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A note on extending Bondy's meta-conjecture

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Abstract

A graph of order $n \geq 3$ is said to be *pancyclic* if it contains a cycle of each length from 3 to n. A *chord* of a cycle is an edge between two nonadjacent vertices of the cycle. A *chorded cycle* is a cycle containing at least one chord. We define a graph of order $n \geq 4$ to be *chorded pancyclic* if it contains a chorded cycle of each length from 4 to n. In this article, we prove the following: If G is a graph of order $n \geq 4$ with $\deg_G(x) + \deg_G(y) \geq n$ for each pair of nonadjacent vertices x, y in G, then G is chorded pancyclic, or $G = K_{n/2,n/2}$, or G is one particular small order exception. We also show this result is sharp, both in terms of the degree sum condition and in terms of the number of chords we can guarantee exist per cycle. We further extend Bondy's meta-conjecture on pancyclic graphs to a meta-conjecture on chorded pancyclic graphs.

1 Introduction

We consider only simple graphs in this paper. For terms not defined here see [3]. The problem of determining conditions that imply the existence of a particular structure within a graph is fundamental in graph theory. One common approach to such problems is to control the degrees of the vertices of the graph in some way. The minimum degree sum of all pairs of nonadjacent vertices is denoted by $\sigma_2(G)$ (see for example [4], [5]). One important early question was to determine if a graph was *Hamiltonian*, that is, contained a cycle spanning the vertex set. This question spurred hundreds of papers, but one of the earliest and most fundamental results was the following theorem of Ore [6].

Theorem 1. (*Ore,* [6]) *If G is a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$, then G is Hamiltonian.*

A stronger property is that of being *pancyclic*, that is, containing cycles of all lengths from 3 to $|V(G)| = n$. In 1971, Bondy (see [1], [2]) proposed his now famed meta-conjecture: "Almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic." Bondy further allowed "There may be a simple family of exceptional graphs". He provided several results to support his meta-conjecture including the following extension of Ore's theorem. Here the complete bipartite graphs $K_{n/2,n/2}$ form the family of exceptions, as they contain only even length cycles.

Theorem 2. (*Bondy,* [1]) *If G is a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$, then G is pancyclic or $G = K_{n/2,n/2}$.*

In this paper, we wish to extend Bondy's meta-conjecture. To do so, we need the following idea. A *chord* of a cycle is an edge between two nonadjacent vertices of the cycle. A *chorded cycle* is a cycle containing at least one chord. We define a graph of order $n \geq 4$ to be *chorded pancyclic* if it contains a chorded cycle of each length 4 to n. A cycle of length k is called a *k-cycle.* Note that by default, a chorded 4-cycle contains a 3-cycle as a subgraph, so the graph is pancyclic. Also note that there are graphs that are pancyclic that are not chorded pancyclic (see Figure 1). We will also need the graph G_6 of Figure 2 which is not chorded pancyclic.

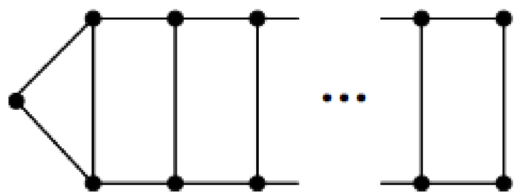

Figure 1: An infinite class of pancyclic but not chorded pancyclic graphs.

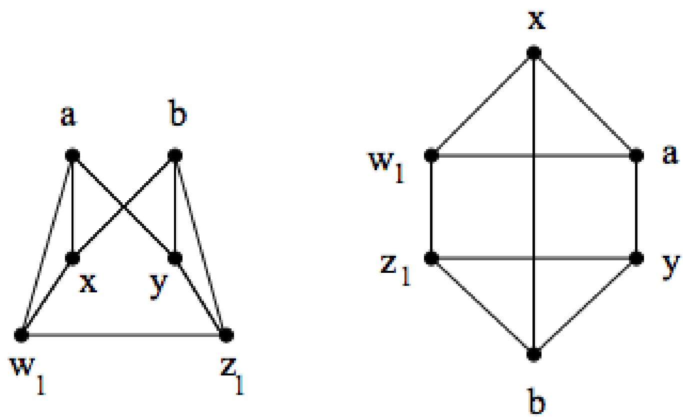

Figure 2: The graph G_6 shown two ways.

Our extension of Bondy's meta-conjecture is the following: Almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is chorded pancyclic. There may be a simple family of exceptional graphs as well as a finite number of small order exceptions. As support for our extension, we prove the following extension of Theorems 1 and 2. Here the complete bipartite graphs are again a simple family of exceptional graphs and G_6 is a small order exception.

Theorem 3. *Let G be a graph of order $n \geq 4$. If $\sigma_2(G) \geq n$, then G is chorded pancyclic, or $G = K_{n/2,n/2}$, or $G = G_6$.*

We note that the graph $G = 2K_{(n-1)/2} + K_1$ serves as a sharpness example for the degree condition of Theorems 1, 2, and 3, as G is not Hamiltonian. We also note that the graph H which can be obtained from $K_{n/2,n/2}$ by adding one edge in one of the partite sets, satisfies the conditions of Theorem 3, and H is chorded pancyclic, but we cannot ask for more than one chord per cycle, as all of the 4-cycles in H contain at most one chord.

We denote the set $N_G(v) = \{x \in V(G) \mid vx \in E(G)\}$, called the *neighborhood* of the vertex v in a graph G, and $\deg_G(v) = |N_G(v)|$. If H and S are subsets of $V(G)$, then we denote by $N_H(S)$ the set of vertices in H which are adjacent to some vertex in S. In particular, if $S = \{v\}$, then we denote $N_H(v) = N_H(\{v\})$. We also use the common notation that $\overline{A}$ is the complement of the set A.

2 Proof of Theorem 3

In this section, let G, n, and G_6 be as described in Theorem 3. Note that by Ore's theorem, G must be Hamiltonian. If $n = 4$, then either $G = K_{2,2}$ or it is a 4-cycle with chords and is then chorded pancyclic. If $n = 5$, then the degree sum condition forces at least two independent chords into the Hamiltonian cycle and it is easy to see the graph must be chorded pancyclic. Thus, we now assume $n \geq 6$. By Bondy's theorem, G is either pancyclic or $K_{n/2,n/2}$. Suppose $G \neq K_{n/2,n/2}$ and $G \neq G_6$. Let x and y be a pair of nonadjacent vertices in $V(G)$ with the smallest number of common neighbors. Partition $V(G) - \{x, y\}$ as follows:

$$\begin{aligned} M &= N_G(x) \cap N_G(y), \\ X &= N_G(x) \cap \overline{N_G(y)}, \\ Y &= N_G(y) \cap \overline{N_G(x)}, \\ D &= \overline{N_G(x)} \cap \overline{N_G(y)}. \end{aligned}$$

Note that the degree sum condition implies $|M| \geq 2$. Let $|M| = 2 + r$, $r \geq 0$.

Claim 1. $|D| \leq r$.

Proof. Suppose not, say $|D| = r + t$ for some $t > 0$. Considering the degree sum of the nonadjacent pair of vertices x and y, we have

$$n \leq \sigma_2(G) \leq \deg_G(x) + \deg_G(y) \leq (n-2) - (r+t) + (2+r) = n - t,$$

a contradiction. □

Claim 2. *There exists a chorded n-cycle in G.*

Proof. We may assume that $n \geq 6$ by the above observations. Since G contains a Hamiltonian cycle, say C, it is easy to see that C is a chorded n-cycle by the degree sum condition. □

Claim 3. *There exists a chorded* 4*-cycle in G.*

Proof. Suppose the claim fails to hold. Since $|M| \geq 2$, consider $a, b \in M$. If $ab \in E(G)$, then a, y, b, x, a is a 4-cycle with chord ab, a contradiction. Thus, $ab \notin E(G)$. This implies M is an independent set. By the choice of x and y, $|N_G(a) \cap N_G(b)| \geq 2 + r$. Let $w \in N_G(a) \cap N_G(b)$. If $w \in M$, then a, w, b, x, a is a 4-cycle with chord xw, a contradiction. Hence, $w \notin M$. If $w \in X$, then a, w, b, x, a is a 4-cycle with chord xw, a contradiction. Hence, $w \notin X$ and by symmetry, $w \notin Y$. Therefore, $w \in \{x, y\} \cup D$ and all common neighbors of a and b must be in $\{x, y\} \cup D$. By Claim 1, we obtain $|D| = r$. Then note that $N_D(M) = D$, and if $D \neq \emptyset$, then D is an independent set, otherwise, when $|D| \geq 2$, there exists a chorded 4-cycle in the subgraph induced by $M \cup D$.

If $X = \emptyset = Y$, then $G = K_{2+r,2+r} = K_{n/2,n/2}$, a contradiction. Thus, we may assume $X \cup Y \neq \emptyset$ and without loss of generality, that $|X| \geq |Y|$. For the nonadjacent pair a and b and their possible adjacencies to $\{x, y\}$, X, Y and D, we have

$$\begin{aligned} |M| + |\{x, y\}| + |X| + |Y| + |D| = n \leq \sigma_2(G) &\leq \deg_G(a) + \deg_G(b) \\ &\leq 2(|\{x, y\}| + |D|) + |N_{X \cup Y}(a)| + |N_{X \cup Y}(b)|. \end{aligned}$$

Since $|M| = 2 + r$ and $|D| = r$,

$$(2 + r) + 2 + |X| + |Y| + r \leq 2(2 + r) + |N_{X\cup Y}(a)| + |N_{X\cup Y}(b)|,$$

and therefore,

$$|X| + |Y| \leq |N_{X\cup Y}(a)| + |N_{X\cup Y}(b)|. \tag{1}$$

Since common neighbors of a and b are not contained in $X \cup Y$, it follows from (1) that $X \cup Y$ is dominated by a and b.

Let $w_1 \in X$ and without loss of generality, assume $aw_1 \in E(G)$. Note that w_1 cannot be adjacent to any other vertex in X, otherwise, say $w_1v \in E(G)$ for any $v \in X - \{w_1\}$, then w_1, v, x, a, w_1 is a 4-cycle with chord xw_1, a contradiction. Also note that by the definition of X, $w_1y \notin E(G)$. Also, for any $t \in M - \{a\}$, $w_1t \notin E(G)$ or again a chorded 4-cycle would exist. Thus, $N_G(w_1) \subseteq \{a, x\} \cup Y \cup D$.

If $Y = \emptyset$, then for the nonadjacent pair w_1 and y,

$$|M| + |\{x, y\}| + |X| + |D| = n \leq \sigma_2(G) \leq \deg_G(w_1) + \deg_G(y) \leq |\{a, x\}| + |D| + |M|$$

and hence, $|X| \leq 0$, a contradiction. Therefore, $Y \neq \emptyset$. If $N_Y(w_1) = \emptyset$, then similarly, $|X| \leq 0$, again a contradiction. Therefore, $N_Y(w_1) \neq \emptyset$. Let $z_1 \in Y$ and let $w_1z_1 \in E(G)$. Since $X \cup Y$ is dominated by a and b, we have that $bz_1 \in E(G)$, as otherwise, if $az_1 \in E(G)$, then a, y, z_1, w_1, a is a 4-cycle with chord az_1, a contradiction.

We now claim that $|M| = 2$. Suppose this is not the case and let $v \in M - \{a, b\}$. Since M is independent, $av \notin E(G)$. By the same argument as before, $X \cup Y$ is dominated by a and v. Now as az_1 is not an edge, then $vz_1 \in E(G)$. Then v, y, b, z_1, v is a 4-cycle with chord yz_1, a contradiction. Hence, $|M| = 2$. Now by Claim 1, $D = \emptyset$.

We note that $|N_X(u)| \leq 1$ and $|N_Y(u)| \leq 1$ for any $u \in \{a, b\}$, otherwise, there would exist a chorded 4-cycle, a contradiction. If $|X| \geq 3$, then since $X \cup Y$ is dominated by a and b, one of a and b would have at least two adjacencies in X, a contradiction. Hence, $|X| \leq 2$ and similarly, $|Y| \leq 2$.

If $|X \cup Y| = 2$, then $G = G_6$, (see Figure 2) a contradiction. Thus, suppose that $|X \cup Y| \geq 3$. Then, by $|X| \geq |Y|$ which is our previous assumption, $|X| = 2$. Let $w_2 \in X - \{w_1\}$. Then note $bw_2 \in E(G)$ since $aw_2 \notin E(G)$. Suppose $|Y| = 1$. Now $n = 7$. Consider the nonadjacent pair y and w_1. By the degree sum condition, $\deg_G(y) + \deg_G(w_1) \geq n = 7$. On the other hand, since $\deg_G(y) = 3$ and $\deg_G(w_1) = 3$, we have $\deg_G(y) + \deg_G(w_1) = 6$, a contradiction. Thus, $|Y| = 2$. Now, $n = 8$. Let $z_2 \in Y - \{z_1\}$. Then $az_2 \in E(G)$ since $bz_2 \notin E(G)$. If $w_1z_2 \in E(G)$, then a, z_2, w_1, x, a is a 4-cycle with chord aw_1, a contradiction. Hence, $w_1z_2 \notin E(G)$. By the degree sum condition, $\deg_G(y) + \deg_G(w_1) \geq n = 8$. On the other hand, since $\deg_G(y) = 4$ and $\deg_G(w_1) = 3$, we have $\deg_G(y) + \deg_G(w_1) = 7$, a contradiction. This completes the proof of Claim 3. □

Claim 4. *If G contains a chorded 4-cycle, then there exists a chorded 5-cycle in G.*

Proof. Suppose $C = v_1, v_2, v_3, v_4, v_1$ is a 4-cycle in G with chord v_2v_4. Since $n \geq 6$ and G is connected by $\sigma_2(G) \geq n$, there is some $x \in V(G) - V(C)$ such that $xv \in E(G)$ for some $v \in V(C)$. We will consider two cases based on the adjacency of x.

Case I: Suppose $xv_1 \in E(G)$ (or by symmetry, $xv_3 \in E(G)$). If x is adjacent to any other vertex in C, then there exists a chorded 5-cycle. Thus, x is not adjacent to any other vertex in C. Since x and v_2 are nonadjacent, they must share at least two common neighbors, and the common neighbors except v_1 must be off of C. Let $y \in V(G) - V(C) - \{x\}$ be such a common neighbor. Then v_1, x, y, v_2, v_4, v_1 is a 5-cycle with chord v_1v_2.

Case II: Suppose $xv_2 \in E(G)$ (or by symmetry, $xv_4 \in E(G)$). If x is adjacent to v_1 or v_3, then there exists a chorded 5-cycle. Thus, $xv_1 \notin E(G)$ and $xv_3 \notin E(G)$. Since x and v_1 are nonadjacent, they must share at least two common neighbors, and let y be such a common neighbor except v_2. If $y \in V(G) - V(C) - \{x\}$, then there exists a chorded 5-cycle. This implies $y = v_4$ and then $xv_4 \in E(G)$. If $v_1v_3 \in E(G)$, then we easily find a chorded 5-cycle, so we may assume $v_1v_3 \notin E(G)$. Based on the degree sum condition, there exists some $y \in V(G) - V(C) - \{x\}$ such that $yv \in E(G)$ for some $v \in \{v_1, v_3\}$. Without loss of generality, suppose $yv_1 \in E(G)$. Then we are in a case analogous to Case I, and we have completed the proof of Claim 4. □

If $n = 6$, $G \neq K_{3,3}$ and $G \neq G_6$, then G is chorded pancyclic by Claims 2, 3, and 4. Thus, we may assume $n \geq 7$.

Claim 5. *The graph G contains a chorded k-cycle for all $6 \leq k \leq n-1$.*

Proof. Recall that since $G \neq K_{n/2,n/2}$, G is pancyclic. Let $6 \leq k \leq n-1$ and consider a chordless k-cycle $C = v_1, v_2, \ldots, v_k, v_1$ in G. Since C is chordless, v_1 and v_3 are nonadjacent and therefore, they must have a common neighbor in $V(G) - V(C)$, say x. Similarly, v_2 and v_6 are nonadjacent and they must have a common neighbor in $V(G) - V(C)$, say y. If $k = n-1$, then since $x = y$, $v_1, x, v_3, v_4, \ldots, v_k, v_1$ is a k-cycle with chord xv_6. Suppose $6 \leq k \leq n-2$. If $x = y$, then there exists a chorded k-cycle as above. If $x \neq y$, then $v_1, x, v_3, v_2, y, v_6, \ldots, v_k, v_1$ is a k-cycle with chord v_1v_2. □

Claims 2, 3, 4 and 5 imply that G is chorded pancyclic. This completes the proof of Theorem 3. □

3 Conclusion

We end by making a specific conjecture concerning chorded pancyclic graphs.

Conjecture 1. *Let G be a Hamiltonian graph of order $n \geq 4$. If $|E(G)| \geq n^2/4$, then G is chorded pancyclic, or $G = K_{n/2,n/2}$, or $G = G_6$.*

If true, this would extend another result of Bondy's [1].

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