AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **63(1)** (2015), Pages 142–152

Generalized Ramsey theorems for r-uniform hypergraphs

Mark Budden

Department of Mathematics and Computer Science
Western Carolina University
Cullowhee, NC 28723
U.S.A.
mrbudden@email.wcu.edu

Josh Hiller

Department of Mathematics
PO Box 118105, University of Florida
Gainesville, FL 32611-8105
U.S.A.
jphiller1@ufl.edu

Aaron Rapp

Department of Mathematics and Statistics
University of North Carolina Greensboro
Greensboro, NC 27402
U.S.A.
afrapp@uncg.edu

Abstract

We show that several known Ramsey number inequalities can be extended to the setting of r-uniform hypergraphs. In particular, we extend Burr's results on tree-star Ramsey numbers, providing exact evaluations for certain hypergraph Ramsey numbers. Then we turn our attention to proving a general multicolor hypergraph Ramsey number inequality from which generalizations of results due to Chvátal and Harary and Robertson can be obtained. Finally, we consider ways in which one may generalize a more recent multicolor Ramsey number inequality due to Xiaodong, Zheng, Exoo, and Radziszowski.

1 Introduction

One of the aims of Ramsey theory on graphs is the explicit determination of Ramsey numbers, often by making gradual improvements on known upper and lower bounds. Lower bounds are typically found by constructing optimal graphs, while upper bounds require more theoretical approaches. An excellent resource for accessing the current state of knowledge on Ramsey numbers on graphs (and hypergraphs) is Radziszowski's Dynamic Survey [5]. In addition to known bounds for specific Ramsey numbers, it lists numerous inequalities for generalized Ramsey numbers. Our goal is to show how many general Ramsey theorems on graphs can be extended to the setting of r-uniform hypergraphs.

Recall that for $r \geq 2$, an r-uniform hypergraph $H = (V, E)$ consists of a set V of vertices and a set E of different unordered r-tuples of vertices, called hyperedges. A vertex is incident with a hyperedge if it is contained in the hyperedge. As usual, we will write $V(H)$ for V and $E(H)$ for E when we wish to emphasize the specific hypergraph we are working with. The degree of a vertex is the number of hyperedges incident with that vertex. When $r = 2$, these definitions coincide with that of standard graphs.

The complete r-uniform hypergraph $K_n^{(r)}$ is the hypergraph containing n vertices in which every r-subset of the vertices represents a hyperedge. An r-uniform tree $T_m^{(r)}$ is a connected r-uniform hypergraph on m vertices that can be formed hyperedge-by-hyperedge, with each new hyperedge including exactly one vertex from the previous hypergraph. We refer to an r-uniform hyperedge that includes $r-1$ vertices of degree 1 as a leaf. Of course, $K_n^{(r)}$ is unique (up to isomorphism), but there can be many r-uniform trees on a given number of vertices. It is also easily observed that the number of hyperedges in $K_n^{(r)}$ and $T_m^{(r)}$ are $\frac{n!}{r!(n-r)!}$ and $\frac{m-1}{r-1}$, respectively.

An r-uniform hypergraph H is called bipartite if $V(H)$ can be partitioned into two disjoint subsets V_1 and V_2 with every hyperedge including vertices from both V_1 and V_2. The complete bipartite r-uniform hypergraph $K_{m,n}^{(r)}$ has vertex sets V_1 and V_2 with cardinalities m and n, respectively, and includes all r-uniform hyperedges that include vertices from both V_1 and V_2. In particular, we call the hypergraph $K_{1,n}^{(r)}$ a star and note that it contains $\frac{n!}{(r-1)!(n-r+1)!}$ hyperedges. When $r = 2$, we write K_n, T_m, and $K_{m,n}$ in place of $K_n^{(2)}$, $T_m^{(2)}$, and $K_{m,n}^{(2)}$.

For any r-uniform hypergraph H, define the weak chromatic number $\chi_w(H)$ to be the minimum number of colors needed to color the vertices of H so that no hyperedge is monochromatic. The strong chromatic number $\chi_s(H)$ is the minimum number of colors needed to color the vertices of H so that all adjacent vertices (contained within a common hyperedge) have different colors. It is easily observed that for any r-uniform hypergraph H,

$$\chi_w(H) \leq \chi_s(H),$$

and whenever $r = 2$, $\chi_w = \chi_s = \chi$, where χ is the chromatic number for graphs.

For any finite collection $H_1, H_2, \ldots, H_t$ of r-uniform hypergraphs, define the Ramsey number

$$R(H_1, H_2, \ldots H_t; r)$$

to be the least $n \in \mathbb{N}$ such that every coloring of the hyperedges of $K_n^{(r)}$ using t colors results in a subhypergraph isomorphic to H_i for some color $i \in \{1, 2, \ldots, t\}$. One can find a comprehensive overview of the current state of knowledge on hypergraph Ramsey numbers in Section 7 of Radziszokski's dynamic survey [5]. If $H_1 = H_2 = \cdots = H_t$, then we write $R_t(H_1; r)$ for the corresponding Ramsey number. It is also standard to write $R(k_1, k_2, \ldots, k_t; r)$ whenever $H_i = K_{k_i}^{(r)}$ for all $i \in \{1, 2, \ldots, t\}$. When $r = 2$, it is standard to reduce the notation to $R(G_1, G_2, \ldots, G_t)$ for graphs $G_1, G_2, \ldots, G_t$, or to just $R(k_1, k_2, \ldots, k_t)$ when $G_i = K_{k_i}$ for all $i \in \{1, 2, \ldots, t\}$. Finally, we denote by $\lceil x \rceil$ and $\lfloor x \rfloor$ the ceiling and floor functions for $x \in \mathbb{R}$, respectively.

This paper focuses on extending numerous results in Ramsey theory that involve the construction of graphs with known maximal complete subgraphs to the setting of r-uniform hypergraphs. In Section 2, we focus on the r-uniform tree-star Ramsey numbers $R(T_m^{(r)}, K_{1,n}^{(r)}; r)$ and provide exact evaluations under certain divisibility assumptions, generalizing the work of Burr [1]. In Section 3, we introduce a new theorem giving lower bounds for multicolor hypergraph Ramsey numbers, which immediately implies several classical inequalities due to Chvátal and Harary [3], Chvátal [2], and Robertson [6]. We also extend a constructive result of Xiaodong, Zheng, Exoo, and Radziszowski [7], which we then use to find new lower bounds for some diagonal 3-uniform Ramsey numbers of the type $R_k(5; 3)$ for $2 \leq k \leq 9$.

2 r-Uniform Tree-Star Ramsey Numbers

In 1974, Burr [1] proved that when $m - 1$ divides $n - 1$,

$$R(T_m, K_{1,n}) = m + n - 1, \tag{2.1}$$

for any tree T_m on m vertices. We extend this result to r-uniform hypergraphs in the following two theorems, and corollary.

Theorem 2.1 *If $r \geq 2$, $k \geq 1$, and $T_m^{(r)}$ is any r-uniform tree on $m \geq r$ vertices, then*

$$R(T_m^{(r)}, K_{1,k(m-1)+r-1}^{(r)}; r) \geq (k + 1)(m - 1) + 1.$$

Proof: Form a 2-coloring of the hyperedges in $K_{(k+1)(m-1)}^{(r)}$ by taking $k + 1$ copies of $K_{m-1}^{(r)}$. Let all of the hyperedges in each copy of $K_{m-1}^{(r)}$ be colored red and all interconnecting hyperedges colored blue. No red $T_m^{(r)}$ has been formed since $T_m^{(r)}$ has m vertices and the largest connected component in the hypergraph spanned by the red hyperedges has order $m - 1$. When considering the largest value of t for which there exists a blue $K_{1,t}^{(r)}$, note that if x is the vertex that is alone in its bipartite vertex set, then at most $r-2$ other vertices in the same copy of $K_{m-1}^{(r)}$ can be included in the other vertex set. Thus, our coloring includes a blue $K_{1,k(m-1)+r-2}^{(r)}$, but not a blue $K_{1,k(m-1)+r-1}^{(r)}$, resulting in the lower bound stated in the theorem. □

Note that in the special case in which $n-1$ is divisible by $m-1$, we can let $k=\frac{n-1}{m-1}$ to obtain the lower bound

$$R(T_m^{(r)}, K_{1,n+r-2}; r) \geq n+m-1. \tag{2.2}$$

This result agrees with the lower bound necessary to prove (2.1) when $r=2$. Now we turn our attention to finding an upper bound for r-uniform tree-star Ramsey numbers.

Theorem 2.2 *If $t+1 \geq r \geq 2$, and $T_m^{(r)}$ is any r-uniform tree on m vertices, then*

$$R(T_m^{(r)}, K_{1,t}^{(r)}; r) \leq m+t-(r-1).$$

Proof: Let $m = r+\ell(r-1)$ (that is, $\ell+1$ is the number of hyperedges in $T_m^{(r)}$). We proceed by induction on $\ell \geq 0$. In the case $\ell=0$, it is easily seen that

$$R(T_r^{(r)}, K_{1,t}^{(r)}; r) = t+1 = r+t-(r-1).$$

Now assume that the inequality is true for the $\ell-1$ case:

$$R(T_{m-(r-1)}^{(r)}; K_{1,t}; r) \leq m+t-2(r-1),$$

for all r-uniform trees on $m-(r-1)$ vertices. For a given r-uniform tree $T_m^{(r)}$, let T' be the tree formed by removing a single leaf (the hyperedge and the $r-1$ vertices of degree 1 incident with that hyperedge) and let x be the vertex in T' that was incident with the removed leaf. Consider a red/blue coloring of the hyperedges in $K_{m+t-(r-1)}^{(r)}$. By the inductive hypothesis, this coloring contains either a red T' or a blue $K_{1,t}^{(r)}$. Assume the former case and note that besides the vertices in T', the graph $K_{m+t-(r-1)}^{(r)}$ contains

$$m+t-(r-1)-(m-(r-1)) = t$$

other vertices. Now consider the hyperedges that include x along with all $r-1$ subsets of vertices from the t not included in T'. If any one of these hyperedges is red, we obtain a red copy of $T_m^{(r)}$. Otherwise, they are all blue, and we have a blue $K_{1,t}^{(r)}$. □

If we assume that $n-1$ is divisible by $m-1$ and let $t=n+r-2$, then combining (2.2) with Theorem 2.2, we obtain the following corollary.

Corollary 2.3 *If $t+1 \geq r \geq 2$, $T_m^{(r)}$ is any tree on m vertices, and $m-1$ divides $t-(r-1)$, we have that*

$$R(T_m^{(r)}; K_{1,t}^{(r)}; r) = m+t-(r-1).$$

3 Generalizations of Some Classical Ramsey Theory Results

In 1972, Chvátal and Harary [3] proved a general Ramsey inequality for graphs:

$$R(G_1, G_2) \geq (c(G_1) - 1)(\chi(G_2) - 1) + 1, \tag{3.1}$$

where $c(G_1)$ is the order of the largest connected component of G_1 and $\chi(G_2)$ is the chromatic number of G_2. Using this result, Chvátal [2] was then able to prove the explicit Ramsey number

$$R(T_m, K_n) = (m-1)(n-1) + 1, \tag{3.2}$$

where T_m is any tree on m vertices. Forty years later, Robertson (Theorem 2.1, [6]) proved that if $t \geq 3$ and $k_i \geq 3$ for $i = 1, 2, \ldots, r$, then

$$R(k_1, k_2, \ldots, k_t) \geq (k_1 - 1)(R(k_2, \ldots, k_r) - 1) + 1. \tag{3.3}$$

His result followed from a "Turán-type" coloring of a complete graph and implied four improved lower bounds for diagonal multicolor Ramsey numbers:

$$R_5(4;2) \geq 1372, \quad R_5(5;2) \geq 7329, \quad R_4(6;2) \geq 5346, \text{ and } R_4(7;2) \geq 19261.$$

While all of these bounds have since been improved (see [5]) Robertson's theorem remains as an excellent way to build multicolor bounds from known results on smaller graphs. In this section, we show that both of these results can be thought of as consequences of the following theorem.

Theorem 3.1 *Let $r \geq 2$, $t \geq 3$, and $H_1, H_2, \ldots, H_t$ be r-uniform hypergraphs with $H_2, \ldots, H_t$ connected. Then*

$$R(H_1, H_2, \ldots, H_t; r) \geq (\chi_w(H_1) - 1)(R(H_2, \ldots, H_t; r) - 1) + 1.$$

Proof: Let $n = R(H_2, \ldots, H_t; r) - 1$ and consider a coloring of the hyperedges of $K^{(r)}_{(\chi_w(H_1)-1)n}$ formed by considering $(\chi_w(H_1) - 1)$ copies of $K_n^{(r)}$. Within each copy of $K_n^{(r)}$, the hyperedges are colored with colors 2 through t such that no monochromatic copy of H_j appears for any color $2 \leq j \leq t$. Color all of the hyperedges that interconnect the different copies of $K_n^{(r)}$ with color 1. If there were a monochromatic H_1 of color 1, then there would be a copy of H_1 where each hyperedge contained vertices in at least two different copies of $K_n^{(r)}$. But coloring vertices according to which of the $(\chi_w(H_1) - 1)$ copies of $K_n^{(r)}$ they are in would yield a weak coloring of H_1 with $(\chi_w(H_1) - 1)$ colors, which is impossible. Thus, we find that

$$R(H_1, H_2, \ldots, H_t; r) > (\chi_w(H_1) - 1)n,$$

from which the result follows. $\square$

Using Theorem 3.1, we obtain the following generalization of Chvátal and Harary's result (3.1) to r-uniform hypergraphs.

Corollary 3.2 *Let H_1 and H_2 be r-uniform hypergraphs with $r \geq 2$. Then*

$$R(H_1, H_2; r) \geq (c(H_1) - 1)(\chi_w(H_2) - 1) + 1,$$

where $c(H_1)$ is the order of the largest connected component of H_1 and $\chi_w(H_2)$ is the weak chromatic number of H_2.

Proof: It is easy to see that $R(H_1, K_r^{(r)}; r) \geq c(H_1)$. Theorem 3.1 thus gives us that

$$\begin{aligned} R(H_1, H_2; r) = R(H_1, H_2, K_r^{(r)}; r) &\geq (\chi_w(H_2) - 1)(R(H_1, K_r^{(r)}; r) - 1) + 1 \\ &\geq (\chi_w(H_2) - 1)(c(H_1) - 1) + 1, \end{aligned}$$

completing the proof of the corollary. □

In addition to Theorem 3.1, we will make use of the following lemma to prove generalizations of the theorems of Chvátal [2] and Robertson [6].

Lemma 3.3 *If $n \geq r \geq 2$, it follows that $\chi_w(K_n^{(r)}) = \lceil \frac{n}{r-1} \rceil$ and $\chi_s(K_n^{(r)}) = n$.*

Proof: Every weak coloring of $K_n^{(r)}$ contains at most $r-1$ vertices of a given color. Thus, $\lceil \frac{n}{r-1} \rceil$ colors are necessary and sufficient. For a strong coloring, no two distinct vertices can have the same color since there exists some hyperedge that includes both vertices. □

We now exploit Corollary 3.2, along with an inductive argument similar to that of Chvátal [2], to find upper and lower bounds for r-uniform tree-complete hypergraph Ramsey numbers.

Theorem 3.4 *If $n \geq r \geq 2$ and $T_m^{(r)}$ is any r-uniform tree on m vertices, then*

$$(m-1)\left(\left\lceil \frac{n}{r-1} \right\rceil - 1\right) + 1 \leq R(T_m^{(r)}, K_n^{(r)}; r) \leq (m-1)(n-1) + 1.$$

Proof: Letting $H_1 = T_m^{(r)}$ and $H_2 = K_n^{(r)}$ in Theorem 3.2 and using the weak chromatic number result from Lemma 3.3, we obtain the first inequality

$$(m-1)\left(\left\lceil \frac{n}{r-1} \right\rceil - 1\right) + 1 \leq R(T_m^{(r)}, K_n^{(r)}; r).$$

To prove the second inequality, consider a 2-coloring of the edges on $K_k^{(r)}$, where $k = (m-1)(n-1) + 1$. First, we handle the base cases in which $m = r$ or $n = r$. If $m = r$, then $T_m^{(r)}$ consists of a single hyperedge and it is easily seen that

$$R(T_m^{(r)}, K_n^{(r)}; r) = n \leq (r-1)(n-1) + 1.$$

If $n = r$, then $K_n^{(r)}$ consists of a single hyperedge and we have

$$R(T_m^{(r)}, K_n^{(r)}; r) = m \leq (m-1)(r-1)+1.$$

Now we proceed by using strong induction on $m+n$. Assume that

$$R(T_{m'}^{(r)}, K_{n'}^{(r)}; r) \leq (m'-1)(n'-1)+1$$

for all $m'+n' < m+n$ and any r-uniform tree $T_{m'}^{(r)}$ on m' vertices. Now, for a fixed r-uniform tree $T_m^{(r)}$ on m vertices, form the r-uniform tree T' by removing a single "leaf." That is, for some hyperedge containing only a single vertex of degree greater than 1, remove the hyperedge and the $r-1$ vertices of degree 1, resulting in T' having order $m-(r-1)$. Call the one remaining vertex from the removed leaf x. By the inductive hypothesis, we have that the red/blue coloring of the hyperedges of $K_k^{(r)}$ contains either a red T' or a blue $K_n^{(r)}$. In the latter case, we are done, so assume the former case. Now, consider the red/blue coloring of the edges of $K_{k-(m-(r-1))}^{(r)}$ formed by removing the $m-(r-1)$ vertices in the red T'-subgraph from the original $K_k^{(r)}$. It is easily confirmed that

$$k-(m-(r-1)) \geq (m-1)(n-2)+1$$

from which we obtain a red/blue coloring of the edges of $K_{(m-1)(n-2)+1}^{(r)}$. Applying the inductive hypothesis again, we find that this hypergraph contains either a red $T_m^{(r)}$ or a blue $K_{n-1}^{(r)}$. In the former case, we are done, so assume the latter case. Thus, the original red/blue coloring of the edges of $K_k^{(r)}$ contains a red T' and a blue $K_{n-1}^{(r)}$ that are disjoint. Consider the possible colors that can be assigned to the hyperedges that contain x and $r-1$ vertices from the $K_{n-1}^{(r)}$ subgraph. If any of them are red, then there exists a red $T_m^{(r)}$. Otherwise, all of them are blue and there exists a blue $K_n^{(r)}$. Hence,

$$R(T_m^{(r)}, K_n^{(r)}; r) \leq (m-1)(n-1)+1,$$

completing the proof of the theorem. □

From this theorem, it seems that the exact value in equation (3.2) found by Chvátal [2] for the $r=2$ case was due to the fact that the weak and strong chromatic numbers agree in this setting. In general, finding explicit values in the higher-uniformity setting seems to be much more difficult.

We now use Theorem 3.1 along with Lemma 3.3 to prove the following generalization of Robertson's Theorem (Theorem 2 in [6]).

Corollary 3.5 *Let $q, r \geq 2$, $n \geq 3$, and suppose that $H_2, \ldots, H_t$ are connected r-uniform hypergraphs. Then*

$$R(K_{(r-1)q+q'}^{(r)}, H_2, \ldots, H_t; r) \geq q(R(H_2, \ldots, H_t; r)-1)+1,$$

for all $1 \leq q' \leq r-1$.

PROOF: By Lemma 3.3, $\chi_w(K^{(r)}_{(r-1)q+q'}) = q+1$. Applying this to Theorem 3.1 completes the proof. □

Of course, for fixed r and q, Corollary 3.5 is strongest when $q'=1$. When $q'=1$, $r=2$, $q+1=k_1$, and $H_i = K_{k_i}$ for $2 \le i \le t$, Theorem 3.5 reduces to Robertson's inequality (3.3). As an example of the utility of Corollary 3.5, the following lower bounds follow immediately from the explicit lower bounds given in Section 7.1 of Radziszowski's dynamic survey [5]:

$$\begin{aligned}
R(K_4^{(3)}, K_4^{(3)}; 3) = 13 &\implies R(K_{2q+1}^{(3)}, K_4^{(3)}, K_4^{(3)}; 3) \ge 12q+1,\\
R(K_4^{(3)}, K_5^{(3)}; 3) \ge 33 &\implies R(K_{2q+1}^{(3)}, K_4^{(3)}, K_5^{(3)}; 3) \ge 32q+1,\\
R(K_5^{(3)}, K_5^{(3)}; 3) \ge 82 &\implies R(K_{2q+1}^{(3)}, K_5^{(3)}, K_5^{(3)}; 3) \ge 81q+1,\\
R(K_5^{(4)}, K_5^{(4)}; 4) \ge 34 &\implies R(K_{3q+1}^{(4)}, K_5^{(4)}, K_5^{(4)}; 4) \ge 33q+1,\\
R(K_4^{(3)} - e, K_4^{(3)} - e; 3) = 7 &\implies R(K_{2q+1}^{(3)}, K_4^{(3)} - e, K_4^{(3)} - e; 3) \ge 6q+1,\\
R(K_4^{(3)} - e, K_5^{(3)}; 3) \ge 14 &\implies R(K_{2q+1}^{(3)}, K_4^{(3)}, K_5^{(3)}; 3) \ge 13q+1,\\
R(K_4^{(3)}, K_4^{(3)}, K_4^{(3)}; 3) \ge 56 &\implies R(K_{2q+1}^{(3)}, K_4^{(3)}, K_4^{(3)}, K_4^{(3)}; 3) \ge 55q+1.
\end{aligned}$$

The next corollary is proved by induction on the number of colors. To simplify the statement, we define the notation $R^k(K_m^{(r)}, H_1, H_2; r)$ to denote the k-color r-uniform hypergraph Ramsey number for $k-2$ copies of $K_m^{(r)}$ along with nonempty r-uniform hypergraphs H_1 and H_2. Note that we are using superscripts to denote these semi-diagonal Ramsey numbers, in contrast to using subscripts for their diagonal counterparts.

Corollary 3.6 *If $k \ge 3$ and $q \ge 2$, then*

$$R^k(K^{(r)}_{(r-1)q+1}, H_1, H_2; r) > q^{k-2}(R(H_1, H_2; r) - 1).$$

PROOF: The proof of Corollary 3.6 follows from a simple inductive argument on k. For the $k=3$ case, Corollary 3.5 implies

$$R^3(K_t^{(r)}, H_1, H_2; r) > q(R(H_1, H_2; r) - 1),$$

where $t = (r-1)q+1$. Assume now that

$$R^k(K_t^{(r)}, H_1, H_2; r) \ge q^{k-2}(R(H_1, H_2; r) - 1) + 1,$$

for $k \ge 3$. Applying Corollary 3.5 again, we have

$$\begin{aligned}
R^{k+1}(K_t^{(r)}, H_1, H_2; r) &\ge q(R^k(K_t^{(r)}, H_1, H_2; r) - 1) + 1\\
&\ge q(q^{k-2}(R(H_1, H_2; r) - 1)) + 1,
\end{aligned}$$

implying the statement of the corollary. □

Of course, when $H_1 = H_2 = K^{(r)}_{(r-1)q+1}$, we obtain the following diagonal case:

$$R_k(K^{(r)}_{(r-1)q+1}; r) > q^{k-2}(R_2(K^{(r)}_{(r-1)q+1}; r) - 1).$$

Now we turn our attention to a 2004 result of Xiaodong, Zheng, Exoo, and Radziszowski. In Theorem 2 of [7], they proved the following multicolor Ramsey number inequality for graphs:

$$R(k_1, k_2, \dots, k_t) \geq (R(k_1, k_2 \dots, k_i) - 1)(R(k_{i+1}, \dots, k_t) - 1) + 1, \tag{3.4}$$

for $k_j \geq 2$, $1 \leq j \leq t$, and $2 \leq i \leq t - 2$. Their proof was constructive and described a method for coloring the edges in K_{mn} with t colors, avoiding the necessary monochromatic subgraphs, where

$$m = R(k_1, k_2 \dots, k_i) - 1 \quad \text{and} \quad n = R(k_{i+1}, \dots, k_t) - 1.$$

Although their approach does not easily generalize to hypergraphs, the following theorem makes use of the constructive method used in [7] to provide a new multicolor Ramsey number inequality for hypergraphs.

Theorem 3.7 *Let $r \geq 2$ and $t - 2 \geq i \geq 3$. Then*

$$R((r-1)^2 + 1, k_2, \dots, k_t; r) \geq (R(k_2, \dots, k_i; r) - 1)(R(k_{i+1}, \dots, k_t; r) - 1) + 1.$$

Proof: Let

$$m = R(k_2, \dots, k_i; r) - 1 \quad \text{and} \quad n = R(k_{i+1}, \dots, k_t; r) - 1$$

and form a t-coloring of the hyperedges in $K^{(r)}_{mn}$ by considering m copies of $K^{(r)}_n$. Color the hyperedges within each copy of $K^{(r)}_n$ with colors $i+1$ through t so that no copy of $K^{(r)}_{k_j}$ exists in color j for any $i + 1 \leq j \leq t$. The remaining hyperedges are those that interconnect the different copies of $K^{(r)}_n$. Give color 1 to the hyperedges that have at least two vertices within a common copy of $K^{(r)}_n$. So, all hyperedges in color 1 include at most $r - 1$ vertices from any given copy of $K^{(r)}_n$ and can include vertices from at most $r - 1$ different copies of $K^{(r)}_n$. Thus, the maximum clique in color 1 has order $(r - 1)^2$. Finally, the remaining hyperedges are those whose vertices are all in different copies of $K^{(r)}_n$. If we identify the vertices in $K^{(r)}_m$ with the distinct copies of $K^{(r)}_n$, we can form a coloring of the remaining hyperedges with colors 2 through i that avoids a copy of $K^{(r)}_{k_j}$ in color j for all $2 \leq j \leq i$. Thus, our t-coloring of the hyperedges of $K^{(r)}_{mn}$ has avoided all of the necessary monochromatic subhypergraphs. □

This theorem can been seen to be a true generalization of Xiaodong, Zheng, Exoo, and Radziszowski's Theorem as it reduces to (3.4) when $r = 2$ since

$$R(2, k_1, \ldots, k_t; r) = R(k_1 \ldots, k_t; r).$$

The following proposition follows as an application of Corollary 3.5 and Theorem 3.7 and describes lower bounds for diagonal 3-uniform hypergraph Ramsey numbers of the form $R_k(5; 3)$.

Proposition 3.8 *The following inequalities hold:*

1. $R_2(5; 3) \geq 82$,
2. $R_3(5; 3) \geq 163$,
3. $R_4(5; 3) \geq 131,073$,
4. $R_5(5; 3) \geq 262,145$,
5. $R_6(5; 3) \geq 524,289$,
6. $R_7(5; 3) \geq 10,616,833$,
7. $R_8(5; 3) \geq 21,233,665$,
8. $R_9(5; 3) \geq 17,179,869,185$.

Proof: The first inequality (1) can be found in [5] and (3) follows from Theorem 2 in [4]. The other inequalities follow as direct applications of Corollary 3.5 and Theorem 3.7. □

4 Conclusion

As we have demonstrated, many general Ramey theorems that use both constructive and theoretical (usually inductive) arguments can be extended to the setting of r-uniform hypergraphs. While we have initiated a study of such generalizations, there are many Ramsey results left to consider for such extensions. We conclude with an additional conjectured extension of Theorem 2 of [7].

Conjecture 4.1 *If $r \geq 2$ and $t > i + 1 > 2$, then*

$$R(k_1, k_2, \ldots, k_t; r) \geq \left\lfloor \frac{R(k_1, k_2, \ldots, k_i; r) - 1}{r - 1} \right\rfloor (R(k_{i+1}, \ldots, k_t; r) - 1) + 1.$$

To provide some support for our conjecture, consider the following construction. Let

$$m = R(k_1, k_2, \ldots, k_i; r) - 1, \qquad n = R(k_{i+1}, \ldots, k_t; r) - 1,$$

$a = \lfloor \frac{m}{r-1} \rfloor$, and form a t-coloring of the hyperedges of $K_{an}^{(r)}$ using a copies of $K_n^{(r)}$. Within each copy of $K_n^{(r)}$, color the hyperedges with colors $i+1$ through t so that no copy of $K_{k_j}^{(r)}$ exists in color j for any $i+1 \leq j \leq t$. The remaining hyperedges each have at most $r-1$ vertices within a single copy of $K_n^{(r)}$, forming a clique of order at most

$$\left\lfloor \frac{m}{r-1} \right\rfloor (r-1) \leq m.$$

It is clear that for any choice of $r-1$ vertices from each copy of $K_n^{(r)}$, the resulting $K^{(r)}_{\lfloor \frac{m}{r-1} \rfloor (r-1)}$ has an i-coloring of the hyperedges that lack a copy of $K_{k_j}^{(r)}$ in color j for all $1 \leq j \leq i$. Of course, it is not clear whether or not this can be done in a well-defined manner for all choices of $r-1$ vertices from each $K_n^{(r)}$. We welcome the reader to determine whether such a coloring is possible.

References

[1] S. Burr, Generalized Ramsey theory for graphs—a survey, *Graphs and Combinatorics*, Lec. Notes in Math., Vol. 406, Springer, Berlin, 1974, pp. 52–75.

[2] V. Chvátal, Tree-complete graph Ramsey numbers, *J. Graph Theory* 1 (1977), 93.

[3] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs III, small off-diagonal numbers, *Pacific J. Math.* 41 (1972), 335–345.

[4] D. Conlon, J. Fox and B. Sudakov, An improved bound for the stepping-up lemma, *Discrete Appl. Math.* 161 (2013), 1191–1196.

[5] S. Radziszowski, Small Ramsey numbers—revision 14, *Elec. J. Combin.* DS1.14 (2014), 1–94.

[6] A. Robertson, New lower bound formulas for multicolored Ramsey numbers, *Elec. J. Combin.* 9 (2002), # R13.

[7] X. Xiaodong, X. Zheng, G. Exoo and S. Radziszowski, Constructive lower bounds on classical multicolor Ramsey numbers, *Elec. J. Combinatorics* 11 no. 1 (2004), #R35.

(Received 10 Feb 2015)