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The nonexistence of distance-regular graphs with intersection arrays $\{27, 20, 10; 1, 2, 18\}$ and $\{36, 28, 4; 1, 2, 24\}$

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Abstract

Locally, a distance-regular graph with '$\mu = 2$' carries the structure of a partial linear space. Using this, we show that there are no distance-regular graphs with intersection array $\{27, 20, 10; 1, 2, 18\}$ or $\{36, 28, 4; 1, 2, 24\}$ (on, respectively, 448 or 625 vertices).

1 Introduction

Let Γ be a distance-regular graph (for definitions, see Section 2 below) such that any two vertices at distance 2 have precisely 2 common neighbours. Let Δ be the subgraph of Γ induced on the neighbours of a fixed vertex. Then in Δ any two nonadjacent vertices have at most one common neighbour. It follows that each edge of Δ lies in a unique maximal clique, so that Δ is the collinearity graph of a partial linear space of girth at least 5 (see [2, Section 2]).

In [2] the following theorem was proved.

Theorem 1.1 *A connected partial linear space with girth at least* 5 *and more than one line in which every point has* λ *neighbours, contains* $k \geq \lambda(\lambda + 3)/2$ *points. Equality holds only in the case* $\lambda = 2$*,* $k = 5$*.*

Applying this to a putative distance-regular graph with intersection array $\{27, 20, 10; 1, 2, 18\}$, we see that the connected components of a local partial linear space have $k \le 27$ points, where each point has $\lambda = 6$ neighbours, and we arrive at a contradiction: each connected component must be a single line of size 7, but 7 does not divide 27. We found

Proposition 1.2 *There is no distance-regular graph with intersection array* $\{27, 20, 10; 1, 2, 18\}$.

Next, consider the intersection array $\{36, 28, 4; 1, 2, 24\}$. Here $\lambda = 7$, $k = 36$ and the inequality is not violated. But there is not much room, and a closer inspection rules out this case as well, see Section 3 below. We find

Proposition 1.3 *There is no distance-regular graph with intersection array* $\{36, 28, 4; 1, 2, 24\}$.

2 Distance-regular graphs

For the concept of distance-regular graph, and any unexplained notation, cf. [1]. Here we very briefly repeat the definitions of distance-regular graph and intersection array.

Let Γ be an undirected graph, without loops or multiple edges. Assume that Γ is connected, with diameter d. If u, v are two vertices, then the distance $d(u, v) = d_\Gamma(u, v)$ between u and v in Γ is the length of a shortest path from u to v. Thus, $d(u, v) = 0$ means $u = v$, and $d(u, v) = 1$ precisely when u, v are adjacent. Define $\Gamma_i(x) = \{y \mid d(x, y) = i\}$ so that $\Gamma(x) := \Gamma_1(x)$ is the set of neighbours of x. The graph Γ is called *distance-regular* when there are constants p^h_{ij} $(0 \le h, i, j \le d)$ such that $|\Gamma_i(x) \cap \Gamma_j(y)| = p^h_{ij}$ for every two vertices x, y with $d(x, y) = h$. For a distance-regular graph with parameters p^h_{ij} one defines constants $a_i := p^i_{1i}$, $b_i := p^i_{1,i+1}$, $c_i := p^i_{1,i-1}$, and further $k := b_0$, $\lambda := a_1$, $\mu := c_2$. Now Γ is regular of valency k, and $a_i + b_i + c_i = k$ for all i, $0 \le i \le d$. Every local graph $\Gamma(x)$ is regular of valency λ. Two vertices at distance 2 have μ common neighbours. All parameters p^h_{ij} can be expressed in terms of the b_i and c_i. The *intersection array* of Γ is by definition $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$.

In the tables at the end of [1], the existence of distance-regular graphs with intersection arrays $\{27, 20, 10; 1, 2, 18\}$ and $\{36, 28, 4; 1, 2, 24\}$ is shown as undecided. Much happened since the publication of that book, and [3] is a survey of all that is new. But the tables at the end of this survey show no progress on these existence questions. In this note we show that there are no such graphs.

3 Nonexistence of a partial linear space

A *partial linear space* is a geometry with points and lines, where two lines meet in at most one point. Let xIL denote that the point x is incident with the line L. The *girth* g of a partial linear space is the length of a shortest cycle $x_0 I L_1 I x_1 I L_2 \ldots L_g I x_0$ of mutually distinct points and lines. Thus, girth at least 5 means that the partial linear space does not contain triangles or quadrangles.

Consider a partial linear space of girth at least 5 on 36 vertices, where each point is collinear with 7 other points. We shall derive a contradiction.

Fix a point x_0, and let there be m_i lines of size i (say, i-lines, $i \geq 2$) on x_0. The valency condition says $\sum m_i(i-1) = 7$. At distance 2 from x_0 there are $\sum m_i(i-1)(8-i)$ vertices, so this expression is not larger than $36-1-7=28$. Let us call a line *long* if it has size at least 5. If x_0 is not on any long lines, then we can bound $8-i \geq 4$ and find $28 = 4\sum m_i(i-1) \leq \sum m_i(i-1)(8-i) \leq 28$. It follows that equality holds everywhere and x_0 is on 4-lines only. But 3 does not divide 7, this is impossible.

We proved that each point is on a (unique) long line, so that the vertex set is partitioned by the long lines.

Consider a long line L of size m, $m = 5, 6, 7$. Then L has $m(8-m)$ neighbours, each on a different long line. So $5m(8-m)+m \leq 36$, a contradiction. It follows that each point is on an 8-line. But 8 does not divide 36, and we reach the final contradiction.

References

[1] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, 1989.

[2] A. E. Brouwer and A. Neumaier, A remark on partial linear spaces of girth 5 with an application to strongly regular graphs, *Combinatorica* **8** (1988), 57–61.

[3] E. R. van Dam, J. H. Koolen and H. Tanaka, Distance-regular graphs, *Electr. J. Combin.* (2016) #DS22.

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