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List backbone coloring of paths and cycles

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Abstract

Let H be a subgraph of G in a graph pair (G, H). A backbone k-coloring for (G, H) is a proper coloring of G by the set of colors $\{1, 2, \ldots, k\}$, adding a condition that colors assigned to adjacent vertices in H must differ by at least two. A list assignment L is a mapping that assigns a set of positive integers $L(v)$ to each vertex v in G. A k-assignment L of G is a list assignment L with $|L(v)| = k$ for each vertex v. If there is a backbone coloring c of G such that $c(v) \in L(v)$, then (G, H) is backbone L-colorable. The backbone choice number of (G, H), denoted by $ch_{BB}(G, H)$, is the smallest integer k such that G is backbone L-colorable for each k-assignment L. The concept of a backbone choice number is a generalization of both the choice number and the $L(2, 1)$-choice number.

The result of Bu, Finbow, Liu and Zhu [*Discrete Appl. Math.* 167 (2014), 45–51] implies that for a path or a cycle G, the upper bound of $ch_{BB}(G, H)$ is 9 where every component H is unicyclic. In this paper, we show that the maximum possible value of $ch_{BB}(G, H)$ is 5. Moreover, we obtain exact values of $ch_{BB}(G, H)$ where G is a path or a cycle in all possible structures of subgraphs H of G.

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1 Introduction

Backbone coloring of graphs has its origin from the problem of channel assignment by Hale [10]. In the channel assignment problem, each channel must be assigned to a set of transmitters without interference. Interferences are separated into two types: weak and strong. The channels assigned to two transmitters with weak interference should be distinct while the channels assigned to two transmitters with strong interference should be far apart. A graph G is constructed with the transmitters represented by the vertices of G, and two vertices are adjacent in G when two corresponding transmitters interfere with each another. Let H be a subgraph of G such that an edge of H is formed from two vertices with strong interference. We call a subgraph of H a *backbone* of G, and (G, H) a *graph pair*.

Let G be a graph and H be a subgraph of G. We say that (G, H) *contains* (G', H') if G' is a subgraph of G and H' is a subgraph of H. In a graph pair (G, H), a vertex coloring $c : V(G) \to \{1, 2, \ldots, k\}$ such that $|c(u) - c(v)| \geq 1$ if $uv \in E(G) \setminus E(H)$, and $|c(u) - c(v)| \geq 2$ if $uv \in E(H)$ is called a *backbone k-coloring* for (G, H). The smallest integer k for which there exists a backbone k-coloring for (G, H) is called the *backbone chromatic number* of (G, H), denoted by $\chi_{BB}(G, H)$.

Backbone coloring of graphs was first studied by Broersma et al. [1]. Denote the chromatic number of a graph G by $\chi(G)$. Obviously, $\chi_{BB}(G, H) \leq 2\chi(G) - 1$ for any graph pair (G, H). This upper bound is sharp as shown in [2] that there is a graph G and a spanning tree H of G with $\chi(G) = n$ and $\chi_{BB}(G, H) = 2\chi(G) - 1$ for any positive integer n. It is shown in [15] that for a graph G with maximum degree Δ, $\chi_{BB}(G, H) \leq \Delta + d + 1$ when H is a d-degenerate subgraph of G, and $\chi_{BB}(G, M) \leq \Delta + 1$ when M is a matching of G. This topic has been investigated rather extensively in recent years (see [3, 4, 5, 7, 11, 14, 17]).

Furthermore, if $E(H) = \emptyset$, then $\chi_{BB}(G, H) = \chi(G)$. Backbone coloring is a generalization of $L(2, 1)$-labeling by setting $G = H^2$, that is, $V(G) = V(H)$ and $uv \in E(G)$ when $uv \in E(H)$ or there is a path with length 2 from u to v in H. Thus, a backbone coloring of (G, H) is equivalent to an $L(2, 1)$-labeling of H. The topic of $L(2, 1)$-labeling was introduced by Griggs and Yeh [9] and has been investigated widely by many authors (see [12, 13, 19]).

A mapping L is called a *list assignment* of a graph G if L assigns a set $L(v)$ of positive integers to each vertex v. A *k-assignment* L of a graph G is a list assignment L such that $|L(v)| = k$ for each $v \in V(G)$. We say that c is a *list coloring* or an *L-coloring* of a graph G if c is a proper coloring of G such that $c(v) \in L(v)$ for each $v \in V(G)$. If a graph G has an L-coloring, then we say that G is *L-colorable*. A graph G is *k-choosable* if every k-assignment of G gives a list coloring. The *choice number* of a graph G, denoted by $ch(G)$, is the smallest integer k such that G is k-choosable. List coloring of graphs has been studied widely by many authors (see [8, 16]).

In 2014, Bu et al. [6] introduced the concept of list backbone coloring of graphs. Consider a graph pair (G, H). We say that c is a *backbone L-coloring* of (G, H) if c

is a backbone coloring of (G,H) such that $c(v) \in L(v)$ for each $v \in V(G)$. If (G,H) has a backbone L-coloring, then we say that (G,H) is *backbone L-colorable*. We say that (G,H) is *backbone k-choosable* if every k-assignment L of G gives a backbone L-coloring. The *backbone choice number* of (G,H), denoted by $ch_{BB}(G,H)$, is the smallest integer k such that G is backbone k-choosable. For $V(G) = \{v_1,\dots,v_n\}$, a list assignment L of G is of *order* $(k_1,\dots,k_n)$ if $|L(v_i)| = k_i$ for all $i = 1,\dots,n$. We say that (G,H) is *backbone* $(k_1,\dots,k_n)$-*choosable* if (G,H) is backbone L-colorable for every list assignment L of order $(k_1,\dots,k_n)$. For convenience, we use $(a,b,c^{[n]},d)$ to denote the sequence $(a,b,\overbrace{c,\dots,c}^{n},d)$. Obviously, $ch_{BB}(G,H) = ch(G)$ if $E(H) = \emptyset$.

From now on, we say that $(G, P_{m_1} \cup P_{m_2} \cup \dots \cup P_{m_k})$ is a graph $G = P_n$ or C_n containing path backbones $P_{m_1}, P_{m_2},\dots,P_{m_k}$ with $n = m_1 + m_2 + \dots + m_k$ such that $V(G) = \{v_1,\dots,v_n\}$, $V(P_{m_1}) = \{v_1,\dots,v_{m_1}\}$, and $V(P_{m_i}) = \{v_{A+1}, v_{A+2}, v_{A+3}, \dots, v_B\}$ where $A = \sum_{j=1}^{i-1} m_j$ and $B = \sum_{j=1}^{i} m_j$ for $2 \le i \le k$. Note that m_i can be 1 for each $i \in \{1,\dots,k\}$ to make this well-defined. For convenience, we use $(G, P_a \cup P_b^{[m]} \cup P_c)$ to denote the $(G, P_a \cup \overbrace{P_b \cup \dots \cup P_b}^{m} \cup P_c)$.

Let L be a list assignment of a graph G. If a vertex v has been colored by $p \in L(v)$, then we define $L'(u) = L(u) \setminus \{p\}$ if $uv \in E(G) \setminus E(H)$, $L'(u) = L(u) \setminus \{p-1,p,p+1\}$ if $uv \in E(H)$, and $L'(u) = L(u)$ if $uv \notin E(G)$. A list assignment L' is called the *residual list assignment* (*RLA* for abbreviation) for the graph $G-v$. This concept can be extended to the situation when many vertices have been colored. Other standard notations follow West [18].

For convenience, sometimes we use $v_i v_{i+1} \dots v_k$ to denote a path with a vertex set $\{v_i, v_{i+1},\dots,v_k\}$, and $v_1 v_2 \dots v_n v_1$ to denote a cycle with a vertex set $\{v_1, v_2,\dots,v_n\}$.

For a cycle or a path G, the results of Bu et al. [6] implies that the upper bound of $ch_{BB}(G,H)$ is 9 where every component H is unicyclic. The aim of this paper is to find exact values of $ch_{BB}(G,H)$ where G is a path or a cycle in all possible structures of subgraphs H of G. As a consequence, we show that the maximum possible value of $ch_{BB}(G,H)$ is 5.

2 Preliminaries

Bu et al. [6] provided several upper bounds for $ch_{BB}(G,H)$ in terms of the choice number of a k-choosable graph G and the structure of H in the following results:

$$ch_{BB}(G,H) \le \begin{cases} 2k, & \text{if } H \text{ is a matching;} \\ 2k+1, & \text{if } H \text{ is a disjoint union of paths with length at most 2;} \\ 3k, & \text{if each component of } H \text{ is unicyclic.} \end{cases}$$

The next remark presents results that will be useful.

Remark 2.1. *Let G be a graph and H be a subgraph of G.*

(i) ([8]) *If a connected graph* G *is neither complete nor an odd cycle, then* $ch(G) \leq \Delta(G)$.

(ii) $\chi(G) \leq ch(G) \leq \Delta(G) + 1$.

(iii) ([2]) $\chi_{BB}(G, H) \leq 2\chi(G) - 1$.

(iv) *If* $E(H) \neq \emptyset$, *then* $ch_{BB}(G, H) \geq 3$.

(v) *If* $E(H) = \emptyset$, *then* $ch_{BB}(G, H) = ch(G)$.

If G is a path or an even cycle, then $ch(G) = 2$, by Remarks 2.1(i, ii). So G is 2-choosable. Then

$$ch_{BB}(G, H) \leq \begin{cases} 4, & \text{if } H \text{ is a matching;} \\ 5, & \text{if } H \text{ is a disjoint union of paths with length at most 2;} \\ 6, & \text{otherwise.} \end{cases}$$

Similarly, if G is an odd cycle, then G is 3-choosable, and

$$ch_{BB}(G, H) \leq \begin{cases} 6, & \text{if } H \text{ is a matching;} \\ 7, & \text{if } H \text{ is a disjoint union of paths with length at most 2;} \\ 9, & \text{otherwise.} \end{cases}$$

From these results, we can see that the value of $ch_{BB}(G, H)$ is at most 9 where G is a path or a cycle. In this paper, we show that $ch_{BB}(G, H) \leq 5$ where G is a path or cycle in all possible structures of subgraphs H of G.

3 Main Results

In this paper, we assume that a graph G is a path or a cycle with n vertices unless the context suggests otherwise and $V(G) = \{v_1, \ldots, v_n\}$ when a vertex set is not identified. The main result of this paper is the following.

Main Theorem. *Let* G *be a cycle or a path and* H *be a spanning subgraph of* G. *Then* $ch_{BB}(G, H) =$

$$\begin{cases} 1, & \textit{when} \quad G = H = P_1; \\ 2, & \textit{when} \quad E(H) = \emptyset \textit{ and } G \textit{ is not an odd cycle;} \\ 5, & \textit{when} \quad G = H \textit{ and } G \textit{ is an odd cycle;} \\ 4, & \textit{when} \quad \textit{(i) } G = H \textit{ and } G \textit{ is an even cycle} \\ & \textit{or} \quad \textit{(ii) } H \textit{ contains a path with length at least } 3 \\ & \textit{or} \quad \textit{(iii) } G \textit{ is a cycle and } H \textit{ is a disjoint union of paths} \\ & \qquad \textit{with length at least } 1 \textit{ and at least one path has length } 2 \\ & \textit{or} \quad \textit{(iv) } (G, H) \textit{ contains } (P_{2k+6}, P_3 \cup P_2^{[k]} \cup P_3) \textit{ where } k \geq 0; \\ 3, & \textit{otherwise.} \end{cases}$$

Throughout the proof, we repeatedly use the following easy observation.

Proposition 3.1. *If* (G, H) *contains* (G', H')*, then* $ch_{BB}(G, H) \geq ch_{BB}(G', H')$.

Proof. Let $k = ch_{BB}(G, H)$. Assume that L is any k-assignment of G'. Note that (G, H) is backbone k-choosable. Since $E(H') \subseteq E(H)$ and $E(G') \subseteq E(G)$, (G', H') is also backbone k-choosable. Hence $k \geq ch_{BB}(G', H')$. □

Our proof of the Main Theorem is split into numerous cases, so it is helpful to provide a brief outline here. (We defer the complete proof to the end of the paper.) Remark 2.1(*ii*) implies that $ch_{BB}(G, H) = 1$ if and only if G has no edges. Remarks 2.1(*iv*,*v*) imply that $ch_{BB}(G, H) = 2$ if and only if H has no edges and G is a path with length at least 1 or an even cycle. The case when G is a cycle and $H = G$ requires significant work, so we defer it to Section 3.3. When $H \neq C_n$, we always have the upper bound $ch_{BB}(G, H) \leq ch_{BB}(C_n, P_n) \leq 4$; the first inequality is from Proposition 3.1 and we prove the second inequality in Lemma 3.2. Thus, the remainder of the proof consists of determining for which pairs (G, H) we have $ch_{BB}(G, H) \geq 4$ (Section 3.1) and for which we have $ch_{BB}(G, H) \leq 3$ (Section 3.2). It is easy to show that $ch_{BB}(P_4, P_4) = 4$, as we do in Lemma 3.2. Thus, we need only consider the case when H consists of disjoint paths of length at most 2. Similarly, if $E(H) \neq \emptyset$ and I is a set of vertices isolated in H, then we have $ch_{BB}(G, H) = ch_{BB}(G - I, H - I)$, as we show in Lemma 3.8. Thus, we can assume that H consists of disjoint paths, each of length 1 or 2. Now, as stated in the Main Theorem, we have $ch_{BB}(G, H) = 4$ if either H contains at least two paths of length two (see Corollary 3.5) or G is a cycle and H contains a single path of length two (see Corollary 3.7). Otherwise $ch_{BB}(G, H) = 3$, as we show in Theorems 3.11 and 3.14.

3.1 H is a union of paths and $ch_{BB}(G, H) = 4$

Lemma 3.2. $ch_{BB}(C_n, P_n) = 4$ *for* $n \geq 3$ *and* $ch_{BB}(P_n, P_n) = 4$ *for* $n \geq 4$.

Proof. We begin by showing that $ch_{BB}(C_3, P_3) \geq 4$ and $ch_{BB}(P_4, P_4) \geq 4$. After this, by Proposition 3.1 it suffices to show that $ch_{BB}(C_n, P_n) \leq 4$.

First, let $C_3 = v_1v_2v_3v_1$ and let $L(v_1) = L(v_2) = L(v_3) = \{1, 2, 3\}$. Suppose that (C_3, P_3) has a backbone L-coloring φ. We must have $|\varphi(v_1) - \varphi(v_2)| = 2$ and $|\varphi(v_2) - \varphi(v_3)| = 2$. But this implies $\varphi(v_1) = \varphi(v_3)$, a contradiction; hence $ch_{BB}(C_3, P_3) \geq 4$. Now instead, let $P_4 = v_1v_2v_3v_4$, let $L(v_1) = L(v_4) = \{2, 3, 4\}$, and let $L(v_2) = L(v_3) = \{1, 2, 3\}$. Suppose that (P_4, P_4) has a backbone L-coloring φ. By symmetry, we may assume that $\varphi(v_2) = 3$ and $\varphi(v_3) = 1$. However, now we have no color available for v_1, a contradiction; hence $ch_{BB}(P_4, P_4) \geq 4$.

Now we show that $ch_{BB}(C_n, P_n) \leq 4$, for all n. Let $C_n = v_1v_2v_3 \ldots v_nv_1$. Assume that L is any 4-assignment of C_n. Since $|L(v_1)| = 4$, we can assign color a_1 to v_1 such that $a_1 \neq \min L(v_1)$ and $a_1 \neq \max L(v_1)$. For $i = 2, 3, \ldots, n$, we can assign color $a_i \in L(v_i)$ to v_i such that $a_i \notin \{a_{i-1} - 1, a_{i-1}, a_{i-1} + 1\}$. If $a_1 \neq a_n$, then we have a backbone L-coloring.

Assume that $a_1 = a_n$. Define $c(v_i) = a_i$ for $i \in \{2, 3, \ldots, n\}$. If there is $b \in L(v_1) \setminus \{a_1, a_2 - 1, a_2, a_2 + 1\}$, then we assign $c(v_1) = b$ to obtain a backbone coloring c of (C_n, P_n). Otherwise, $L(v_1) = \{a_1, a_2 - 1, a_2, a_2 + 1\}$. Because $a_2 - 1, a_2$, and $a_2 + 1$ are consecutive numbers, we have that $a_1 = \min L(v_1)$ or $a_1 = \max L(v_1)$. This contradicts to the choice of a_1. Hence we can assign $c(v_1) \in L(v_1)$ to obtain a backbone coloring c of (C_n, P_n). Therefore $ch_{BB}(C_n, P_n) \leq 4$. □

Theorem 3.3. *Let H be a spanning subgraph of G such that H is a disjoint union of paths with any length. If some component of H has length at least 3, then $ch_{BB}(G, H) = 4$.*

Proof. Assume that there is a component of H with length at least 3. By Proposition 3.1 and Lemma 3.2, we have

$$4 = ch_{BB}(P_4, P_4) \leq ch_{BB}(G, H) \leq ch_{BB}(C_n, P_n) = 4.$$

Therefore $ch_{BB}(G, H) = 4$. □

Lemma 3.4. *For $k \geq 0$, $ch_{BB}(P_{2k+6}, P_3 \cup P_2^{[k]} \cup P_3) = 4$.*

Proof. Let $k \geq 0$ and let $P_{2k+6} = v_1 v_2 v_3 \ldots v_{2k+6}$. Let $L(v_1) = L(v_{2k+6}) = \{2, 3, 4\}$ and $L(v_i) = \{1, 2, 3\}$ for all other i. We show that $(P_{2k+6}, P_3 \cup P_2^{[k]} \cup P_3)$ has no backbone L-coloring. Assume, to the contrary, that it has a backbone L-coloring φ. Since no v_i is isolated in $P_3 \cup P_2^{[k]} \cup P_3$, we must have $\varphi(v_i) \neq 2$ for all $i \in \{2, \ldots, 2k+5\}$. Thus these v_i must alternate the colors 1 and 3. By symmetry, we can assume that $\varphi(v_{2i}) = 1$ and $\varphi(v_{2i+1}) = 3$ for all $i \in \{1, \ldots, 2k+2\}$. But now $\varphi(v_{2k+5}) = 3$, so v_{2k+6} has no allowable color. Hence $ch_{BB}(P_{2k+6}, P_3 \cup P_2^{[k]} \cup P_3) \geq 4$. By Proposition 3.1 and Lemma 3.2, $ch_{BB}(P_{2k+6}, P_3 \cup P_2^{[k]} \cup P_3) \leq ch_{BB}(C_{2k+6}, P_{2k+6}) = 4$. Therefore $ch_{BB}(P_{2k+6}, P_3 \cup P_2^{[k]} \cup P_3) = 4$. □

Corollary 3.5. *Let H be a spanning subgraph of P_n such that H is a disjoint union of paths, each with length 1 or 2. If at least two components of H have length 2, then $ch_{BB}(P_n, H) = 4$.*

Proof. Assume that H contains two components with length 2. Then (P_n, H) contains $(P_{2k+6}, P_3 \cup P_2^{[k]} \cup P_3)$ where $k \geq 0$. By Proposition 3.1 and Lemma 3.4, we have that $ch_{BB}(P_n, H) \geq ch_{BB}(P_{2k+6}, P_3 \cup P_2^{[k]} \cup P_3) = 4$. From Proposition 3.1 and Lemma 3.2, we have that $ch_{BB}(P_n, H) \leq ch_{BB}(C_n, P_n) = 4$. Therefore $ch_{BB}(P_n, H) = 4$. □

Lemma 3.6. *For $k \geq 0$, $ch_{BB}(C_{2k+3}, P_3 \cup P_2^{[k]}) = 4$.*

Proof. Let $C_{2k+3} = v_1 v_2 v_3 \ldots v_{2k+3} v_1$. If $k = 0$ then we have that $ch_{BB}(C_{2k+3}, P_3 \cup P_2^{[k]}) = ch_{BB}(C_3, P_3) = 4$ by Lemma 3.2. Now, we assume that $k \geq 1$. Without loss of generality, we consider C_{2k+3} with lists of colors such that $L(v_i) = \{1, 2, 3\}$ for $i = 1, \ldots, 2k + 3$. Note that we cannot assign 2 to v_i for $i = 1, \ldots, 2k + 3$ to obtain a backbone coloring. We must color each vertex either 1 or 3. However, then the number of vertices we can color is at most $2\left\lfloor\frac{2k+3}{2}\right\rfloor < 2k+3$, a contradiction. Hence $ch_{BB}(C_{2k+3}, P_3 \cup P_2^{[k]}) \geq 4$. By Proposition 3.1 and Lemma 3.2, we have $ch_{BB}(C_{2k+3}, P_3 \cup P_2^{[k]}) \leq ch_{BB}(C_{2k+3}, P_{2k+3}) = 4$. Therefore $ch_{BB}(C_{2k+3}, P_3 \cup P_2^{[k]}) = 4$. □

Corollary 3.7. *Let H be a spanning subgraph of C_n such that H is a disjoint union of paths, each with length 1 or 2. If some component of H has length 2, then $ch_{BB}(C_n, H) = 4$.*

Proof. Assume that there is a component of H with length 2. We consider two cases.

CASE 1: There is exactly one component of H with length 2. By Lemma 3.6, we obtain $ch_{BB}(C_n, H) = 4$.

CASE 2: There is at least two components of H with length 2. By Proposition 3.1, Lemma 3.2, and Corollary 3.5, we obtain that $4 = ch_{BB}(P_n, H) \leq ch_{BB}(C_n, H) \leq ch_{BB}(C_n, P_n) = 4$. Therefore $ch_{BB}(C_n, H) = 4$.

From both cases, we obtain $ch_{BB}(C_n, H) = 4$. □

3.2 H is a union of paths and $ch_{BB}(G, H) = 3$

Recall that $ch_{BB}(P_4, P_4) = 4$, so $ch_{BB}(G, H) \geq 4$ if $P_4 \subseteq H$. Thus, in this section, we consider the case when H is a disjoint union of paths, each with length at most two. The bounds in this section rely on the following important lemma.

Lemma 3.8. *Let H be a spanning subgraph of G and I be the set of all isolated vertices in H. If $E(H - I) \neq \emptyset$, then $ch_{BB}(G, H) = ch_{BB}(G - I, H - I)$.*

Proof. Let $k = ch_{BB}(G - I, H - I)$. Let L be any k-assignment of G. Because $E(H - I) \neq \emptyset$, $ch_{BB}(G - I, H - I) \geq 3$ by Remark 2.1(iv). Consider a k-assignment L of (G, H). We can assign an available color in $L(v)$ to v for each $v \in V(G - I)$ to give an L-backbone coloring. Since G is a path or a cycle, $d_G(u) \leq 2$ for each $u \in I$. We can assign an available color in $L(u)$ to u for each $u \in I$ because $k \geq 3$ and $d_G(u) \leq 2$. Then we obtain an L-backbone coloring for (G, H). So, $ch_{BB}(G, H) \leq ch_{BB}(G - I, H - I)$. Since (G, H) contains $(G - I, H - I)$, we obtain that $ch_{BB}(G, H) \geq ch_{BB}(G - I, H - I)$ by Proposition 3.1. Therefore $ch_{BB}(G, H) = ch_{BB}(G - I, H - I)$. □

Note that for any graph pair (G, H), we have $ch_{BB}(G-I, H-I) \leq ch_{BB}(G, H) \leq \max(ch_{BB}(G - I, H - I), 1 + \max_{v\in I} d_G(v))$. The lemma above is the special case where $1 + \max_{v\in I} d_G(v) = 3$, so both inequalities hold with equality.

As a result of Lemma 3.8, it suffices to consider the case when each component of H is a path of length 1 or 2. In the next few lemmas, we consider the case where each component of H is a path of length 1.

Lemma 3.9. *For $n \geq 1$, a graph pair $(P_{2n}, P_2^{[n]})$ is backbone $(2, 3^{[2n-1]})$-choosable. Thus, $ch_{BB}(P_{2n}, P_2^{[n]}) = 3$.*

Proof. Let $P_{2n} = v_1 v_2 \ldots v_{2n}$. Assume that L is a list assignment of P_{2n} of order $(2, 3^{[2n-1]})$. We show that $(P_{2n}, P_2^{[n]})$ is backbone $(2, 3^{[2n-1]})$-choosable by induction on n. Let $P(n)$ be the proposition that $(P_{2n}, P_2^{[n]})$ is backbone $(2, 3^{[2n-1]})$-choosable.

Base case: $k = 1$. We show that $P(1)$ is true. We have that $P_2 = v_1 v_2$ and L is a list assignment of P_2 of order $(2, 3)$. Color v_1 with a color $a \in L(v_1)$ such that $L(v_2) \neq \{a-1, a, a+1\}$. Now color v_2. Thus, (P_2, P_2) is backbone L-colorable, so (P_2, P_2) is backbone $(2, 3)$-choosable.

Inductive step: $k \geq 2$. Assume that $P(k-1)$ is true. Then $(P_{2k-2}, P_2^{[k-1]})$ is backbone $(2, 3^{[2k-3]})$-choosable. We can obtain a backbone coloring by assigning color $a_i \in L(v_i)$ to v_i for $i = 1, 2, \ldots, 2k-2$. Consider the path $P = v_{2k-2} v_{2k-1} v_{2k}$. Since we assign a_{2k-2} to v_{2k-2}, the order of the RLA for the path $P - v_{2k-2} = v_{2k-1} v_{2k}$ is $(2, 3)$. From case $k = 1$, we obtain $(P_{2k}, P_2^{[k]})$ is backbone $(2, 3^{[2k-1]})$-choosable.

Therefore $(P_{2n}, P_2^{[n]})$ is backbone $(2, 3^{[2n-1]})$-choosable. Furthermore, we obtain that $ch_{BB}(P_{2n}, P_2^{[n]}) \leq 3$. By Remark 2.1($iv$), we have that $ch_{BB}(P_{2n}, P_2^{[n]}) \geq 3$. Therefore $ch_{BB}(P_{2n}, P_2^{[n]}) = 3$. □

Lemma 3.10. *Let A and B be sets, each consisting of three distinct integers. If $A \neq B$, then there exist x, y, p, q, r, where $x \neq y$ and $p \neq q$, such that*

1. *$x, y \in A$ and $p, q, r \in B$, or*
2. *$x, y \in B$ and $p, q, r \in A$*

which satisfy $|x - p| \geq 2$, $|x - q| \geq 2$, and $|y - r| \geq 2$.

Proof. Assume that A and B are sets consisting of three distinct integers such that $A \neq B$. Let $m = \min(A \cup B)$ and $M = \max(A \cup B)$. We consider three cases.

CASE 1: $m, M \in A \cap B$. Since $A \neq B$, we may assume that there are $k \in A$ and $l \in B$ such that $m < k < l < M$ and $A = \{m, k, M\}$ and $B = \{m, l, M\}$. We choose $x = m$, $y = k$, $p = M$, $q = l$, and $r = M$ to make $|x - p| \geq 2$, $|x - q| \geq 2$, and $|y - r| \geq 2$.

CASE 2: $m \in A \setminus B$. Since $|B| = 3$, there are distinct $k, l \in B$ with $|k - m| \geq 2$ and $|l - m| \geq 2$. We choose $x = m, p = k$, and $q = l$. Let $s \in A \setminus \{m\}$. If there is $t \in B$ such that $|s - t| \geq 2$, then we choose $y = s$ and $r = t$. Otherwise, $B = \{s-1, s, s+1\}$. There is $w \in A \setminus \{m, s\}$ such that $|w - b| \geq 2$ for some $b \in B$. We choose $y = w$ and $r = b$.

CASE 3: $M \in A \setminus B$. This case is similar to *CASE 2*.

From these three cases, the proof is completed. □

Now we use Lemma 3.10 to strengthen the bound in Lemma 3.9 to the case when $G = C_{2n}$.

Theorem 3.11. $ch_{BB}(C_{2n}, P_2^{[n]}) = 3$.

Proof. Let $C_{2n} = v_1v_2v_3 \dots v_{2n}v_1$. For each $i = 1, 2, 3, \dots, n$, we may assume that v_{2i-1} and v_{2i} are endpoints of an edge e_i. Let L be any 3-assignment of C_{2n}. We show that $ch_{BB}(C_{2n}, P_2^{[n]}) \leq 3$ by considering three cases.

CASE 1: There is $j \in \{1, 2, \dots, n\}$ such that $L(v_{2j-1}) \neq L(v_{2j})$. Without loss of generality, we may assume that $L(v_1) \neq L(v_2)$. By Lemma 3.10, we may assume further that there are $x, y \in L(v_1)$ and $p, q, r \in L(v_2)$ such that $|x-p| \geq 2$, $|x-q| \geq 2$, and $|y-r| \geq 2$ where $x \neq y$ and $p \neq q$. We can assign y to v_1 and r to v_2. Consider the path $P = v_2v_3v_4 \dots v_{2n-1}v_{2n}$. Then the order of the RLA for the path $P - v_2$ is $(2, 3^{[2n-3]})$. Note that in forming this list assignment, we deleted from $L(v_3)$ the color r used on v_2, but we did not delete from $L(v_{2n})$ the color y used on v_1, since $v_1 \notin V(P)$. From Lemma 3.9, we can extend this coloring to P_{2n} to obtain a backbone coloring by assigning color a_i to v_i for $i = 3, 4, \dots, 2n$. If $y \neq a_{2n}$, then we obtain a desired backbone coloring of $(C_{2n}, P_2^{[n]})$. If $y = a_{2n}$ and $p \neq a_3$, then we can assign x to v_1 and p to v_2. If $y = a_{2n}$ and $p = a_3$, then we can assign x to v_1 and q to v_2. We obtain a backbone coloring of $(C_{2n}, P_2^{[n]})$.

Next, we assume that $L(v_{2j-1}) = L(v_{2j})$ where $e_j = v_{2j-1}v_{2j}$ for all $j = 1, 2, \dots, n$.

CASE 2: All lists of endpoints of e_j have the same list for $j = 1, 2, \dots, n$. Without loss of generality, we may assume that $L(v_i) = \{p, q, r\}$ where $p < q < r$ for all $i = 1, 2, 3, \dots, 2n$. Then $r - p \geq 2$. We obtain a desired backbone coloring by assigning colors p to v_{2j-1} and r to v_{2j}, for all $j = 1, 2, \dots, n$.

CASE 3: There are e_k and e_l where $k \neq l$ and $k, l \in \{1, 2, \dots, n\}$ such that $L(v_{2k}) \neq L(v_{2l})$. Without loss of generality, we may assume that $k = 1$ and $l = n$ and $L(v_1) = L(v_2) = L_1$ and $L(v_{2n-1}) = L(v_{2n}) = L_2$ where $L_1 \neq L_2$. Let $M_1 = \max L_1$, $m_1 = \min L_1$, $M_2 = \max L_2$, and $m_2 = \min L_2$. Since $|L_1| = 3$ and $|L_2| = 3$, $M_1 - m_1 \geq 2$ and $M_2 - m_2 \geq 2$.

CASE 3.1: $M_1 \notin L_2$. We assign M_1 to v_1 and m_1 to v_2. Consider the path $P = v_2v_3v_4 \dots v_{2n-3}v_{2n-2}$. Then the order of the RLA for the path $P - v_2$ is $(2, 3^{[2n-5]})$. From Lemma 3.9, we can extend a coloring to P_{2n-2} to obtain a backbone coloring by assigning color a_i to v_i for $i = 3, 4, \dots, 2n-2$. If $a_{2n-2} \neq M_2$, then we can assign M_2 to v_{2n-1} and m_2 to v_{2n}. Otherwise, we can assign m_2 to v_{2n-1} and M_2 to v_{2n}. Since $M_1 \notin L_2$, we obtain a desired backbone coloring.

CASE 3.2: $m_1 \notin L_2$. This case is similar to *CASE 3.1*.

CASE 3.3: $M_1, m_1 \in L_2$. Recall that $L_1 = \{m_1, b, M_1\}$ where $m_1 < b < M_1$.

CASE 3.3.1: $L_1 \neq \{b-1, b, b+1\}$. Then $m_1 \leq b - 2$ or $M_1 \geq b + 2$. We can assign color b to v_1 and color m_1 or M_1 to v_2 to make $b - m_1 \geq 2$ or $M_1 - b \geq 2$. Consider the path $P = v_2v_3v_4 \dots v_{2n-3}v_{2n-2}$. Then the order of the RLA for the path $P - v_2$ is $(2, 3^{[2n-5]})$. From Lemma 3.9, we can extend a coloring to P_{2n-2} to obtain a backbone coloring by assigning color a_i to v_i for $i = 3, 4, \dots, 2n-2$. If $a_{2n-2} \neq M_2$, then we can assign M_2 to v_{2n-1} and m_2 to v_{2n}. Otherwise, we can assign m_2 to v_{2n-1} and M_2 to v_{2n}. Since $L_1 \neq L_2$, we have $b \notin L_2$. Then $b \neq M_2$ and $b \neq m_2$. Hence, we obtain a desired backbone coloring.

CASE 3.3.2: $L_1 = \{b-1, b, b+1\}$. Then $M_1 = b+1$ and $m_1 = b-1$. Since $L_1 \neq L_2$, $L_2 = \{b-1, b+1, c\}$ where $c \geq b+2$ or $c \leq b-2$. Without loss of generality, we may assume that $c \geq b+2$. Then $M_2 = c$ and $m_2 = b-1$. We can assign $b+1$ to v_1 and $b-1$ to v_2. Consider the path $P = v_2 v_3 v_4 \ldots v_{2n-3} v_{2n-2}$. Then the order of the RLA for the path $P - v_2$ is $(2, 3^{[2n-5]})$. From Lemma 3.9, we can extend a coloring to P_{2n-2} to obtain a backbone coloring by assigning color a_i to v_i for $i = 3, 4, \ldots, 2n-2$. If $a_{2n-2} \neq c$, then we can assign c to v_{2n-1} and $b-1$ to v_{2n}. Otherwise, we can assign $b+1$ to v_{2n-1} and $b-1$ to v_{2n}. Hence, we obtain a desired backbone coloring.

From three cases, we obtain that $ch_{BB}(C_{2n}, P_2^{[n]}) \leq 3$. By Remark 2.1($iv$), we have that $ch_{BB}(C_{2n}, P_2^{[n]}) \geq 3$. Therefore $ch_{BB}(C_{2n}, P_2^{[n]}) = 3$. □

Next, we consider the case where G is a path and H has a single component that is a path of length 2. Recall that we considered the analogous case for $G = C_{2n+1}$ in the previous section, where we showed that $ch_{BB}(C_{2k+3}, P_3 \cup P_2^{[k]}) = 4$.

Lemma 3.12. *$ch_{BB}(P_3, P_3) = 3$.*

Proof. Assume that $P_3 = v_1 v_2 v_3$. Let L be any 3-assignment of P_3. Assign color a_2 to v_2 such that $a_2 \in L(v_2)$ where $L(v_1) \neq \{a_2 - 1, a_2, a_2 + 1\} \neq L(v_3)$. Then we obtain a backbone coloring of (P_3, P_3). We obtain that $ch_{BB}(P_3, P_3) \leq 3$. By Remark 2.1(iv), we have that $ch_{BB}(P_3, P_3) \geq 3$. Therefore $ch_{BB}(P_3, P_3) = 3$. □

Lemma 3.13. *Let H be a spanning subgraph of $P_n (n \geq 2)$ such that H is a disjoint union of paths, each with length at most 2. If at most one component of H has length 2, then $ch_{BB}(P_n, H) = 3$.*

Proof. By Lemma 3.8, we can assume that H has no isolated vertices, so each path in H has length 1 or 2. Let $P_n = v_1 v_2 \ldots v_n$. If H has exactly one component, then $H = P_2$ or $H = P_3$. By Lemmas 3.9 and 3.12, we obtain that $ch_{BB}(P_n, H) = 3$. Assume that H has at least two components. Let L be any 3-assignment of P_n. We consider two cases.

CASE 1: There is exactly one component of H with length 2, say P_3. Then other components of H are paths with length 1. Let $V(P_3) = \{v_i, v_{i+1}, v_{i+2}\}$ for some $1 \leq i \leq n-2$. Note that in this case n and i are odd. By Lemma 3.12, we have that $ch_{BB}(P_3, P_3) = 3$. Then we can obtain a backbone coloring to v_i, v_{i+1}, and v_{i+2} as follows. Consider the path $P = v_1 \ldots v_i$. Since we assign a_i to v_i, the order of the RLA for the path $P - v_i$ is $(3^{[i-2]}, 2)$. By Lemma 3.9, $(P - v_i, P_2^{\left[\frac{i-1}{2}\right]})$ is backbone $(3^{[i-2]}, 2)$-choosable. Similarly, $(P^* - v_{i+2}, P_2^{\left[\frac{n-i-2}{2}\right]})$ is backbone $(2, 3^{[n-i-3]})$-choosable where $P^* = v_{i+2} \ldots v_n$. Hence we obtain a backbone coloring of (P_n, H).

CASE 2: Each component of H is a path with length 1. Then n is even and $H = P_2^{\left[\frac{n}{2}\right]}$. By Lemma 3.9, we obtain that $ch_{BB}(P_n, P_2^{\left[\frac{n}{2}\right]}) = 3$.

From both cases, we have that $ch_{BB}(P_n, H) \leq 3$. We obtain that $ch_{BB}(P_n, H) \geq 3$ by Remark 2.1(iv). Therefore $ch_{BB}(P_n, H) = 3$. □

Our next theorem summarizes most of the results of this section (all of those for which H has at least one isolated vertex).

Theorem 3.14. *Let H be a spanning subgraph of G such that H is a disjoint union of paths, each with length at most 2, and let I be the set of all isolated vertices in H. Assume that $I \neq \emptyset$ and $E(H - I) \neq \emptyset$. Let $G - I = \bigcup_{i \in \Lambda} G_i$, where G_i is a component of $G - I$ and $H_i = G_i \cap H$. If for each $i \in \Lambda$, H_i contains at most one component with length 2, then $ch_{BB}(G, H) = 3$. Otherwise, there is some $j \in \Lambda$ such that H_j contains at least two components with length 2, and then $ch_{BB}(G, H) = 4$.*

Proof. Let $i \in \Lambda$. Assume that H_i contains at most one component with length 2. By Lemma 3.13, $ch_{BB}(G_i, H_i) = 3$. Consequently, $ch_{BB}(G - I, H - I) \leq 3$. Since $E(H - I) \neq \emptyset$, $ch_{BB}(G - I, H - I) \geq 3$. Therefore $ch_{BB}(G - I, H - I) = 3$. By Lemma 3.8, $ch_{BB}(G, H) = ch_{BB}(G - I, H - I) = 3$.

Otherwise, there is $j \in \Lambda$ such that H_j contains at least two components with length 2. By Corollary 3.5, $ch_{BB}(G_j, H_j) = 4$. Since $(G - I, H - I)$ contains (G_j, H_j), $ch_{BB}(G - I, H - I) \geq ch_{BB}(G_j, H_j) = 4$ by Proposition 3.1. Since (C_n, P_n) contains $(G - I, H - I)$, we have that $4 = ch_{BB}(C_n, P_n) \geq ch_{BB}(G - I, H - I)$ by Proposition 3.1 and Lemma 3.2. Hence, $ch_{BB}(G - I, H - I) = 4$. By Lemma 3.8, $ch_{BB}(G, H) = ch_{BB}(G - I, H - I) = 4$. □

3.3 H is a cycle

In this section we consider the case when $H = G = C_n$. We first need the following two lemmas about sets of integers.

Lemma 3.15. *If A and B are each a set consisting of five distinct integers, then there exists x, p, q such that*

1. *$x \in A$ and $p, q \in B$, or*
2. *$x \in B$ and $p, q \in A$*

which satisfy $|x - p| \geq 2$, $|x - q| \geq 2$, and $|p - q| \geq 3$.

Proof. Assume that A and B are sets consisting of five distinct integers. Let $m = \min(A \cup B)$ and $M = \max(A \cup B)$.

CASE 1: $m, M \in A \cap B$. Assume that $A = \{m, a_1, a_2, a_3, M\}$ and $B = \{m, b_1, b_2, b_3, M\}$ so that $m < a_1 < a_2 < a_3 < M$ and $m < b_1 < b_2 < b_3 < M$. We choose $x = a_2$, $p = m$, and $q = M$ to obtain $|x - p| \geq 2$, $|x - q| \geq 2$, and $|p - q| \geq 3$.

CASE 2: $m \in A \setminus B$. Assume that $A = \{m, a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$ so that $m < a_1 < a_2 < a_3 < a_4$ and $b_1 < b_2 < b_3 < b_4 < b_5$. So we have that $m < b_1 < b_2 < b_3 < b_4 < b_5$. We choose $x = m$, $p = b_2$, and $q = b_5$ to obtain $|x - p| \geq 2$, $|x - q| \geq 2$, and $|p - q| \geq 3$.

CASE 3: $M \in A \setminus B$. We can apply *CASE 2* to the sets $-A$ and $-B$.

This completes the proof. □

Lemma 3.16. *Let u and v be two adjacent vertices in C_n and L be any k-assignment of C_n such that $k \geq 4$. The pair (C_n, C_n) has a backbone L-coloring if there are x, p, q such that*

1. $x \in L(u)$ *and* $p, q \in L(v)$*, or*
2. $x \in L(v)$ *and* $p, q \in L(u)$

satisfy $|x - p| \geq 2$*,* $|x - q| \geq 2$*, and* $|p - q| \geq 3$*.*

Proof. Assume that $C_n = v_1v_2v_3 \dots v_nv_1$. Let $k \geq 4$. It suffices to show only the case $k = 4$. Let L be any 4-assignment of C_n. We may assume that $x \in L(v_1)$ and $p, q \in L(v_n)$ such that $|x - p| \geq 2$, $|x - q| \geq 2$, and $|p - q| \geq 3$. We assign $a_1 = x$ to v_1. For $i = 2, 3, \dots, n-1$, we can assign color $a_i \in L(v_i)$ to v_i such that $a_i \notin \{a_{i-1} - 1, a_{i-1}, a_{i-1} + 1\}$. If $|p - a_{n-1}| \leq 1$ and $|q - a_{n-1}| \leq 1$, then $|p - q| \leq 2$, a contradiction. So $|p - a_{n-1}| \geq 2$ or $|q - a_{n-1}| \geq 2$. Then we can assign p or q to v_n. This yields a backbone L-coloring of (C_n, C_n). □

Theorem 3.17. $ch_{BB}(C_{2n+1}, C_{2n+1}) = 5$*.*

Proof. We first observe that $ch_{BB}(C_{2n+1}, C_{2n+1}) \geq 5$. We assign to each vertex the list $\{1, 2, 3, 4\}$. Suppose that (C_{2n+1}, C_{2n+1}) has a backbone L-coloring. Note that if a vertex has its color in $\{1, 2\}$, then neither of its neighbors has its color in $\{1, 2\}$. The same is true for $\{3, 4\}$. Thus, at most $\left\lfloor \frac{2n+1}{2} \right\rfloor$ vertices are colored with 1 or 2, and at most $\left\lfloor \frac{2n+1}{2} \right\rfloor$ vertices are colored with 3 or 4, a contradiction. Thus, $ch_{BB}(C_{2n+1}, C_{2n+1}) \geq 5$.

Assume that $C_{2n+1} = v_1v_2v_3 \dots v_{2n+1}v_1$. Let L be any 5-assignment of C_{2n+1}. By Lemma 3.15, we may assume further that there are $x \in L(v_1)$ and $p, q \in L(v_{2n+1})$ such that $|x - p| \geq 2$, $|x - q| \geq 2$, and $|p - q| \geq 3$. Lemma 3.16 yields a desired backbone coloring. So $ch_{BB}(C_{2n+1}, C_{2n+1}) \leq 5$. Therefore $ch_{BB}(C_{2n+1}, C_{2n+1}) = 5$. □

Theorem 3.18. $ch_{BB}(C_{2n}, C_{2n}) = 4$*.*

Proof. Assume that $C_{2n} = v_1v_2v_3 \dots v_{2n}v_1$. Let L be any 4-assignment of C_{2n}. We consider two cases.

CASE 1: Each vertex has the same list. We may assume that $L(v_i) = \{a, b, c, d\}$ where $a < b < c < d$ for all $i = 1, \dots, 2n$. Then $d - a \geq 2$. We obtain a desired backbone coloring by assigning a to v_{2j-1} and d to v_{2j} for all $j = 1, \dots, n$.

CASE 2: There are two adjacent vertices with different lists. Without loss of generality, we may assume that $L(v_1) \neq L(v_{2n})$. Let $M = \max(L(v_1) \cup L(v_{2n}))$ and $m = \min(L(v_1) \cup L(v_{2n}))$.

CASE 2.1: $M, m \in L(v_1) \cap L(v_{2n})$. Since $L(v_1) \neq L(v_{2n})$, there are $a, b, c \in L(v_1) \cup L(v_{2n})$ such that $m < a < b < c < M$. Without loss of generality, we may assume that $b \in L(v_1)$. Then $M - b \geq 2$, $b - m \geq 2$, and $M - m \geq 3$. By Lemma 3.16, there is a backbone L-coloring of (C_{2n}, C_{2n}).

CASE 2.2: $M \in L(v_1) \setminus L(v_{2n})$. We consider 2 subcases.

CASE 2.2.1: $m \in L(v_1)$. Assume that $L(v_1) = \{m, a_1, b_1, M\}$ and $L(v_{2n}) = \{a_2, b_2, c_2, d_2\}$ such that $m < a_1 < b_1 < M$ and $a_2 < b_2 < c_2 < d_2$. Then $m \le a_2 < b_2 < c_2 < d_2 < M$. We obtain $M - c_2 \ge 2$, $c_2 - m \ge 2$, and $M - m \ge 3$. By Lemma 3.16, there is a backbone L-coloring of (C_{2n}, C_{2n}).

CASE 2.2.2: $m \notin L(v_1)$. Assume that $L(v_1) = \{a_1, b_1, c_1, M\}$ and $L(v_{2n}) = \{m, a_2, b_2, c_2\}$ such that $a_1 < b_1 < c_1 < M$ and $m < a_2 < b_2 < c_2$. Then $a_1 \ge m + 1$ and $c_2 \le M - 1$. If $a_1 \ge m + 2$, then $a_1 - m \ge 2$, $M - m \ge 2$, and $M - a_1 \ge 3$. By Lemma 3.16, there is a backbone L-coloring of (C_{2n}, C_{2n}). If $c_2 \le M - 2$, then $M - c_2 \ge 2$, $M - m \ge 2$, and $c_2 - m \ge 3$. By Lemma 3.16, there is a backbone L-coloring of (C_{2n}, C_{2n}).

Let $a_1 = m + 1$ and $c_2 = M - 1$. If $M - b_1 \ge 3$, then $M - m \ge 2$ and $b_1 - m > b_1 - (m+1) = b_1 - a_1 \ge 1$. By Lemma 3.16, there is a backbone L-coloring of (C_{2n}, C_{2n}). If $b_2 - m \ge 3$, then $M - m \ge 2$ and $M - b_2 > (M - 1) - b_2 = c_2 - b_2 \ge 1$. By Lemma 3.16, there is a backbone L-coloring of (C_{2n}, C_{2n}). Suppose that $M - b_1 = 2$ and $b_2 - m = 2$. Hence $L(v_1) = \{m + 1, M - 2, M - 1, M\}$ and $L(v_{2n}) = \{m, m + 1, m + 2, M - 1\}$.

It remains only to consider the list L with the property that for each i, if $L(v_i) = \{a, b, c, d\}$, where $a < b < c < d$, then $L(v_{i+1 \bmod 2n})$ is one of $\{a, b, c, d\}$, $\{a-1, a, a+1, d-1\}$, and $\{a+1, d-1, d, d+1\}$. Now, assign $\min L(v_i)$ to v_i for each odd i and $\max L(v_i)$ to v_i for each even i. We show that this assignment is a backbone L-coloring of (C_{2n}, C_{2n}). Consider $v_i v_{i+1}$. If i is odd then we assign a to v_i. Three possible colors for v_{i+1} are $d - 1$, d, and $d + 1$. Since $d - a \ge 3$, we have that $(d - 1) - a \ge 2$ and $(d + 1) - a \ge 4$. If i is even then we assign d to v_i. Three possible colors for $v_{i+1 \bmod 2n}$ are $a - 1$, a, and $a + 1$. Since $d - a \ge 3$, we have that $d - (a + 1) \ge 2$ and $d - (a - 1) \ge 4$. Thus, we have a backbone L-coloring of (C_{2n}, C_{2n}).

CASE 2.3: $m \in L(v_1) \setminus L(v_{2n})$. This case is similar to *CASE 2.2*.

From both cases, we find that $ch_{BB}(C_{2n}, C_{2n}) \le 4$. By Proposition 3.1 and Lemma 3.2, we have that $ch_{BB}(C_{2n}, C_{2n}) \ge ch_{BB}(C_{2n}, P_{2n}) = 4$. Therefore $ch_{BB}(C_{2n}, C_{2n}) = 4$. □

3.4 Proof of Main Theorem

Now we are ready to prove the **Main Theorem**.

Proof. Obviously, $ch_{BB}(G, H) = 1$ when $G = H = P_1$.

Assume that $E(H) = \emptyset$ and G is not an odd cycle. We obtain that $ch_{BB}(G, H) = 2$ by Remarks 2.1(i, ii, v).

By Theorem 3.17, we find that $ch_{BB}(G, H) = 5$ when $G = H$ and G is an odd cycle.

Suppose that $G = H$ with G and H are even cycles or H contains a path with length at least 3 or G is a cycle and H is a disjoint union of paths with length at least 1 and at least one path has length 2 or (G, H) contains $(P_{2k+6}, P_3 \cup P_2^{[k]} \cup P_3)$

where $k \geq 0$. We obtain that $ch_{BB}(G, H) = 4$ by Theorems 3.3, 3.14, 3.18, and Corollary 3.7.

For the remaining cases, we have

(i) G is an odd cycle and $E(H) \neq \emptyset$ or

(ii) G is a cycle and H is a disjoint union of paths with length 1 or

(iii) each G_i is a path and H_i is a disjoint union of paths such that each path has length 1 except that possibly one path has length 2 where I is the set of all isolated vertices in H such that $I \neq \emptyset$ and $E(H - I) \neq \emptyset$ and $G - I = \bigcup_{i \in \Lambda} G_i$ where G_i is a component of $G - I$ and $H_i = G_i \cap H$.

By Remarks 2.1(ii, v), $ch_{BB}(C_{2n+1}, H) = ch(C_{2n+1}) = 3$ where $E(H) = \emptyset$. For (ii) and (iii), we apply Theorems 3.11, 3.14, and Lemma 3.13 to complete the proof. □

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