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Magic rectangle sets of odd order

Dalibor Froncek

Department of Mathematics and Statistics
University of Minnesota Duluth
Duluth, MN
U.S.A.
dalibor@d.umn.edu

To the memory of Dan Archdeacon

Abstract

A magic rectangle set $\mathcal{M} = \mathrm{MRS}(a, b; c)$ is a collection of c arrays $(a \times b)$ with entries $1, 2, \ldots, abc$, each appearing once, with all row sums in every rectangle equal to a constant ρ and all column sums in every rectangle equal to a constant σ. It was proved by the author [*AKCE Int. J. Graphs Comb.* 10 (2013), 119–127] that if an $\mathrm{MRS}(a, b; c)$ exists, then $a \equiv b \pmod 2$. It was also proved there that if $a \equiv b \equiv 0 \pmod 2$ and $b \geq 4$, then an $\mathrm{MRS}(a, b; c)$ exists for every c, and if $a \equiv b \equiv 1 \pmod 2$ and an $\mathrm{MRS}(a, b; c)$ exists, then $c \equiv 1 \pmod 2$. For a, b, c not all relatively prime, the existence of an $\mathrm{MRS}(a, b; c)$ follows from Hagedorn's construction of a 3-dimensional magic rectangle 3-MR(a, b, c) [T.R. Hagedorn, *Discrete Math.* 207 (1999), 53–63]. We prove that if $a \leq b$ and both a, b are odd, then an $\mathrm{MRS}(a, b; c)$ exists if and only if $3 \leq a$ and c is any odd positive integer. This completely settles the existence of magic rectangle sets.

1 Introduction

Magic squares are among the oldest known mathematical structures, having been studied for thousands of years. The first known magic square originates in the 4^{th} century BC. A *magic square* of order n is an $n \times n$ array with entries $1, 2, \ldots, n^2$, each appearing once, such that the sum of each row, column, and both main diagonals is equal to $n(n^2 + 1)/2$. For a comprehensive survey of magic squares, see Chapter 34 in [1]. Magic rectangles are a natural generalization of magic squares. An $a \times b$ *magic rectangle* is an $a \times b$ array with entries $1, 2, \ldots, ab$, each appearing once, such that the sum of each row is equal to $b(ab + 1)/2$ and the sum of each column is equal to $a(ab + 1)/2$. Finally, an n-dimensional magic rectangle is a natural generalization of magic rectangles to n dimensions. The author has introduced another, weaker generalization of magic rectangles, namely the magic rectangle sets [2, 3]. A *magic rectangle set* $\mathrm{MRS}(a, b; c)$ is a collection of c arrays $(a \times b)$ with entries $1, 2, \ldots, abc$, each appearing once, with all row sums in every rectangle equal to a constant ρ and all column sums in every rectangle equal to a constant σ.

The author proved in [2] that an $\mathrm{MRS}(a, b; c)$ can exist only when $a \equiv b \pmod 2$ (and $a, b > 1$) and showed that for a and b both even an $\mathrm{MRS}(a, b; c)$ exists for any positive integer c. For a, b odd it was shown in [2] that c must be odd as well. A partial solution of that case follows from Hagedorn's result on 3-dimensional magic rectangles [4].

Magic rectangle sets can be used to construct handicap distance antimagic graphs, which in turn are models of incomplete handicap tournaments (see [2, 3]). A *handicap incomplete tournament* of n teams with known strengths ranked $1, 2, \ldots, n$ is a tournament in which every team plays the same number of games and the sum of rankings of opponents of the strongest team, ranked number 1, is the lowest, while the sum of rankings of opponents of the weakest team, ranked number n, is the highest. In other words, a stronger team plays stronger opponents than a weaker team.

In this paper, we prove the existence of $\mathrm{MRS}(a, b; c)$ for all admissible triples of odd numbers a, b, c.

2 Definitions and known results

Definition 2.1 *A* magic rectangle $\mathrm{MR}(a, b)$ *is an* $a \times b$ *array whose entries are* $\{1, 2, \ldots, ab\}$, *each appearing once, with all its row sums equal to a constant* ρ *and all column sums equal to a constant* σ.

The sum of all entries in the array is $ab(ab + 1)/2$; it follows that

$$\rho = \sum_{j=1}^{b} m_{ij} = b(ab + 1)/2 \text{ for all } i$$

and

$$\sigma = \sum_{i=1}^{a} m_{ij} = a(ab + 1)/2 \text{ for all } j.$$

Hence a and b must have the same parity. An example is shown in Figure 1.

15	2	14	4	5
8	10	7	9	6
1	12	3	11	13

Figure 1: Magic rectangle $\mathrm{MR}(3, 5)$ with $\rho = 40$ and $\sigma = 24$

The following existence result was proved by Harmuth [5, 6] more than 130 year ago.

Theorem 2.2 [5, 6] *A magic rectangle* MR(a, b) *exists if and only if* $a, b > 1$, $ab > 4$, *and* $a \equiv b \pmod 2$.

Hagedorn introduced an n-dimensional version of magic rectangles in [4].

Definition 2.3 *An* n-dimensional magic rectangle n-MR$(a_1, a_2, \ldots, a_n)$ *is an* $a_1 \times a_2 \times \cdots \times a_n$ *array with entries* $d_{i_1,i_2,\ldots,i_n}$ *which are elements of* $\{1, 2, \ldots, a_1a_2 \ldots a_n\}$, *each appearing once, such that all sums in the* k*-th direction are equal to a constant* σ_k. *That is, for every* k, $1 \le k \le n$, *and every selection of indices* $i_1, i_2, \ldots, i_{k-1}$, $i_{k+1}, \ldots, i_n$, *we have*

$$\sum_{j=1}^{a_k} d_{i_1,i_2,\ldots,i_{k-1},j,i_{k+1},\ldots,i_n} = \sigma_k,$$

where $\sigma_k = a_k(a_1a_2 \ldots a_n + 1)/2$.

The following existence results were also proved by Hagedorn in [4].

Theorem 2.4 [4] *If there exists an* n*-dimensional magic rectangle* n-MR$(a_1, a_2, \ldots, a_n)$, *then* $a_1 \equiv a_2 \equiv \cdots \equiv a_n \pmod 2$.

For $a_1 \equiv a_2 \equiv \cdots \equiv a_n \equiv 0 \pmod 2$, Hagedorn found a complete existence characterization.

Theorem 2.5 [4] *An* n*-dimensional magic rectangle* n-MR$(a_1, a_2, \ldots, a_n)$ *with* $a_1 \le a_2 \le \cdots \le a_n$ *and all* a_i *even exists if and only if* $2 \le a_1$ *and* $4 \le a_2 \le \cdots \le a_n$.

For odd dimensions, only the following 3-dimensional result is known so far.

Theorem 2.6 [4] *A* 3*-dimensional magic rectangle* 3-MR(a_1, a_2, a_3) *with* $3 \le a_1 \le a_2 \le a_3$ *exists whenever* $\gcd(a_i, a_j) > 1$ *for some* $i, j \in \{1, 2, 3\}$.

The author introduced the notion of magic rectangle sets [2], which is a less restrictive generalization of magic rectangles. Magic rectangle sets can be viewed as 3-dimensional $a \times b \times c$ arrays in which each horizontal layer (an $a \times b$ rectangle) has all row and column sums equal, while there is no restriction on vertical column sums.

Such sets are useful tools for constructions of handicap tournaments as mentioned above. Hence, we want to know whether they can be found even for triples (a, b, c) for which 3-dimensional magic rectangles 3-MR(a, b, c) do not exist.

Definition 2.7 *A* magic rectangle set $\mathcal{M} = $ MRS$(a, b; c)$ *is a collection of* c *arrays* $(a \times b)$ *whose entries are elements of* $\{1, 2, \ldots, abc\}$, *each appearing once, with all row sums in every rectangle equal to a constant* $\rho = b(abc+1)/2$ *and all column sums in every rectangle equal to a constant* $\sigma = a(abc + 1)/2$.

The existence of magic rectangle sets for a, b both even was completely settled by the author in [2].

Theorem 2.8 [2] *If* $a \equiv b \equiv 0 \pmod 2$ *and* $b \geq 4$, *then a magic rectangle set* $\mathrm{MRS}(a, b; c)$ *exists for every* c.

Notice that while a 3-dimensional magic rectangle 3-MR(a_1, a_2, a_3) forms a set $\mathcal{M} = \mathrm{MRS}(a_i, a_j; a_k)$ for every permutation of indices $1, 2, 3$, it does not exist when a_1, a_2, a_3 do not all have the same parity.

While for a and b both even the number of rectangles in a magic rectangle set $\mathrm{MRS}(a, b; c)$ can be either even or odd, for odd values of a or b the number c must be odd as well. This follows from an easy observation, which was also made in [2].

Theorem 2.9 [2] *If* a *or* b *is odd and* abc *is even, then the magic rectangle set* $\mathrm{MRS}(a, b; c)$ *does not exist.*

In the following section, we prove the existence of such sets with a, b, c odd, even for the cases that do not follow from Hagedorn's result on 3-dimensional magic rectangles.

In the proof we use the notion of Kotzig arrays, which share some properties of Latin and magic rectangles.

Definition 2.10 *A* Kotzig array $\mathrm{KA}(a, c)$ *is an* $a \times c$ *array with entries from the set* $\{1, 2, \ldots, c\}$, *each of them appearing exactly once in every row, such that the sum of every column is equal to* $a(c + 1)/2$.

An example of a 3×7 Kotzig array KA$(3, 7)$ is shown in Figure 2.

1	2	3	4	5	6	7
7	5	3	1	6	4	2
4	5	6	7	1	2	3

Figure 2: Kotzig array KA$(3, 7)$

Kotzig arrays are known to exist except when a is odd and c is even (see [7]).

Theorem 2.11 [7] *An* $a \times c$ *Kotzig array* $\mathrm{KA}(a, c)$ *exists if and only if* $a > 1$ *and* $a(c - 1)$ *is even.*

It is obvious that the sum of every row is equal to $c(c+1)/2$ and therefore KA(a, c) has the "magic-like" property with respect to the sums of rows and columns, but at the same time allows entry repetitions in columns.

3 New results

Now we prove that magic rectangle sets with a, b, c all odd and both a, b greater than one always exist.

Theorem 3.1 *Let a, b, c be positive odd integers such that $1 < a \leq b$. Then a magic rectangle set* MRS$(a, b; c)$ *exists.*

Proof: We construct an MRS$(a, b; c)$ with rectangles $W^1, W^2, \ldots, W^c$ in several steps. First we take a magic rectangle MR(a, b) with entries m_{ij}, row sums ρ and column sums σ. Then we create an $a \times b$ *underlying rectangle* $U(a, b)$ with entries u_{ij} defined as $u_{ij} = c(m_{ij} - 1)$. The sum of each row is then

$$\rho_U = \sum_{j=1}^{b} c(m_{ij} - 1) = cb(ab - 1)/2 \text{ for all } i$$

and the sum of each column is

$$\sigma_U = \sum_{i=1}^{a} c(m_{ij} - 1) = ca(ab - 1)/2 \text{ for all } j.$$

We present several figures to illustrate an example of how a magic rectangle set MRS$(3, 5; 7)$ is constructed. An example of an underlying rectangle $U(3, 5)$ is shown in Figure 3. It is based on the magic rectangle MR$(3, 5)$ presented in Figure 1.

105	14	98	28	35
56	70	49	63	42
7	84	21	77	91

Figure 3: Underlying rectangle $U(3, 5)$ with $\rho_U = 280$ and $\sigma_U = 168$

Now we construct an $a \times c$ Kotzig array KA(a, c) with entries k_{ij}. Based on this Kotzig array, we build c different $a \times b$ *residual rectangles* $R^s(a, b)$ with entries r^s_{ij} for $1 \leq s \leq c$ as follows. In the first column of a given $R^s(a, b)$, we place the s-th column of KA(a, c). That is, $r^s_{i1} = k_{is}$. In the following $a - 1$ columns, we place a circulant array constructed from the first column. More formally, we have $r^s_{ij} = r^s_{i+j-1\ 1}$ for $j = 2, 3, \ldots, s$ with the addition in the first subscript performed modulo a. We observe that the first a columns of $R^s(a, b)$ form a Latin square-like array with the entries of $LS(a)$ replaced by entries of the s-th column of KA(a, c). Notice that unlike in a Latin square, an entry can appear repeatedly in each of the first a columns, when the s-th column of KA(a, c) contained repeated entries. It should be obvious that the sum of every column (denoted σ_R) and every partial sum of the first a entries of each row is equal to $a(c + 1)/2$ in each $R^s(a, b)$. When $b > a$, the remaining $b - a$ columns will be filled as follows. All even columns will be the same as the first column, while the odd columns will be filled with the complements of the entries in the previous column with respect to $c+1$. Formally, $r^s_{ij} = r^s_{i1}$ when j is even and $j > a$, and $r^s_{ij} = c+1-r^s_{i1}$ when j is odd and $j > a$. Then the sum of every even column is again $a(c+1)/2 = \sigma_R$ and for the odd columns we have the sum $a(c+1) - a(c+1)/2 = a(c+1)/2 = \sigma_R$ as well. The sum of each row is now

$$\rho_R = \sum_{j=1}^{c} r^s_{ij} = a(c+1)/2 + (b-a)(c+1)/2 = b(c+1)/2.$$

First three residual rectangles $R^1(3,5), R^2(3,5), R^3(3,5)$ based on the Kotzig array KA$(3,7)$ presented in Figure 2 are shown in Figure 4.

1	7	4	1	7
7	4	1	7	1
4	1	7	4	4

2	5	5	2	6
5	5	2	5	3
5	2	5	5	3

3	3	6	3	5
3	6	3	3	5
6	3	3	6	2

Figure 4: Residual rectangles $R^1(3,5)$, $R^2(3,5)$, $R^3(3,5)$

Each rectangle W^s with entries w^s_{ij} in our MRS$(a,b;c)$ is now constructed by adding the entries of the residual rectangle $R^s(a,b)$ to the entries of the underlying rectangle $U(a,b)$, that is, $w^s_{ij} = u_{ij} + r^s_{ij}$ for $1 \le i \le a, 1 \le j \le b, 1 \le s \le c$. The row and column sums are

$$\rho = \rho_U + \rho_R = bc(ab-1)/2 + b(c+1)/2 = (ab^2c - bc + bc + b)/2 = b(abc+1)/2$$

and

$$\sigma = \sigma_U + \sigma_R = ac(ab-1)/2 + a(c+1)/2 = (a^2bc - ac + ac + a)/2 = a(abc+1)/2$$

as desired.

First three rectangles W^1, W^2, W^3 of a magic rectangle set MRS$(3,5;7)$ are shown in Figure 5.

106	21	102	29	42
63	74	50	70	43
11	85	28	81	95

107	19	103	30	41
61	75	51	68	45
12	86	26	82	94

108	17	104	31	40
59	76	52	66	47
13	87	24	83	93

Figure 5: Rectangles W^1, W^2, W^3 of MRS$(3,5;7)$

We still need to verify that every number $1, 2, \ldots, abc$ appears exactly once among the entries of our set $W^1, W^2, \ldots, W^c$. First we observe that for any pairs $(i,j) \neq (i',j')$ we have $|u_{ij} - u_{i'j'}| \ge c$ and $|r^s_{ij} - r^t_{i'j'}| \le c-1$. Therefore, if $w^s_{ij} = w^t_{i'j'}$, we must have $(i,j) = (i',j')$. Suppose that $w^s_{ij} = w^t_{ij}$ and $s \neq t$. Then we have $w^s_{ij} - w^t_{ij} = 0$ and

$$w^s_{ij} - w^t_{ij} = (u_{ij} + r^s_{ij}) - (u_{ij} + r^t_{ij}) = r^s_{ij} - r^t_{ij} = 0.$$

Hence $r^s_{ij} = r^t_{ij} = r$ and because both entries are in the same row, they both correspond to the same entry in the first column. That is, if $j = 1$ or $j > a$ and j is even, then $r = r^s_{i1} = r^t_{i1}$. Recall that r^s_{i1} is an entry of the Kotzig array KA(a, b), namely k_{is}, while $r^t_{i1} = k_{it}$. Because $s \neq t$, we have two identical entries in the i-th row of KA(a, b), which is impossible.

The case of $j > a$ and j odd is very similar. We have $r = c + 1 - k_{is} = c + 1 - k_{it}$, implying $k_{is} = k_{it}$, a contradiction. Finally, if $2 \leq j \leq a$, then $r = r^s_{l1} = r^t_{l1}$ for some $l \neq i$ and we have $k_{ls} = k_{lt}$, the same contradiction as above. This completes the proof. □

As we mentioned in Section 2, the case when $a \equiv b \equiv 0 \pmod 2$ was completely solved in [2], similarly as the non-existence of certain classes of magic rectangle squares with at least one of a, b odd. Together with Theorem 2.8 we now get a complete solution, summarized in the following theorem. The assertion follows directly from Theorems 2.8, 2.9, and 3.1.

Theorem 3.2 *Let a, b, c be positive integers such that $1 < a \leq b$. Then a magic rectangle set* MRS$(a, b; c)$ *exists if and only if either a, b, c are all odd, or a and b are both even, c is arbitrary, and $(a, b) \neq (2, 2)$.*

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