), 56 pages.

[21] G. B. Khosrovshahi and R. Laue, t-Designs wth $t \geq 3$, in *The CRC Handbook of Combinatorial Designs*, 2nd edition (Eds. C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton (2007), 79–101.

[22] V. Linek and B. Stevens, Octahedral designs, *J. Combin. Des.*, 18 (2010), 319–327.

[23] J. X. Lu, On large sets of disjoint Steiner triple systems I-III, *J. Combin. Theory A*, 34 (1983), 140–182.

[24] J. X. Lu, On large sets of disjoint Steiner triple systems IV-VI, *J. Combin. Theory A*, 37 (1984), 136–192.

[25] J. X. Lu, On large sets of disjoint Steiner triple systems VII, unpublished manuscript, 1984.

[26] M. Meszka and A. Rosa, Decomposing complete 3-uniform hypergraphs into Hamiltonian cycles, *Australis. J. Combin.*, 45 (2009) 291–302.

[27] W.H. Mills, On the covering of triples by quadruples, *Congr. Numer.*, 10 (1974) 563–581.

[28] W.H. Mills, On the existence of H designs, *Congr. Numer.*, 79 (1990) 129–141.

[29] H. Mohácsy and D.K. Ray-Chaudhuri, Candelabra systems and designs, *J. Statist. Plann. Inference.*, 106 (2002) 419–448.

[30] L. Teirlinck, A completion of Lu's determination of the spectrum for large sets of disjoint Steiner triple systems, *J. Combin. Theory A*, 57 (1991), 302–305.

[31] R. Yuster, Decomposing hypergraphs into simple hypertrees, *Combinatorica*, 20 (2000), 119–140.

[32] X. Zhang and G. Ge, A new existence proof for Steiner quadruple systems, *Des. Codes Cryptogr.*, 69 (2013) 65–76.

(Received 14 May 2014; revised 29 July 2014)