AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **62(3)** (2015), Pages 211–227

Generalization of the degree distance of the tensor product of graphs

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Abstract

In this paper, the exact formulae for the generalized degree distance, the degree distance and the reciprocal degree distance of the tensor product of a connected graph and of the complete multipartite graph with partite sets of sizes $m_0, m_1, \ldots, m_{r-1}$ are obtained. In addition we show that the result given by Wang and Kang [*J. Comb. Optim.*, online June 2014] on the degree distance of the tensor product of graphs is incorrect, and the corrected version of this result is the corollary of our main theorem.

1 Introduction

All the graphs considered in this paper are simple and connected. For a graph G, and for vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. We also let $d_G(v)$ be the degree of a vertex $v \in V(G)$. For two simple graphs G and H, their *tensor product*, denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever g_1g_2 is an edge in G and h_1h_2 is an edge in H. Note that if G and H are connected graphs, then $G \times H$ is connected only if at least one of the graphs is nonbipartite. The tensor product of graphs has been extensively studied in relation to areas such as graph colorings, graph recognition, decompositions of graphs, and design theory; see [1, 3, 4, 18, 21].

A *topological index* of a graph is a real number related to the graph; it does not depend on any labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [10]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index.

Let G be a connected graph. The *Wiener index* of G is defined as $W(G) = \frac{1}{2}\sum_{u,\,v\,\in\,V(G)} d_G(u,v)$, with the summation over all pairs of distinct vertices of G. This definition can be further generalized in the following way:

$$W_\lambda(G) = \frac{1}{2}\sum_{u,\,v\,\in\,V(G)} d_G^\lambda(u,v),$$

where $d_G^\lambda(u,v) = (d_G(u,v))^\lambda$ and λ is a real number [11, 12]. If $\lambda = -1$, then $W_{-1}(G) = H(G)$, where $H(G)$ is the Harary index of G. In the chemical literature both $W_{\frac{1}{2}}$ [28] as well as the general case W_λ were examined [8, 13].

Dobrynin and Kochetova [6] and Gutman [9] independently proposed a vertex-degree-weighted version of the Wiener index called the *degree distance* or the *Schultz molecular* topological index, which is defined for a connected graph G as $DD(G) = \frac{1}{2}\sum_{u,v\in V(G)} (d_G(u) + d_G(v))d_G(u,v)$, where $d_G(u)$ is the degree of the vertex u in G. Note that the degree distance is a degree-weight version of the Wiener index. In the literature, many results on the degree distance $DD(G)$ have been put forward in past decades, and they mainly deal with extreme properties of $DD(G)$. Tomescu [25] showed that the star is the unique graph with minimum degree distance within the class on n-vertex connected graphs. Tomescu [26] deduced properties of graphs with minimum degree distance in the class of n-vertex connected graphs with $m \geq n-1$ edges. For other related results along this line, see [5, 16, 19].

The additively weighted Harary index(H_A) or the reciprocal degree distance (RDD) is defined in [2] as $H_A(G) = \mathrm{RDD}(G) = \frac{1}{2}\sum_{u,v\in V(G)} \frac{(d_G(u)+d_G(v))}{d_G(u,v)}$. In [14], Hamzeh et al. recently introduced the concept of the generalized degree distance of graphs. Hua and Zhang [17] have obtained lower and upper bounds for the reciprocal degree distance of a graph in terms of other graph invariants, including the degree distance, Harary index, the first Zagreb index, the first Zagreb coindex, pendent vertices, independence number, chromatic number and vertex- and edge-connectivity. Pattabiraman and Vijayaragavan [22, 23] have obtained the reciprocal degree distance of the join, tensor product, strong product and wreath product of two connected graphs in terms of other graph invariants. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [2, 20, 24].

The generalized degree distance, denoted by $H_\lambda(G)$, is defined as $H_\lambda(G) = \frac{1}{2}\sum_{u,v\in V(G)} (d_G(u) + d_G(v))d_G^\lambda(u,v)$, where λ is a any real number. If $\lambda = 1$ then $H_\lambda(G) = DD(G)$, and if $\lambda = -1$ then $H_\lambda(G) = \mathrm{RDD}(G)$. The generalized degree distance of unicyclic and bicyclic graphs are studied by Hamzeh et al. [14, 15]. Also they have given the generalized degree distance of the Cartesian product, join, symmetric difference, composition and disjunction of two graphs. It is well-known that many graphs arise from simpler graphs via various graph operations. Hence it is important to understand how certain invariants of such product graphs are related to the corresponding invariants of the original graphs. In this paper, the exact formulae for the generalized degree distance, degree distance and reciprocal degree distance of the tensor product $G \times K_{m_0, m_1, \ldots, m_{r-1}}$, where $K_{m_0, m_1, \ldots, m_{r-1}}$ is the complete multipartite graph with partite sets of sizes m_0, m_1, $\ldots$, m_{r-1} are obtained. We show in Section 2 that the major result proved in the paper [27] is incorrect, and the corrected version of this result is a corollary of Theorem 2.5 proved by us. In addition, we have given a counterexample to justify our claim.

The *first Zagreb index* is defined as $M_1(G) = \sum\limits_{u \in V(G)} d_G(u)^2$. In fact, one can rewrite the first Zagreb index as $M_1(G) = \sum\limits_{uv \in E(G)} (d_G(u) + d_G(v))$. Zagreb indices are found to have applications in Quantitative Structure Property Relationship (QSPR) and Quantitative Structure Activity Relationship (QSAR) studies as well; see [7].

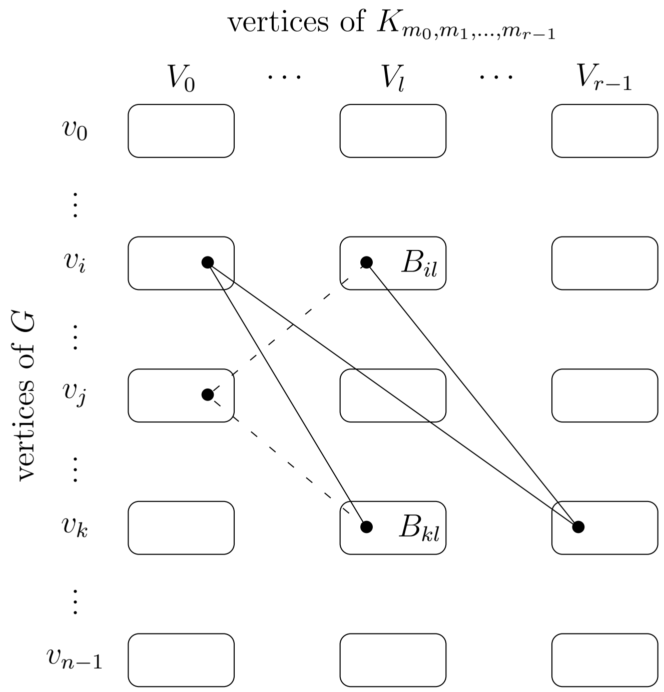
If $m_0 = m_1 = \cdots = m_{r-1} = s$ in $K_{m_0, m_1, \ldots, m_{r-1}}$ (the complete multipartite graph with partite sets of sizes m_0, m_1, $\ldots$, m_{r-1}), then we denote this graph by $K_{r(s)}$. For $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph of G induced by S. For two subsets $S, T \subset V(G)$, not necessarily disjoint, by the notation $d_G(S,T)$ we mean the sum of the distances in G from each vertex of S to every vertex of T; that is, $d_G(S,T) = \sum\limits_{s \in S, t \in T} d_G(s,t)$.

2 Generalized degree distance of tensor product of graphs

Let G be a connected graph with $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and let $K_{m_0, m_1, \ldots, m_{r-1}}$, $r \geq 3$, be the complete multiparite graph with partite sets V_0, V_1, $\ldots$, V_{r-1} with $|V_i| = m_i$, $0 \leq i \leq r-1$. In the graph $G \times K_{m_0, m_1, \ldots, m_{r-1}}$, let $B_{ij} = v_i \times V_j, v_i \in V(G)$ and $0 \leq j \leq r-1$. For our convenience, we write

$$\begin{aligned} V(G) \times V(K_{m_0, m_1, \ldots, m_{r-1}}) &= \bigcup_{i=0}^{n-1} \left\{ v_i \times \bigcup_{j=0}^{r-1} V_j \right\} \\ &= \bigcup_{i=0}^{n-1} \{\{v_i \times V_0\} \cup \{v_i \times V_1\} \cup \cdots \cup \{v_i \times V_{r-1}\}\} \\ &= \bigcup_{i=0}^{n-1} \left\{ B_{i0} \cup B_{i1} \cup \cdots \cup B_{i(r-1)} \right\}, \text{ where } B_{ij} = v_i \times V_j \\ &= \bigcup_{\substack{i=0 \\ j=0}}^{\substack{r-1 \\ n-1}} B_{ij}. \end{aligned}$$

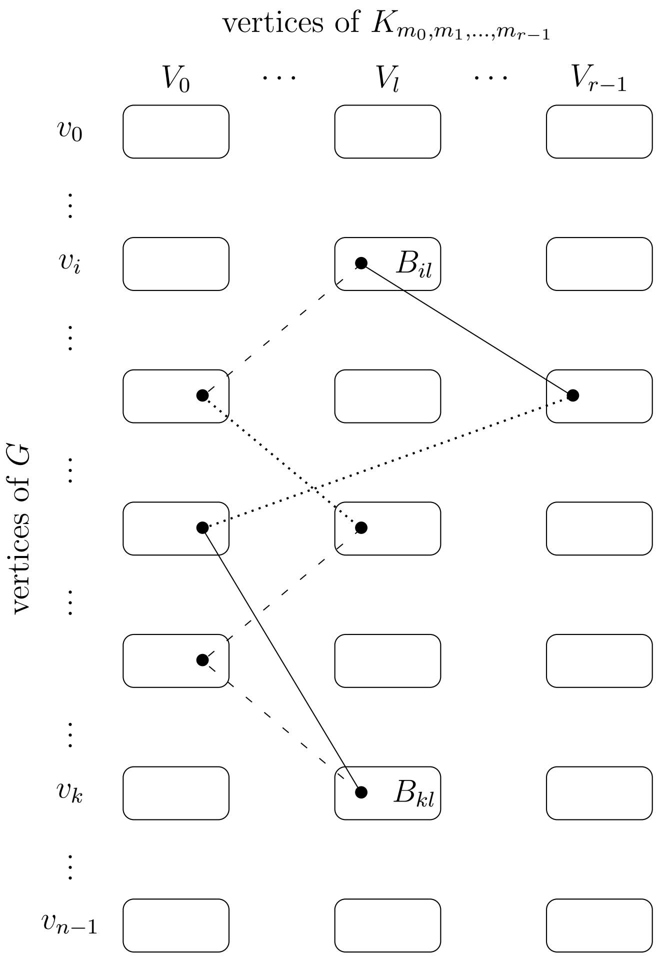
Let $\mathscr{B} = \{B_{ij}\}_{\substack{i=0,1,\ldots,n-1 \\ j=0,1,\ldots,r-1}}$. We call $X_i = \bigcup_{j=0}^{r-1} B_{ij}$ a *layer* and $Y_j = \bigcup_{i=0}^{n-1} B_{ij}$ a *column* of $G \times K_{m_0,m_1,\ldots,m_{r-1}}$; see Figures 1 and 2. Clearly, a layer (respectively, column) is an independent set in $G \times K_{m_0,m_1,\ldots,m_{r-1}}$; in particular, B_{ij} is an independent set. Further, if $v_iv_k \in E(G)$, then the subgraph $\langle B_{ij} \cup B_{kp}\rangle$ of $G \times K_{m_0,m_1,\ldots,m_{r-1}}$ is isomorphic to $K_{|V_j||V_p|}$ or a totally disconnected graph according as $j \neq p$ or $j = p$. This is used in the proof of the next lemma.



If v_iv_k is on a triangle $v_iv_jv_k$ in G, then the distance from a vertex in B_{il} to a vertex in B_{kl} is 2 and a shortest path is shown with broken edges. If v_iv_k is an edge but not on a triangle in G, then the distance from a vertex in B_{il} to a vertex in B_{kl} is 3 and a shortest path is shown with solid edges.

Fig. 1

The proof of the following lemma follows easily from the properties and structure of $G \times K_{m_0,m_1,\ldots,m_{r-1}}$ and the paths as shown in Figures 1 and 2.



If the distance d from v_i to v_k in G is even (resp. odd), then the distance from a vertex in B_{il} to a vertex in B_{kl} is d (resp. d) and a shortest path is shown with broken (resp. solid) edges.

Fig. 2

Lemma 2.1. *Let G be a connected graph on $n \geq 2$ vertices and let B_{ij}, $B_{kp} \in \mathscr{B}$ of the graph $G \times K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$, where $r \geq 3$.*

(i) The distance between any two distinct vertices in B_{ij} is 2.

(ii) The distance between any two vertices one from B_{ij} and another from B_{ip}, $j \neq p$, is 2.

(iii) The distance between any two vertices one from B_{ij} and another from B_{kj}, $i \neq k$, is 2 or 3 according as v_iv_k lies on a triangle in G or $v_iv_k \in E(G)$ and v_iv_k does not lies on a triangle in G.

(iv) If $v_iv_k \in E(G)$, then the distance between two vertices, one in B_{ij} and the other in B_{kp}, $i \neq k,\ j \neq p$, is 1.

(v) If $v_iv_k \notin E(G)$, then the distance between two vertices, one in B_{ij} and the other in B_{kp}, is $d_G(v_i, v_k)$.

The proof of the following lemma follows easily from Lemma 2.1, and hence it is left to the reader. This lemma is used in the proof of the main theorem of this section.

Lemma 2.2. *Let G be a connected graph on $n \geq 2$ vertices and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \times K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 3$.*

(i) If $v_iv_k \in E(G)$, then

$$d^{\lambda}_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_jm_p, & \text{if } j \neq p, \\ 2^{\lambda}m_j^2, & \text{if } j = p \text{ and } v_iv_k \text{ is on a triangle of } G, \\ 3^{\lambda}m_j^2, & \text{if } j = p \text{ and } v_iv_k \text{ is not on a triangle of } G. \end{cases}$$

(ii) If $v_iv_k \notin E(G)$, then $d^{\lambda}_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_jm_p\ d^{\lambda}_G(v_i, v_k), & if\ j \neq p, \\ m_j^2\ d^{\lambda}_G(v_i, v_k), & if\ j = p. \end{cases}$

(iii) $d^{\lambda}_{G'}(B_{ij}, B_{ip}) = \begin{cases} 2^{\lambda}m_j(m_j - 1), & if\ j = p, \\ 2^{\lambda}m_jm_p, & if\ j \neq p. \end{cases}$

Proof. (i) Let $v_iv_k \in E(G)$.

If $j \neq p$, then the distance between a vertex of B_{ij} and a vertex of B_{kp} is 1 in G' and there are m_jm_p pairs of vertices between B_{ij} and B_{kp}; hence $d^{\lambda}_{G'}(B_{ij}, B_{kp}) = m_jm_p$.

If $j = p$ and v_iv_k lie in a triangle (respectively, not in a triangle), then the distance between a vertex of B_{ij} and a vertex of B_{kj} is 2^{λ} (respectively, 3^{λ}) in G', and there are m_j^2 pairs of vertices between B_{ij} and B_{kj}; hence $d^{\lambda}_{G'}(B_{ij}, B_{kj}) = 2^{\lambda}m_j^2$ (respectively, $3^{\lambda}m_j^2$).

(ii) If $v_iv_k \notin E(G)$, then the distance from a vertex of B_{ij} to a vertex of B_{kj} (respectively, B_{kp}) is $d^{\lambda}_G(v_i, v_k)$ in G' and there are m_jm_p (respectively, m_j^2) pair of vertices between B_{ij} and B_{kp} (respectively, B_{kj}); hence $d^{\lambda}_{G'}(B_{ij}, B_{kp}) = m_jm_pd^{\lambda}_G(v_i, v_k)$ (respectively, $d^{\lambda}_{G'}(B_{ij}, B_{kj}) = m_j^2d^{\lambda}_G(v_i, v_k)$).

(iii) The distance between a vertex of B_{ij} and a vertex of B_{ip} (respectively, B_{ij}) is 2^{λ} (respectively, 2^{λ}) in G', and there are $m_j(m_j - 1)$ (respectively, m_jm_p) pairs of vertices between B_{ij} and B_{ip} (respectively, B_{ij}); hence $d^{\lambda}_{G'}(B_{ij}, B_{ij}) = 2^{\lambda}m_j(m_j - 1)$ (respectively, $d^{\lambda}_{G'}(B_{ij}, B_{ip}) = 2^{\lambda}m_jm_p$). ■

Lemma 2.3. *Let G be a connected graph and let B_{ij} be in $G' = G \times K_{m_0, m_1, \ldots, m_{r-1}}$. Then the degree of a vertex $(v_i, u_j) \in B_{ij}$ in G' is $d_{G'}((v_i, u_j)) = d_G(v_i)(n_0 - m_j)$, where $n_0 = \sum_{j=0}^{r-1} m_j$.* ■

Remark 2.4. *The following sums hold:* $\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} m_j m_p = 2q$, $\sum_{j=0}^{r-1} m_j^2 = n_0^2 - 2q$, $\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} m_j^2 m_p = n_0^3 - 2n_0 q - \sum_{j=0}^{r-1} m_j^3 = \sum_{\substack{j,p=0\\ j\neq p}}^{r-1} m_j m_p^2$ *and*

$$\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} m_j^3 m_p = n_0 \sum_{j=0}^{r-1} m_j^3 - \sum_{j=0}^{r-1} m_j^4 = \sum_{\substack{j,p=0\\ j\neq p}}^{r-1} m_j m_p^3,$$

where $n_0 = \sum_{j=0}^{r-1} m_j$ *and* q *is the number of edges of* $K_{m_0, m_1, \ldots, m_{r-1}}$. ■

Next we determine the generalized degree distance of $G \times K_{m_0, m_1, \ldots, m_{r-1}}$.

Theorem 2.5. *Let G be a connected graph with $n \geq 2$ vertices and m edges and let E_2 be the set of edges of G which do not lie on any C_3 of it. If n_0 and q are the numbers of vertices and edges of $K_{m_0, m_1, \ldots, m_{r-1}}$, $r \geq 3$, respectively, then*

$$H_\lambda(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 2n_0 q\, H_\lambda(G) + 2^{\lambda+2} m q (n_0 - 1) +$$

$$\left((2^\lambda - 1) M_1(G) + \frac{(3^\lambda - 2^\lambda)}{2} \sum_{v_i v_k \in E_2} \Big(d_G(v_i) + d_G(v_k)\Big)\right) \Big(n_0^3 - 2q n_0 - \sum_{j=0}^{r-1} m_j^3\Big).$$

Proof. Let $G' = G \times K_{m_0, m_1, \ldots, m_{r-1}}$. Clearly,

$$\begin{aligned}
H_\lambda(G') &= \frac{1}{2} \sum_{B_{ij}, B_{kp} \in \mathscr{B}} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kp})\Big) d_{G'}^\lambda(B_{ij}, B_{kp}) \\
&= \frac{1}{2}\Bigg(\sum_{i=0}^{n-1} \sum_{\substack{j,p=0\\ j\neq p}}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{ip})\Big) d_{G'}^\lambda(B_{ij}, B_{ip}) \\
&\quad + \sum_{\substack{i,k=0\\ i\neq k}}^{n-1} \sum_{j=0}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kj})\Big) d_{G'}^\lambda(B_{ij}, B_{kj}) \\
&\quad + \sum_{\substack{i,k=0\\ i\neq k}}^{n-1} \sum_{\substack{j,p=0\\ j\neq p}}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kp})\Big) d_{G'}^\lambda(B_{ij}, B_{kp}) \\
&\quad + \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{ij})\Big) d_{G'}^\lambda(B_{ij}, B_{ij}) \Bigg) \\
&= \frac{1}{2}\{A_1 + A_2 + A_3 + A_4\},
\end{aligned} \tag{2.1}$$

where A_1 to A_4 are the sums of the above terms, in order.

We shall calculate A_1 to A_4 of (2.1) separately.

($\mathbf{A_1}$) First we compute $\sum\limits_{i=0}^{n-1}\sum\limits_{\substack{j,p=0\\ j\neq p}}^{r-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{ip})\Big)d^{\lambda}_{G'}(B_{ij},B_{ip})$.

$$\begin{aligned}
&\sum_{i=0}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{ip})\Big)d^{\lambda}_{G'}(B_{ij},B_{ip})\\
&= \sum_{i=0}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}\Big((n_0-m_j)d_G(v_i)+(n_0-m_p)d_G(v_i)\Big)\,2^{\lambda}\,m_jm_p,\\
&\quad \text{by Lemmas 2.1, 2.2 and 2.3}\\
&= \sum_{i=0}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}2^{\lambda}\Big(2n_0-m_j-m_p\Big)d_G(v_i)m_jm_p\\
&= 2^{\lambda+2}\,m\Big(4n_0q-n_0^3+\sum_{j=0}^{r-1}m_j^3\Big),\ \text{by Remark 2.4.}
\end{aligned}$$

($\mathbf{A_2}$) Next we compute $\sum\limits_{j=0}^{r-1}\sum\limits_{\substack{i,k=0\\ i\neq k}}^{n-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{kj})\Big)d^{\lambda}_{G'}(B_{ij},B_{kj})$. For this, initially we calculate $\sum\limits_{\substack{i,k=0\\ i\neq k}}^{n-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{kj})\Big)d^{\lambda}_{G'}(B_{ij},B_{kj})$.

Let $E_1=\{uv\in E(G)\,|\,uv \text{ is on a } C_3 \text{ in } G\}$ and $E_2=E(G)-E_1$.

$$\begin{aligned}
&\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{kj})\Big)d^{\lambda}_{G'}(B_{ij},B_{kj})\\
&= \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{kj})\Big)d^{\lambda}_{G'}(B_{ij},B_{kj})
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\in E_1}}^{n-1} \Big(d_{G'}(B_{ij})+d_{G'}(B_{kj})\Big) d^{\lambda}_{G'}(B_{ij},B_{kj}) \\
&+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\in E_2}}^{n-1} \Big(d_{G'}(B_{ij})+d_{G'}(B_{kj})\Big) d^{\lambda}_{G'}(B_{ij},B_{kj}) \\
= &\sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\notin E(G)}}^{n-1} (n_0-m_j)(d_G(v_i)+d_G(v_k))m_j^2\ d^{\lambda}_G(v_i,v_k) \\
&+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\in E_1}}^{n-1} (n_0-m_j)(d_G(v_i)+d_G(v_k))2^{\lambda}\, m_j^2 \\
&+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\in E_2}}^{n-1} (n_0-m_j)(d_G(v_i)+d_G(v_k))3^{\lambda}\, m_j^2, \quad \text{by Lemmas 2.2 and 2.3} \\
= &\sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\notin E(G)}}^{n-1} (n_0-m_j)\Big(d_G(v_i)+d_G(v_k)\Big)m_j^2\ d^{\lambda}_G(v_i,v_k) \\
&+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\in E_1}}^{n-1} (n_0-m_j)\Big(d_G(v_i)+d_G(v_k)\Big)\Big(2^{\lambda}\, m_j^2+m_j^2-m_j^2\Big) \\
&+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\in E_2}}^{n-1} (n_0-m_j)\Big(d_G(v_i)+d_G(v_k)\Big)\Big(3^{\lambda}\, m_j^2+m_j^2-m_j^2\Big)
\end{aligned}
$$

$$= \Bigg(\sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k \notin E(G)}}^{n-1} (n_0 - m_j)\Big(d_G(v_i) + d_G(v_k)\Big)m_j^2\ d_G^{\lambda}(v_i, v_k)$$

$$+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k \in E_1}}^{n-1} (n_0 - m_j)\Big(d_G(v_i) + d_G(v_k)\Big)m_j^2\ d_G^{\lambda}(v_i, v_k)$$

$$+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k \in E_2}}^{n-1} (n_0 - m_j)\Big(d_G(v_i) + d_G(v_k)\Big)m_j^2\ d_G^{\lambda}(v_i, v_k)\Bigg)$$

$$+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k \in E_1}}^{n-1} (n_0 - m_j)\Big(d_G(v_i) + d_G(v_k)\Big)(2^{\lambda} - 1)m_j^2$$

$$+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k \in E_2}}^{n-1} (n_0 - m_j)\Big(d_G(v_i) + d_G(v_k)\Big)(3^{\lambda} - 1)m_j^2,$$

since $d_G^{\lambda}(v_i, v_k) = 1$ if $v_iv_k \in E_1$ and $v_iv_k \in E_2$

$$= \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} (n_0 - m_j)\Big(d_G(v_i) + d_G(v_k)\Big)m_j^2\ d_G^{\lambda}(v_i, v_k)$$

$$+\Bigg(\sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k \in E_1}}^{n-1} (n_0 - m_j)\Big(d_G(v_i) + d_G(v_k)\Big)(2^{\lambda} - 1)m_j^2$$

$$+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k \in E_2}}^{n-1} (n_0 - m_j)\Big(d_G(v_i) + d_G(v_k)\Big)(2^{\lambda} - 1)m_j^2\Bigg)$$

$$+ \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k \in E_2}}^{n-1} (n_0 - m_j)\Big(d_G(v_i) + d_G(v_k)\Big)(3^{\lambda} - 2^{\lambda})m_j^2$$

$$= (n_0 - m_j)m_j^2\Big(2H_{\lambda}(G) + 2M_1(G)(2^{\lambda} - 1) + (3^{\lambda} - 2^{\lambda}) \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k \in E_2}}^{n-1} (d_G(v_i) + d_G(v_k))\Big), \quad (2.2)$$

where $M_1(G)$ is the first Zagreb index of G. Note that each edge v_iv_k of G is being counted twice in the sum, namely, v_iv_k and v_kv_i.

Now summing (2.2) over $j = 0, 1, \ldots, r-1$, we get

$$\sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \Big(d_{G'}(B_{ij})) + d_{G'}(B_{kj})\Big) d^{\lambda}_{G'}(B_{ij}, B_{kj}) = \Big(n_0^3 - 2qn_0 - \sum_{j=0}^{r-1} m_j^3\Big)$$
$$\times \Big(2H_\lambda(G) + 2(2^\lambda - 1)M_1(G) + (3^\lambda - 2^\lambda) \sum_{\substack{i,k=0 \\ i \neq k \\ v_i v_k \in E_2}}^{n-1} \big(d_G(v_i) + d_G(v_k)\big)\Big), \qquad (2.3)$$

by Remark 2.4.

$(\mathbf{A_3})$ Next we compute $\sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kp})\Big) d^{\lambda}_{G'}(B_{ij}, B_{kp})$.

$$\sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kp})\Big) d^{\lambda}_{G'}(B_{ij}, B_{kp})$$
$$= \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} \Big((n_0 - m_j)d_G(v_i) + (n_0 - m_p)d_G(v_k)\Big) m_j m_p \; d^{\lambda}_G(v_i, v_k),$$

by Lemmas 2.2 and 2.3

$$= \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} \Big((n_0 m_j m_p - m_j^2 m_p) d_G(v_i) d^{\lambda}_G(v_i, v_k) + (n_0 m_j m_p - m_j m_p^2) d_G(v_k) d^{\lambda}_G(v_i, v_k)\Big)$$
$$= \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} n_o m_j m_p \Big(d_G(v_i) + d_G(v_k)\Big) d^{\lambda}_G(v_i, v_k) - m_j^2 m_p d_G(v_i) d^{\lambda}_G(v_i, v_k)$$
$$- m_j m_p^2 d_G(v_k) d^{\lambda}_G(v_i, v_k),$$

by Remark 2.4

$$= 2qn_0(2H_\lambda(G)) - (n_0^3 - 2qn_0 - \sum_{j=0}^{r-1} m_j^3)(2H_\lambda(G))$$
$$= \Big(4n_0 q - n_0^3 + \sum_{j=0}^{r-1} m_j^3\Big) 2H_\lambda(G). \qquad (2.4)$$

($\mathbf{A_4}$) Finally, we compute $\sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{r-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{ij})\Big)d^{\lambda}_{G'}(B_{ij},B_{ij})$.

$$\begin{aligned}
&\sum_{i=0}^{n-1}\sum_{j=0}^{r-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{ij})\Big)d^{\lambda}_{G'}(B_{ij},B_{ij})\\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{r-1}2^{\lambda+1}(n_0-m_j)d_G(v_i)m_j(m_j-1), \quad \text{by Lemma 2.2}\\
&= \Big(\sum_{i=0}^{n-1}d_G(v_i)\Big)\sum_{j=0}^{r-1}2^{\lambda+1}(n_0-m_j)m_j(m_j-1)\\
&= 2^{\lambda+2}m\Big(n_0^3-2n_0q-2q-\sum_{j=0}^{r-1}m_j^3\Big), \quad \text{by Remark 2.4.} \qquad (2.5)
\end{aligned}$$

Using (2.1) and the sums $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$ and $\mathbf{A_4}$ in (2.2), (2.3), (2.4) and (2.5), respectively, we have,

$$\begin{aligned}
H_\lambda(G') &= 2n_0q\,H_\lambda(G)+2^{\lambda+2}mq(n_0-1)\\
&+\left((2^\lambda-1)M_1(G)+\frac{(3^\lambda-2^\lambda)}{2}\sum_{v_iv_k\in E_2}\Big(d_G(v_i)+d_G(v_k)\Big)\right)\Big(n_0^3-2qn_0-\sum_{j=0}^{r-1}m_j^3\Big).
\end{aligned}$$

■

Using $\lambda=1$ in Theorem 2.5, we have the following corollary, which is a corrected version of the main theorem proved by Wang and Kang [27].

Corollary 2.6. *Let G be a connected graph with $n\geq 2$ vertices and m edges and let E_2 be the set of edges of G which do not lie on any C_3 of it. If n_0 and q are the numbers of vertices and edges of $K_{m_0,m_1,\dots,m_{r-1}}$, $r\geq 3$, respectively, then* $DD(G\times K_{m_0,m_1,\dots,m_{r-1}})=2n_0q\,DD(G)+8mq(n_0-1)+\Big(M_1(G)+\frac{1}{2}\sum\limits_{v_iv_k\in E_2}(d_G(v_i)+d_G(v_k))\Big)\Big(n_0^3-2qn_0-\sum\limits_{j=0}^{r-1}m_j^3\Big)$. ■

Counterexample: One can easily find the degree distance of the graph $K_2\times K_{2,2,2}$ (see Fig. 3) is 960. For this graph the result given by Wang and Kang [27] is not correct.

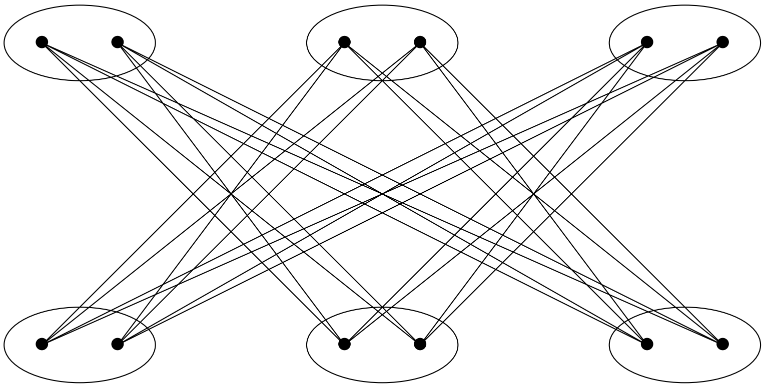


Figure 3 Tensor product of K_2 and $K_{2,2,2}$

Using Corollary 2.6, we have the following corollaries.

Corollary 2.7. *Let G be a connected graph with $n \geq 2$ vertices and m edges. If each edge of G is on a C_3, then* $DD(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 2qn_0\, DD(G) + 8mq(n_0 - 1) + M_1(G)\Big(n_0^3 - 2qn_0 - \sum_{j=0}^{r-1} m_j^3\Big),\ r \geq 3.$ ■

For a triangle free graph, $E_2 = E(G)$ and hence $\sum_{v_iv_k \in E_2} \Big(d_G(v_i) + d_G(v_k)\Big) = M_1(G)$.

Corollary 2.8. *If G is a connected triangle free graph on $n \geq 2$ vertices and m edges, then* $DD(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 2qn_0\, DD(G) + 8mq(n_0 - 1) + \frac{3}{2}M_1(G)\Big(n_0^3 - 2qn_0 - \sum_{j=0}^{r-1} m_j^3\Big),\ r \geq 3.$ ■

If $m_i = s,\ 0 \leq i \leq r-1$, in Theorem 2.5, Corollaries 2.7 and 2.8, we have the following corollaries.

Corollary 2.9. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Let E_2 be the set of edges of G which do not lie on a triangle. Then* $DD(G \times K_{r(s)}) = r^2s^3(r-1)DD(G) + 4mrs^2\Big(r^2s - r - rs + 1\Big) + \Big(M_1(G) + \frac{1}{2}\sum_{v_iv_k \in E_2}\Big(d_G(v_i) + d_G(v_k)\Big)\Big)rs^3(r-1),\ r \geq 3.$ ■

Corollary 2.10. *Let G be a connected graph with $n \geq 2$ vertices and m edges. If each edge of G is on a C_3, then* $DD(G \times K_{r(s)}) = r^2s^3(r-1)DD(G) + M_1(G)rs^3(r-1) + 4mrs^2\Big(r^2s - r - rs + 1\Big),\ r \geq 3.$ ■

Corollary 2.11. *If G is a connected triangle free graph on $n \geq 2$ vertices and m edges, then* $DD(G \times K_{r(s)}) = r^2s^3(r-1)DD(G) + 2M_1(G)rs^3(r-1) + 4mrs^2\Big(r^2s - r - rs + 1\Big),\ r \geq 3.$ ■

If we consider $s = 1$, in Corollaries 2.9, 2.10 and 2.11, we have the following corollaries.

Corollary 2.12. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Let E_2 be the set of edges of G which do not lie on a triangle. Then* $DD(G \times K_r) = r^2(r-1)DD(G) + \Big(M_1(G) + \frac{1}{2}\sum\limits_{v_iv_k \in E_2}(d_G(u_i) + d_G(u_k))\Big) + 4mr(r-1)^2,\ r \geq 3.$ ■

Corollary 2.13. *Let G be a connected graph on $n \geq 2$ vertices with m edges. If each edge of G is on a C_3, then* $DD(G \times K_r) = r^2(r-1)DD(G) + M_1(G)r(r-1) + 4r(r-1)^2m$, *where* $r \geq 3$. ■

Corollary 2.14. *If G is a connected triangle free graph on $n \geq 2$ vertices and m edges, then* $DD(G \times K_r) = r^2(r-1)DD(G) + 2r(r-1)M_1(G) + 4r(r-1)^2m\Big),\ r \geq 3.$ ■

Using $\lambda = -1$, in Theorem 2.5, we obtain the reciprocal degree distance of tensor product of complete multipartite graph $K_{m_0, m_1, \ldots, m_{r-1}}$ and a given connected graph G.

Corollary 2.15. *Let G be a connected graph with $n \geq 2$ vertices and m edges and let E_2 be the set of edges of G which do not lie on any C_3 of it. If n_0 and q are the numbers of vertices and edges of $K_{m_0, m_1, \ldots, m_{r-1}}$, $r \geq 3$, respectively, then* $RDD(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 2n_0q\,\mathrm{RDD}(G) + 2mq(n_0 - 1) - \Bigg(\frac{M_1(G)}{2} + \frac{1}{12}\sum\limits_{v_iv_k \in E_2}\Big(d_G(v_i) + d_G(v_k)\Big)\Bigg)\Big(n_0^3 - 2qn_0 - \sum\limits_{j=0}^{r-1} m_j^3\Big).$ ■

Using Corollary 2.15, we have the following corollaries.

Corollary 2.16. *Let G be a connected graph with $n \geq 2$ vertices and m edges. If each edge of G is on a C_3, then* $RDD(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 2qn_0\,\mathrm{RDD}(G) + 2mq(n_0 - 1) - \frac{M_1(G)}{2}\Big(n_0^3 - 2qn_0 - \sum\limits_{j=0}^{r-1} m_j^3\Big),\ r \geq 3.$ ■

Corollary 2.17. *If G is a connected triangle free graph on $n \geq 2$ vertices and m edges, then* $RDD(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 2qn_0\,\mathrm{RDD}(G) + 2mq(n_0 - 1) - \frac{2M_1(G)}{3}\Big(n_0^3 - 2qn_0 - \sum\limits_{j=0}^{r-1} m_j^3\Big),\ r \geq 3.$ ■

If $m_i = s,\ 0 \leq i \leq r - 1$, in Theorem 2.5 Corollaries 2.16 and 2.17, we have the following corollaries.

Corollary 2.18. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Let E_2 be the set of edges of G which do not lie on a triangle. Then $RDD(G \times K_{r(s)}) = r^2s^3(r-1)\mathrm{RDD}(G) + mrs^2\Big(r^2s - r - rs + 1\Big) - \Big(\frac{M_1(G)}{2} + \frac{1}{12}\sum\limits_{v_iv_k \in E_2}\Big(d_G(v_i) + d_G(v_k)\Big)\Big)rs^3(r-1),\ r \geq 3.$* ■

Corollary 2.19. *Let G be a connected graph with $n \geq 2$ vertices and m edges. If each edge of G is on a C_3, then $RDD(G \times K_{r(s)}) = r^2s^3(r-1)\mathrm{RDD}(G) - \frac{M_1(G)}{2}rs^3(r-1) + mrs^2\Big(r^2s - r - rs + 1\Big),\ r \geq 3.$* ■

Corollary 2.20. *If G is a connected triangle free graph on $n \geq 2$ vertices and m edges, then $RDD(G \times K_{r(s)}) = r^2s^3(r-1)\mathrm{RDD}(G) - \frac{2M_1(G)}{3}rs^3(r-1) + mrs^2\Big(r^2s - r - rs + 1\Big),\ r \geq 3.$* ■

If we consider $s = 1$, in Corollaries 2.18, 2.19 and 2.20, we have the following corollaries.

Corollary 2.21. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Let E_2 be the set of edges of G which do not lie on a triangle. Then $RDD(G \times K_r) = r(r-1)\left(r\mathrm{RDD}(G) - \frac{1}{2}M_1(G) - \frac{1}{12}\sum\limits_{v_iv_k \in E_2}(d_G(u_i) + d_G(u_k)) + (r-1)m\right),\ r \geq 3.$* ■

Corollary 2.22. *Let G be a connected graph on $n \geq 2$ vertices with m edges. If each edge of G is on a C_3, then $RDD(G \times K_r) = r(r-1)\Big(r\mathrm{RDD}(G) - \frac{1}{2}M_1(G) + (r-1)m\Big)$, where $r \geq 3$.* ■

Corollary 2.23. *If G is a connected triangle free graph on $n \geq 2$ vertices and m edges, then $RDD(G \times K_r) = r(r-1)\Big(r\mathrm{RDD}(G) - \frac{2}{3}M_1(G) + (r-1)m\Big),\ r \geq 3.$* ■

References

[1] N. Alon and E. Lubetzky, Independent set in tensor graph powers, *J. Graph Theory,* 54 (2007), 73–87.

[2] Y. Alizadeh, A. Iranmanesh and T. Doslic, Additively weighted Harary index of some composite graphs, *Discrete Math.,* 313 (2013), 26–34.

[3] A.M. Assaf, Modified group divisible designs, *Ars Combin.* 29 (1990), 13–20.

[4] B. Bresar, W. Imrich, S. Klavžar and B. Zmazek, Hypercubes as direct products, *SIAM J. Discrete Math.* 18 (2005), 778–786.

[5] S. Chen and W. Liu, Extremal modified Schultz index of bicyclic graphs, *MATCH Commun. Math. Comput. Chem.* 64 (2010), 767–782.

[6] A.A. Dobrynin and A.A. Kochetova, Degree distance of a graph: a degree analogue of the Wiener index, *J. Chem. Inf. Comput. Sci.* 34 (1994), 1082–1086.

[7] J. Devillers and A.T. Balaban, Eds., *Topological indices and related descriptors in QSAR and QSPR*, Gordon and Breach, Amsterdam, The Netherlands, 1999.

[8] B. Furtula, I.Gutman, Z. Tomovic, A. Vesel and I. Pesek, Wiener-type topological indices of phenylenes, *Indian J. Chem.* 41A (2002), 1767–1772.

[9] I. Gutman, Selected properties of the Schultz molecular topological index, *J. Chem. Inf. Comput. Sci.* 34 (1994), 1087–1089.

[10] I. Gutman and O.E. Polansky, *Mathematical Concepts in Organic Chemistry,* Springer-Verlag, Berlin, 1986.

[11] I. Gutman, A property of the Wiener number and its modifications, *Indian J. Chem.* 36A (1997), 128–132.

[12] I. Gutman, A.A. Dobrynin, S. Klavzar and L. Pavlovic, Wiener-type invariants of trees and their relation, *Bull. Inst. Combin. Appl.* 40 (2004), 23–30.

[13] I. Gutman, D.Vidovic and L. Popovic, Graph representation of organic molecules: Cayley's plerograms vs. his kenograms, *J.Chem. Soc. Faraday Trans.* 94 (1998), 857–860.

[14] A. Hamzeh, A. Iranmanesh, S. Hossein-Zadeh and M.V. Diudea, Generalized degree distance of trees, unicyclic and bicyclic graphs, *Studia Ubb Chemia* LVII 4 (2012), 73–85.

[15] A. Hamzeh, A. Iranmanesh and S. Hossein-Zadeh, Some results on generalized degree distance, *Open J. Discrete Math.* 3 (2013), 143–150.

[16] B. Horoldagva and I. Gutman , On some vertex-degree-based graph invariants, *MATCH Commun. Math. Comput. Chem.* 65 (2011), 723–730.

[17] H. Hua and S. Zhang, On the reciprocal degree distance of graphs, *Discrete Appl. Math.* 160 (2012), 1152–1163.

[18] W. Imrich and S. Klavžar, *Product graphs: Structure and Recognition*, John Wiley, New York (2000).

[19] O. Khormali, A. Iranmanesh, I. Gutman and A. Ahamedi, Generalized Schultz index and its edge versions, *MATCH Commun. Math. Comput. Chem.* 64 (2010), 783–798.

[20] S.C. Li and X. Meng, Four edge-grafting theorems on the reciprocal degree distance of graphs and their applications, *J. Combin. Opt.* (2013); DOI 10.1007/s 10878-013-9649-1.

[21] A. Mamut and E. Vumar, Vertex vulnerability parameters of Kronecker products of complete graphs, *Inform. Process. Lett.* 106 (2008), 258–262.

[22] K. Pattabiraman and M. Vijayaragavan, Reciprocal degree distance of some graph operations, *Trans. Combin.* 2 (2013), 13–24.

[23] K. Pattabiraman and M. Vijayaragavan, Reciprocal degree distance of product graphs, *Discrete Appl. Math.* 179 (2014), 201–213.

[24] G.F. Su, L.M. Xiong, X.F. Su and X.L. Chen, Some results on the reciprocal sum-degree distance of graphs, *J. Combin. Opt.* (2013); DOI 10.1007/s 10878-013-9645-5.

[25] I. Tomescu, Some extremal properties of the degree distance of a graph, *Discrete Appl. Math.* 98 (1999), 159–163.

[26] I. Tomescu, Properties of connected graphs having minimum degree distance, *Discrete Math.* 309 (2008), 2745–2748.

[27] H. Wang and L. Kang, Further properties on the degree distance of graphs, *J. Combin. Opt.*; online June 2014. DOI 10.1007/s10878-014-9757-6.

[28] H.Y. Zhu, D.J. Klenin and I. Lukovits, Extensions of the Wiener number, *J. Chem. Inf. Comput. Sci.* 36 (1996), 420–428.

(Received 26 Aug 2014; revised 30 Apr 2015)