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The total vertex irregularity strength of generalized helm graphs and prisms with outer pendant edges

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Abstract

For a simple graph $G=(V,E)$ with the vertex set V and the edge set E, a vertex irregular total k-labeling $f: V \cup E \to \{1,2,\dots,k\}$ is a labeling

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of vertices and edges of G in such a way that for any two different vertices x and x', their weights $wt_f(x) = f(x) + \sum_{xy \in E} f(xy)$ and $wt_f(x') = f(x') + \sum_{x'y' \in E} f(x'y')$ are distinct. A smallest positive integer k for which G admits a vertex irregular total k-labeling is defined as a total vertex irregularity strength of graph G, denoted by $\mathrm{tvs(G)}$. In this paper, we determine the exact value of the total vertex irregularity strength for generalized helm graphs and for prisms with outer pendant edges.

1 Introduction

Let us consider a connected and undirected graph $G = (V, E)$ without loops and parallel edges. The set of vertices and edges of this graph are denoted by $V(G)$ and $E(G)$, respectively. Wallis [11] (see also [12]) defined a labeling of G as a mapping that carries a set of graph elements into a set of integers, called labels. If the domain of the mapping is either a vertex set, or an edge set, or a union of vertex and edge sets, then the labeling is called a *vertex labelingi*, *edge labelingi*, or *total labeling*, respectively. In his survey, Gallian [6] shows that there are various kinds of labelings on graphs, and one of them is a vertex irregular total labeling.

For a graph G, Bača et al. [5] define a labeling $f : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ to be a *vertex irregular total k-labeling* if for every two different vertices x and y the vertex-weights satisfy $wt_f(x) \neq wt_f(y)$, where the *vertex-weight* $wt_f(x) = f(x) + \sum_{xz \in E} f(xz)$. The minimum k for which G has a vertex irregular total k-labeling is defined as the *total vertex irregularity strength* of G and is denoted by $\mathrm{tvs(G)}$.

For a graph G with p vertices and q edges, Bača et al. [5] gave a lower and an upper bound of the total vertex irregularity strength of G by the form

$$\left\lceil \tfrac{p+\delta}{\Delta+1} \right\rceil \leq \mathrm{tvs}(G) \leq p + \Delta - 2\delta + 1, \tag{1}$$

where δ and Δ are the minimum and the maximum degree of G, respectively. They also determined the exact values of the total vertex irregularity strength for cycles, stars, complete graphs and prisms.

Nurdin et al. [10] proved that for connected graph having n_i vertices of degree i, $i = \delta, \delta+1, \dots, \Delta$, the lower bound on the $\mathrm{tvs(G)}$ is given by the form

$$\mathrm{tvs(G)} \geq \max\left\{ \left\lceil \tfrac{\delta+\mathrm{n}_\delta}{\delta+1} \right\rceil, \mathrm{i} \left\lceil \tfrac{\delta+\mathrm{n}_\delta+\mathrm{n}_{\delta+1}}{\delta+2} \right\rceil, \dots, \left\lceil \tfrac{\delta+\sum_{\mathrm{i}=\delta}^{\Delta} \mathrm{n}_\mathrm{i}}{\Delta+1} \right\rceil \right\}. \tag{2}$$

Furthermore, they posed a conjecture that for any connected graph its total vertex irregularity strength is equal to the lower bound from (2).

The conjecture by Nurdin et al. has been verified for flowers, disjoint union of helm graphs, generalized friendship graphs and web graphs in [1], for quadtrees and banana trees in [9] and for convex polytope graphs in [2]. Anholcer et al. [3] proved that for any tree T with n_1 pendant vertices, no vertex of degree 2 and no isolated vertex, the $\mathrm{tvs(T)} = \lceil (\mathrm{n_1}+1)/2 \rceil$. Further results can be found in [4] and [8].

Motivated by the results on the total vertex irregularity strength of helm graphs (see [1]), we investigate the total vertex irregularity strength of generalized helm graphs H_n^m, $n, m \geq 3$. For $m = 1$ and 2, the total vertex irregularity strength of generalized helm graphs can be found in [7]. Also we investigate the total vertex irregularity strength of prisms with outer pendant edges.

2 Generalized helm graphs

A generalized helm graph, H_n^m, is a graph obtained by inserting m vertices to every pendant edge of helm H_n. A generalized helm graph H_n^m has $(m+2)n+1$ vertices and $(m+3)n$ edges. Let the vertex set of H_n^m be $V(H_n^m) = \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m+1\} \cup \{u_i : 1 \leq i \leq n\} \cup \{w\}$ and the edge set of H_n^m be $E(H_n^m) = \{(v_{i,j}v_{i,j+1}) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{(v_{i,m+1}u_i) : 1 \leq i \leq n\} \cup \{(u_iu_{i+1}) : 1 \leq i \leq n\} \cup \{(wu_i) : 1 \leq i \leq n\}$, where the indices are taken modulo n. In order to obtain the total vertex irregularity strength of H_n^m, firstly we prove the lower bound of this parameter as follows.

Lemma 2.1. *Let H_n^m, $n, m \geq 3$, be the generalized helm graph. Then*

$$\mathrm{tvs(H_n^m)} \geq \left\lceil \tfrac{\mathrm{(m+1)n+1}}{3} \right\rceil.$$

Proof. The graph H_n^m, $n, m \geq 3$, contains n pendant vertices, mn vertices of degree 2, n vertices of degree 4 and one vertex of degree n. For $m \geq 3$, according to (2), we have

$$\begin{aligned}
\mathrm{tvs}(H_3^m) &\geq \max\left\{2, \left\lceil \tfrac{3m+4}{3} \right\rceil, \left\lceil \tfrac{3m+5}{4} \right\rceil, \left\lceil \tfrac{3m+8}{5} \right\rceil\right\} = \left\lceil \tfrac{3m+4}{3} \right\rceil, \\
\mathrm{tvs}(H_4^m) &\geq \max\left\{\left\lceil \tfrac{5}{2} \right\rceil, \left\lceil \tfrac{4m+5}{3} \right\rceil, \left\lceil \tfrac{4m+10}{5} \right\rceil\right\} = \left\lceil \tfrac{4m+5}{3} \right\rceil
\end{aligned}$$

and for $n \geq 5$ we get

$$\mathrm{tvs}(H_n^m) \geq \max\left\{\left\lceil \tfrac{n+1}{2} \right\rceil, \left\lceil \tfrac{(m+1)n+1}{3} \right\rceil, \left\lceil \tfrac{(m+2)n+1}{5} \right\rceil, \left\lceil \tfrac{(m+2)n+2}{n+1} \right\rceil\right\} = \left\lceil \tfrac{(m+1)n+1}{3} \right\rceil.$$

We can see that $\mathrm{tvs(H_n^m)} \geq \lceil (\mathrm{(m+1)n+1})/3 \rceil$ for every $n, m \geq 3$. □

The next theorem presents the exact value of the total vertex irregularity strength of the generalized helm graph H_n^m, $n, m \geq 3$.

Theorem 2.2. *Let H_n^m, $n, m \geq 3$, be the generalized helm graph. Then*

$$\mathrm{tvs(H_n^m)} = \left\lceil \tfrac{\mathrm{(m+1)n+1}}{3} \right\rceil.$$

Proof. Immediately from Lemma 2.1 it follows that $\mathrm{tvs}(\mathrm{H_n^m}) \geq \lceil ((\mathrm{m}+1)\mathrm{n}+1)/3 \rceil$. Put $k = \lceil ((m+1)n+1)/3 \rceil$. To show that k is an upper bound for the total vertex irregularity strength of the generalized helm graph H_n^m i, we describe a total k-labeling $f : V(H_n^m) \cup E(H_n^m) \to \{1, 2, \ldots, k\}$. Let $f(w) = k$ and $f(v_{i,m+1}u_i) = k$ for $1 \leq i \leq n$. Next we will distinguish the following three cases.

Case 1. $m \equiv 0 \pmod 3$, $m \geq 3$, $n \geq 3$.
Define a total labeling f of an element x, $x \in V(H_n^m) \cup E(H_n^m)$ in the following way.

x	$f(x)$		n
$v_{i,j}$	$1,$	$1 \leq i \leq n;\ 1 \leq j \leq \frac{2m}{3}$	≥ 3
	$1,$	$1 \leq i \leq \lceil \frac{n+1}{3} \rceil;\ j = \frac{2m}{3}+1$	
	$i - \lceil \frac{n-2}{3} \rceil,$	$\lceil \frac{n+1}{3} \rceil + 1 \leq i \leq n;\ j = \frac{2m}{3}+1$	
	$\lceil \frac{2n}{3} \rceil,$	$1 \leq i \leq \lceil \frac{n+1}{3} \rceil;\ j = \frac{2m}{3}+2$	
	$i + \lceil \frac{n-3}{3} \rceil,$	$\lceil \frac{n+1}{3} \rceil + 1 \leq i \leq n;\ j = \frac{2m}{3}+2$	$\not\equiv 2 \pmod 3$
	$1 + \lceil \frac{n}{3} \rceil,$	$\lceil \frac{n+1}{3} \rceil + 1 \leq i \leq n;\ j = \frac{2m}{3}+2$	$\equiv 2 \pmod 3$
	$\lceil \frac{n-3}{3} \rceil + i + (j - \frac{2m}{3} - 2)n,$	$1 \leq i \leq n;\ \frac{2m}{3}+3 \leq j \leq m+1$	$\not\equiv 2 \pmod 3$
	$\lceil \frac{n}{3} \rceil + i + (j - \frac{2m}{3} - 2)n,$	$1 \leq i \leq n;\ \frac{2m}{3}+3 \leq j \leq m+1$	$\equiv 2 \pmod 3$
$v_{i,j}v_{i,j+1}$	$\frac{j-1}{2}n + i,$	$1 \leq i \leq n;\ 1 \leq j \leq \frac{2m}{3}, j \equiv 1 \pmod 2$	≥ 3
	$\frac{j}{2}n,$	$1 \leq i \leq n;\ 2 \leq j \leq \frac{2m}{3}, j \equiv 0 \pmod 2$	
	$\frac{j-1}{2}n + i,$	$1 \leq i \leq \lceil \frac{n-2}{3} \rceil;\ j = \frac{2m}{3}+1$	
	$k,$	$\lceil \frac{n-2}{3} \rceil + 1 \leq i \leq n;\ j = \frac{2m}{3}+1$	
	$k,$	$1 \leq i \leq n;\ \frac{2m}{3}+2 \leq j \leq m$	
u_i	$1,$	$1 \leq i \leq 8$	$\equiv 0 \pmod 3$
	$2,$	$9 \leq i \leq n, i \equiv 1 \pmod 2$	
	$1,$	$10 \leq i \leq n, i \equiv 0 \pmod 2$	
	$1,$	$1 \leq i \leq 6$	$\equiv 1 \pmod 3$
	$2,$	$7 \leq i \leq n, i \equiv 1 \pmod 2$	
	$1,$	$8 \leq i \leq n, i \equiv 0 \pmod 2$	
	$1,$	$1 \leq i \leq 4$	$\equiv 2 \pmod 3$
	$2,$	$5 \leq i \leq n, i \equiv 1 \pmod 2$	
	$1,$	$6 \leq i \leq n, i \equiv 0 \pmod 2$	

x	$f(x)$		n
u_iu_{i+1}	$\min\left\{(m+1)\frac{n-3}{3}+m-2+\frac{i+1}{2},k\right\}$,	$1\le i\le n, i\equiv 1 \pmod 2$	$\equiv 0 \pmod 3$
	$1+\frac{i}{2}$,	$2\le i\le 8, i\equiv 0 \pmod 2$	
	$i-3$,	$10\le i\le n, i\equiv 0 \pmod 2$	
	$\min\left\{(m+1)\frac{n-4}{3}+\frac{4m-3}{3}+\frac{i+1}{2},k\right\}$,	$1\le i\le n, i\equiv 1 \pmod 2$	$\equiv 1 \pmod 3$
	$1+\frac{i}{2}$,	$2\le i\le 6, i\equiv 0 \pmod 2$	
	$i-2$,	$8\le i\le n, i\equiv 0 \pmod 2$	
	$\min\left\{(m+1)\frac{n-5}{3}+5\frac{m}{3}+\frac{i+1}{2},k\right\}$,	$1\le i\le n, i\equiv 1 \pmod 2$	$\equiv 2 \pmod 3$
	$1+\frac{i}{2}$,	$2\le i\le 4, i\equiv 0 \pmod 2$	
	$i-1$,	$6\le i\le n, i\equiv 0 \pmod 2$	
wu_i	3,	$i=1$	$n=3$
	$2m$,	$i=1$	$n=6$
	1,	$i=1$	$\equiv 3 \pmod 6$; ≥ 9
	$(m-2)\left\lceil\frac{n}{3}\right\rceil+5$,	$i=1$	$\equiv 0 \pmod 6$; ≥ 12
	k,	$i\ne 1$	≥ 3
	$\frac{4m}{3}$,	$i=1$	$n=4$
	1,	$i=1$	$\equiv 1 \pmod 6$; ≥ 7
	$(m-2)\left\lceil\frac{n}{3}\right\rceil+\frac{15-2m}{3}$,	$i=1$	$\equiv 4 \pmod 6$; ≥ 10
	1,	$i=1$	$\equiv 5 \pmod 6$; ≥ 5
	$(m-2)\left\lceil\frac{n}{3}\right\rceil+\frac{9-m}{3}$,	$i=1$	$\equiv 2 \pmod 6$; ≥ 8

We can see that under the labeling f, all vertex and edge labels are at most k. The vertex-weights of H_n^m are

$$\begin{aligned} wt(v_{i,j}) &= (j-1)n+1+i, && \text{for } 1\le i\le n, 1\le j\le m+1,\\ wt(u_i) &= (m+1)n+1+i, && \text{for } 1\le i\le n. \end{aligned}$$

If $n\equiv 0 \pmod 3$, then

$$wt(w)=\begin{cases} nk+3, & \text{for } n=3,\\ nk+2m, & \text{for } n=6,\\ nk+1, & \text{for } n\ge 9 \text{ odd},\\ nk+(m-2)\frac{n}{3}+5, & \text{for } n\ge 12 \text{ even}. \end{cases}$$

If $n\equiv 1 \pmod 3$, then

$$wt(w)=\begin{cases} nk+\frac{4m}{3}, & \text{for } n=4,\\ nk+1, & \text{for } n\ge 7 \text{ odd},\\ nk+(m-2)\left\lceil\frac{n}{3}\right\rceil+\frac{15-2m}{3}, & \text{for } n\ge 10 \text{ even}. \end{cases}$$

If $n\equiv 2 \pmod 3$, then

$$wt(w)=\begin{cases} nk+1, & \text{for } n\ge 5 \text{ odd},\\ nk+(m-2)\left\lceil\frac{n}{3}\right\rceil+\frac{9-m}{3}, & \text{for } n\ge 8 \text{ even}. \end{cases}$$

Clearly, the weights of the vertices $v_{i,j}$ and u_i form a sequence of consecutive integers from 2 up to $n(m+2)+1$ and the weight of the vertex w is greater than $n(m+2)+1$. It means that the vertex-weights are different for all pairs of distinct vertices. We conclude that f is the vertex irregular total k-labeling.

Case 2. $m \equiv 1 \pmod 3$, $m \geq 4$, $n \geq 3$. Define a total labeling f of an element x, $x \in V(H_n^m) \cup E(H_n^m)$ as follows.

x	$f(x)$		n
$v_{i,j}$	$1,$	$1 \leq i \leq n;\ 1 \leq j \leq \frac{2m-2}{3}$	≥ 3
	$1,$	$1 \leq i \leq \lceil \frac{2n+1}{3} \rceil;\ j = \frac{2m+1}{3}$	
	$i - \lceil \frac{2n+1}{3} \rceil + 1,$	$\lceil \frac{2n+1}{3} \rceil + 1 \leq i \leq n;\ j = \frac{2m+1}{3}$	
	$n - \lceil \frac{2n+1}{3} \rceil + 1,$	$1 \leq i \leq \lceil \frac{2n+1}{3} \rceil;\ j = \frac{2m+4}{3}$	
	$i - \lceil \frac{2n+1}{3} \rceil + \lceil \frac{n}{3} \rceil,$	$\lceil \frac{2n+1}{3} \rceil + 1 \leq i \leq n;\ j = \frac{2m+4}{3}$	
	$2\lceil \frac{n}{3} \rceil - 1 + i + (j - \frac{2m-2}{3} - 3)n,$	$1 \leq i \leq n;\ \frac{2m+7}{3} \leq j \leq m+1$	
$v_{i,j}v_{i,j+1}$	$\min\left\{\frac{j-1}{2}n + i, k\right\},$	$1 \leq i \leq n;\ 1 \leq j \leq 2\lceil \frac{m}{3} \rceil - 1,$ $j \equiv 1 \pmod 2$	≥ 3
	$\frac{jn}{2},$	$1 \leq i \leq n;\ 2 \leq j \leq 2\lceil \frac{m}{3} \rceil - 2,$ $j \equiv 0 \pmod 2$	
	$k,$	$1 \leq i \leq n;\ 2\lceil \frac{m}{3} \rceil \leq j \leq m$	
u_i	$1,$	$1 \leq i \leq 8$	$\equiv 0 \pmod 3$
	$2,$	$9 \leq i \leq n, i \equiv 1 \pmod 2$	
	$1,$	$10 \leq i \leq n, i \equiv 0 \pmod 2$	
	$1,$	$1 \leq i \leq 4$	$\equiv 1 \pmod 3$
	$2,$	$5 \leq i \leq n, i \equiv 1 \pmod 2$	
	$1,$	$6 \leq i \leq n, i \equiv 0 \pmod 2$	
u_i	$1,$	$1 \leq i \leq 6$	$\equiv 2 \pmod 3$
	$2,$	$7 \leq i \leq n, i \equiv 1 \pmod 2$	
	$1,$	$8 \leq i \leq n, i \equiv 0 \pmod 2$	
u_iu_{i+1}	$\min\{(m+1)\frac{n-3}{3} + m-2+\frac{i+1}{2}, k\},$	$1 \leq i \leq n, i \equiv 1 \pmod 2$	$\equiv 0 \pmod 3$
	$1 + \frac{i}{2},$	$2 \leq i \leq 8, i \equiv 0 \pmod 2$	
	$i - 3,$	$10 \leq i \leq n, i \equiv 0 \pmod 2$	
	$\min\{(m+1)\frac{n-4}{3} + \frac{4m-1}{3}+\frac{i+1}{2}, k\},$	$1 \leq i \leq n, i \equiv 1 \pmod 2$	$\equiv 1 \pmod 3$
	$1 + \frac{i}{2},$	$2 \leq i \leq 4, i \equiv 0 \pmod 2$	
	$i - 1,$	$6 \leq i \leq n, i \equiv 0 \pmod 2$	
	$\min\{(m+1)\frac{n-5}{3} + \frac{5m-2}{3}+\frac{i+1}{2}, k\},$	$1 \leq i \leq n, i \equiv 1 \pmod 2$	$\equiv 2 \pmod 3$
	$1 + \frac{i}{2},$	$2 \leq i \leq 6, i \equiv 0 \pmod 2$	
	$i - 2,$	$8 \leq i \leq n, i \equiv 0 \pmod 2$	

x	$f(x)$		n
wu_i	$3,$	$i=1$	$n=3$
	$2m,$	$i=1$	$n=6$
	$1,$	$i=1$	$\equiv 3 \pmod 6;\ \geq 9$
	$(m-2)\frac{n}{3}+5,$	$i=1$	$\equiv 0 \pmod 6;\ \geq 12$
	$k,$	$i \neq 1$	≥ 3
	$\frac{4m-1}{3},$	$i=1$	$n=4$
	$1,$	$i=1$	$\equiv 1 \pmod 6;\ \geq 7$
	$(m-2)\left\lceil \frac{n}{3} \right\rceil + \frac{11-2m}{3},$	$i=1$	$\equiv 4 \pmod 6;\ \geq 10$
	$1,$	$i=1$	$\equiv 5 \pmod 6;\ \geq 5$
	$(m-2)\left\lceil \frac{n}{3} \right\rceil + \frac{13-m}{3},$	$i=1$	$\equiv 2 \pmod 6;\ \geq 8$

It is a routine matter to verify that under the labeling f all vertex and edge labels are at most k and vertex-weights of H_n^m are

$$\begin{aligned} wt(v_{i,j}) &= (j-1)n+1+i, && \text{for } 1 \leq i \leq n, 1 \leq j \leq m+1, \\ wt(u_i) &= (m+1)n+1+i, && \text{for } 1 \leq i \leq n. \end{aligned}$$

If $n \equiv 0 \pmod 3$, then

$$wt(w) = \begin{cases} nk+3, & \text{for } n=3, \\ nk+2m, & \text{for } n=6, \\ nk+1, & \text{for } n \geq 9 \text{ odd}, \\ nk+(m-2)\frac{n}{3}+5, & \text{for } n \geq 12 \text{ even}. \end{cases}$$

If $n \equiv 1 \pmod 3$, then

$$wt(w) = \begin{cases} nk+\frac{4m-1}{3}, & \text{for } n=4, \\ nk+1, & \text{for } n \geq 7 \text{ odd}, \\ nk+(m-2)\left\lceil \frac{n}{3} \right\rceil + \frac{11-2m}{3}, & \text{for } n \geq 10 \text{ even}, \end{cases}$$

and if $n \equiv 2 \pmod 3$, then

$$wt(w) = \begin{cases} nk+1, & \text{for } n \geq 5 \text{ odd}, \\ nk+(m-2)\left\lceil \frac{n}{3} \right\rceil + \frac{13-m}{3}, & \text{for } n \geq 8 \text{ even}. \end{cases}$$

Thus weights of the vertices $v_{i,j}$ and u_i form a set of consecutive integers from 2 up to $n(m+2)+1$ and the weight of the vertex w is greater than $n(m+2)+1$. Therefore, the vertex-weights are pairwise distinct and f is the vertex irregular total k-labeling.

Case 3. $m \equiv 2 \pmod 3$, $m \geq 5$, $n \geq 3$. Define a total labeling f of an element x, $x \in V(H_n^m) \cup E(H_n^m)$ in the following way.

x	$f(x)$		n
$v_{i,j}$	$1,$	$1 \le i \le n;\ 1 \le j \le 2\left\lceil \frac{m}{3} \right\rceil$	≥ 3
	$i,$	$1 \le i \le n;\ j = 2\left\lceil \frac{m}{3} \right\rceil + 1$	
	$(j - 2\left\lceil \frac{m}{3} \right\rceil - 1)n - 1 + i,$	$1 \le i \le n;\ 2\left\lceil \frac{m}{3} \right\rceil + 2 \le j \le m+1$	
u_i	$1,$	$1 \le i \le 8$	≥ 3
	$2,$	$9 \le i \le n, i \equiv 1 \pmod 2$	
	$1,$	$10 \le i \le n, i \equiv 0 \pmod 2$	
$v_{i,j}v_{i,j+1}$	$\frac{j-1}{2}n + i,$	$1 \le i \le n;\ 1 \le j \le \frac{2m-1}{3}$	≥ 3
	$\frac{jn}{2},$	$1 \le i \le n;\ 2 \le j \le 2\left\lceil \frac{m}{3} \right\rceil$	
	$k,$	$1 \le i \le n;\ 2\left\lceil \frac{m}{3} \right\rceil + 1 \le j \le m$	
u_iu_{i+1}	$\min\left\{(n-3)\left\lceil \frac{m}{3} \right\rceil + m + \frac{i-3}{2}, k\right\},$	$1 \le i \le n, i \equiv 1 \pmod 2$	≥ 3
	$1 + \frac{i}{2},$	$2 \le i \le 8, i \equiv 0 \pmod 2$	
	$i - 3,$	$10 \le i \le n, i \equiv 0 \pmod 2$	
wu_i	$3,$	$i = 1$	$n = 3$
	$2m,$	$i = 1$	$n = 6$
	$1,$	$i = 1$	$\equiv 3 \pmod 6;\ \ge 9$
	$(m-2)\frac{n}{3} + 5,$	$i = 1$	$\equiv 0 \pmod 6;\ \ge 12$
	$k,$	$i \ne 1$	≥ 3
	$\left\lceil \frac{4m}{3} \right\rceil,$	$i = 1$	$n = 4$
	$1,$	$i = 1$	$\equiv 1 \pmod 6;\ \ge 7$
	$(m-2)\left\lceil \frac{n}{3} \right\rceil + \frac{19-2m}{3},$	$i = 1$	$\equiv 4 \pmod 6;\ \ge 10$
	$2,$	$i = 1$	$n = 5$
	$(m-2)\left\lceil \frac{n}{3} \right\rceil + \frac{17-m}{3},$	$i = 1$	$\equiv 2 \pmod 6;\ \ge 8$
	$1,$	$i = 1$	$\equiv 5 \pmod 6;\ \ge 11$

Observe that under the labeling f all vertex and edge labels are at most k. It means that f is the total k-labeling. For the vertex-weights of H_n^m we have:

$$\begin{aligned} wt(v_{i,j}) &= (j-1)n + 1 + i, \quad \text{for } 1 \le i \le n, 1 \le j \le m+1, \\ wt(u_i) &= (m+1)n + 1 + i, \quad \text{for } 1 \le i \le n. \end{aligned}$$

If $n \equiv 0 \pmod 3$, then

$$wt(w) = \begin{cases} nk + 3, & \text{for } n = 3, \\ nk + 2m, & \text{for } n = 6, \\ nk + 1, & \text{for } n \ge 9 \text{ odd}, \\ nk + (m-2)\left\lceil \frac{n}{3} \right\rceil + 5, & \text{for } n \ge 12 \text{ even}. \end{cases}$$

If $n \equiv 1 \pmod 3$, then

$$wt(w) = \begin{cases} nk + \left\lceil \frac{4m}{3} \right\rceil, & \text{for } n = 4, \\ nk + 1, & \text{for } n \ge 7 \text{ odd}, \\ nk + (m-2)\left\lceil \frac{n}{3} \right\rceil + \frac{19-2m}{3}, & \text{for } n \ge 10 \text{ even}, \end{cases}$$

and if $n \equiv 2 \pmod 3$, then

$$wt(w) = \begin{cases} nk + 2, & \text{for } n = 5, \\ nk + (m-2)\left\lceil \frac{n}{3} \right\rceil + \frac{17-m}{3}, & \text{for } n \ge 8 \text{ even}, \\ nk + 1, & \text{for } n \ge 11 \text{ odd}. \end{cases}$$

One can see that the vertex-weights of vertices $v_{i,j}$ and u_i attain consecutive integers from 2 up to $n(m+2)+1$ and the vertex-weight of w is greater than $n(m+2)+1$. Therefore, the vertex-weights are different for all vertices. Thus the labeling f is the required vertex irregular total k-labeling. In fact, for every of previous three cases

$$\mathrm{tvs}(H_n^m) \leq \left\lceil \tfrac{(m+1)n+1}{3} \right\rceil. \tag{3}$$

Combining (3) with the lower bound given in Lemma 2.1, we conclude that $\mathrm{tvs}(H_n^m) = \lceil ((m+1)n+1)/3 \rceil$. □

3 A prisms with outer pendant edges

In this part, we study the total vertex irregularity strength for a prism with outer pendant edges. It is a graph derived from a prism D_n, $n \geq 3$, by hanging a leaf from every vertex on the outer-cycle and denoted by $\mathcal{P}_n$. Let $V(\mathcal{P}_n) = \{v_{i,j} : 1 \leq i \leq n, j = 1,2,3\}$ be the vertex set and $E(\mathcal{P}_n) = \{v_{i,j}v_{i+1,j} : 1 \leq i \leq n, j = 2,3\} \cup \{v_{i,1}v_{i,2}, v_{i,2}v_{i,3} : 1 \leq i \leq n\}$ be the edge set of $\mathcal{P}_n$, where indices are taken modulo n. Thus, $\mathcal{P}_n$ has $4n$ edges, n vertices of degree 1, n vertices of degree 3 and n vertices of degree 4.

The next two lemmas determine the upper bound for the total vertex irregularity strength of $\mathcal{P}_n$.

Lemma 3.1. *Let $\mathcal{P}_n$, $n = 4,5,7,10$, be the prism with outer pendant edges. Then*

$$\mathrm{tvs}(\mathcal{P}_\mathrm{n}) \leq \left\lceil \tfrac{3\mathrm{n}+1}{5} \right\rceil.$$

Proof. Let $k = \lceil (3n+1)/5 \rceil$. To prove the upper bound of $\mathrm{tvs}(\mathcal{P}_\mathrm{n})$, it is sufficient to show the existence of a vertex irregular total k-labeling. Define the total labeling f of an element x, $x \in V(\mathcal{P}_n) \cup E(\mathcal{P}_n)$ as in the following table.

It is not difficult to see that under the total labeling f all vertex and edge labels are at most k and that the vertex-weights of $\mathcal{P}_n$ are as follows:

$$\begin{aligned} wt_f(v_{i,1}) &= i+1, && \text{for } 1 \leq i \leq n,\\ wt_f(v_{i,2}) &= 2n+i+1, && \text{for } 1 \leq i \leq n,\\ wt_f(v_{i,3}) &= n+i+1, && \text{for } 1 \leq i \leq n. \end{aligned}$$

Since the vertex-weights form a set of consecutive integers from 2 up to $3n+1$ then the weights of all vertices are pairwise distinct. It proves that $\mathrm{tvs}(\mathcal{P}_\mathrm{n}) \leq \mathrm{k}$. □

x	$f(x)$		n
$v_{i,1}$	$1,$	$1 \le i \le k$	$4,5,7,10$
	$i-k+1,$	$k+1 \le i \le n$	
$v_{i,2}$	$1,$	$1 \le i \le k+1$	$4,5,7,10$
	$i-k,$	$k+2 \le i \le n$	
$v_{i,3}$	$2,$	$i = 1$	$4,5$
	$k,$	$2 \le i \le n$	
	$\lceil \frac{n}{7} \rceil,$	$i = 1$	$7,10$
	$\lceil \frac{n}{7} \rceil + i + 1,$	$2 \le i \le n-k$	
	$k,$	$n-k+1 \le i \le n$	
$v_{i,1}v_{i,2}$	$\min\{i,k\},$	$1 \le i \le n$	$4,5,7,10$
$v_{i,2}v_{i+1,2}$	$k,$	$1 \le i \le n$	
$v_{i,2}v_{i,3}$	$k-1,$	$1 \le i \le k$	$4,7,10$
	k	$k+1, \le i \le n$	
	$k-2,$	$1 \le i \le k$	5
	$k-1,$	$k+1 \le i \le n$	
$v_{i,3}v_{i+1,3}$	$1,$	$i = 1,2,4$	4
	$2,$	$i = 3$	
	$1,$	$1 \le i \le \lceil \frac{n}{3} \rceil$	$5,7$
	$\left\lceil \frac{i - \lceil \frac{n}{3} \rceil + 3}{3} \right\rceil,$	$\lceil \frac{n}{3} \rceil + 1 \le i \le n$	
	$1,$	$1 \le i \le \lceil \frac{n}{3} \rceil$	10
	$\left\lceil \frac{i - \lceil \frac{n}{3} \rceil + 2}{2} \right\rceil,$	$\lceil \frac{n}{3} \rceil + 1 \le i \le \lceil \frac{n}{3} \rceil + 2$	
	$\lceil \frac{i}{2} \rceil - 1,$	$\lceil \frac{n}{3} \rceil + 3 \le i \le n, i$ odd	
	$\lceil \frac{i}{2} \rceil - 2,$	$\lceil \frac{n}{3} \rceil + 3 \le i \le n, i$ even	

Lemma 3.2. *Let $\mathcal{P}_n$, $n \neq 4,5,7,10$, be the prism with outer pendant edges. Then*

$$\mathrm{tvs}(\mathcal{P}_\mathrm{n}) \le \left\lceil \frac{3\mathrm{n}+1}{5} \right\rceil.$$

Proof. Let $k = \lceil (3n+1)/5 \rceil$. Define the total labeling g of an element x, $x \in V(\mathcal{P}_n) \cup E(\mathcal{P}_n)$ as in the table overleaf.

It is a matter for routine checking to see that under the labeling g all vertex and edge labels are at most k and the vertex-weights successively form three sets of consecutive integers

$$\{wt_g(v_{i,1}) = i+1 : 1 \le i \le n\} = \{2,3,\ldots,n+1\},$$
$$\{wt_g(v_{i,3}) = n+i+1 : 1 \le i \le n\} = \{n+2,n+3,\ldots,2n+1\},$$
$$\{wt_g(v_{i,2}) = 2n+i+1 : 1 \le i \le n\} = \{2n+2,2n+3,\ldots,3n+1\}.$$

So the labeling g has the required properties and the existence of an vertex irregular total k-labeling for $\mathcal{P}_n$ is proved. □

x	$g(x)$		n
$v_{i,1}$	$1,$	$1 \leq i \leq k$	$\neq 4,5,7,10$
	$i-k+1,$	$k+1 \leq i \leq n$	
$v_{i,2}$	$1,$	$1 \leq i \leq 2$	3
	$2,$	$i=3$	
	$3\left\lceil \frac{n-15}{15} \right\rceil + 2,$	$i=1$	$\equiv 0 \pmod 3;\ \geq 6$
	$3\left\lceil \frac{n-15}{15} \right\rceil + 1,$	$2 \leq i \leq k$	
	$i-k+3\left\lceil \frac{n-15}{15} \right\rceil + 1,$	$k+1 \leq i \leq n$	
	$3\left\lceil \frac{n-10}{15} \right\rceil + 1,$	$i=1$	$\equiv 1 \pmod 3; \geq 13$
	$3\left\lceil \frac{n-10}{15} \right\rceil,$	$2 \leq i \leq k$	
	$i-k+3\left\lceil \frac{n-5}{15} \right\rceil,$	$k+1 \leq i \leq n$	
	$3\left\lceil \frac{n-5}{15} \right\rceil,$	$i=1$	$\equiv 2 \pmod 3; \geq 8$
	$3\left\lceil \frac{n-5}{15} \right\rceil - 1,$	$2 \leq i \leq k$	
	$i-k+3\left\lceil \frac{n-5}{15} \right\rceil - 1,$	$k+1 \leq i \leq n$	
$v_{i,3}$	$1,$	$i=1$	3
	$2,$	$2 \leq i \leq 3$	
	$1,$	$i=1$	$\equiv 0 \pmod 3; \geq 6$
	$i+n-k-1,$	$2 \leq i \leq n-k-3\left\lceil \frac{n-15}{15} \right\rceil$	
	$k,$	$n-k-3\left\lceil \frac{n-15}{15} \right\rceil + 1 \leq i \leq n$	
	$1,$	$i=1$	$\equiv 1 \pmod 3; \geq 13$
	$i+n-k-1,$	$2 \leq i \leq n-k-2\left\lceil \frac{n-10}{15} \right\rceil - \left\lceil \frac{n-25}{15} \right\rceil$	
	$k,$	$n-k-2\left\lceil \frac{n-10}{15} \right\rceil - \left\lceil \frac{n-25}{15} \right\rceil + 1 \leq i \leq n$	
	$1,$	$i=1$	$\equiv 2 \pmod 3; \geq 8$
	$i+n-k-1,$	$2 \leq i \leq n-k-2\left\lceil \frac{n-20}{15} \right\rceil - \left\lceil \frac{n-5}{15} \right\rceil$	
	$k,$	$n-k-2\left\lceil \frac{n-20}{15} \right\rceil - \left\lceil \frac{n-5}{15} \right\rceil + 1 \leq i \leq n$	
$v_{i,1}v_{i,2}$	$\min\{i,k\},$	$1 \leq i \leq n$	$\neq 4,5,7,10$
$v_{i,2}v_{i+1,2}$	$k,$	$1 \leq i \leq n$	
$v_{i,2}v_{i,3}$	$2,$	$1 \leq i \leq 3$	3
	$k-1,$	$i=1$	$\neq 3,4,5,7,10$
	$k,$	$2 \leq i \leq n$	
$v_{i,3}v_{i+1,3}$	$1,$	$1 \leq i \leq 3$	3
	$1,$	$1 \leq i \leq n-k-2\left\lceil \frac{n-15}{15} \right\rceil - \left\lceil \frac{n-30}{15} \right\rceil$	$\equiv 0 \pmod 3;$ ≥ 6
	$\left\lceil \frac{i-n+k+2\left\lceil \frac{n-15}{15} \right\rceil + \left\lceil \frac{n-30}{15} \right\rceil + 2}{2} \right\rceil,$	$n-k-2\left\lceil \frac{n-15}{15} \right\rceil - \left\lceil \frac{n-30}{15} \right\rceil + 1 \leq i \leq n$	
	$1,$	$1 \leq i \leq n-k-2\left\lceil \frac{n-10}{15} \right\rceil - \left\lceil \frac{n-40}{15} \right\rceil$	$\equiv 1 \pmod 3;$ ≥ 13
	$\left\lceil \frac{i-n+k+2\left\lceil \frac{n-10}{15} \right\rceil + \left\lceil \frac{n-40}{15} \right\rceil + 2}{2} \right\rceil,$	$n-k-2\left\lceil \frac{n-10}{15} \right\rceil - \left\lceil \frac{n-40}{15} \right\rceil + 1 \leq i \leq n$	
	$1,$	$1 \leq i \leq n-k-3\left\lceil \frac{n-20}{15} \right\rceil$	$\equiv 2 \pmod 3;$ ≥ 8
	$\left\lceil \frac{i-n+k+3\left\lceil \frac{n-20}{15} \right\rceil + 2}{2} \right\rceil,$	$n-k-3\left\lceil \frac{n-20}{15} \right\rceil + 1 \leq i \leq n$	

The next theorem proves the exact value of the total vertex irregularity strength for $\mathcal{P}_n$.

Theorem 3.3. *Let $\mathcal{P}_n$, $n \geq 3$ be the prism with outer pendant edges. Then*

$$\mathrm{tvs}(\mathcal{P}_\mathrm{n}) = \left\lceil \tfrac{3\mathrm{n}+1}{5} \right\rceil .$$

Proof. According to (1) and (2), for the graph of $\mathcal{P}_n$ with order $3n$, having minimum degree 1 and maximum degree 4 we have that $\mathrm{tvs}(\mathcal{P}_\mathrm{n}) \geq \lceil (3\mathrm{n}+1)/5 \rceil$, $n \geq 3$. Combining with the upper bound from Lemmas 3.1 and 3.2 we conclude that $\mathrm{tvs}(\mathcal{P}_\mathrm{n}) = \lceil (3\mathrm{n}+1)/5 \rceil$, $n \geq 3$. $\square$

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