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MaxDDBS problem on butterfly networks

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Abstract

A maximum degree-diameter bounded subgraph problem can be seen as a degree-diameter problem restricted to certain host graphs. In this paper, we investigate the MaxDDBS problem when the host graph is a butterfly network. We give constructive lower bounds for subgraphs of maximum degree 4, 3 and 2.

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1 Introduction

A maximum degree-diameter bounded subgraph (MaxDDBS) problem is a generalisation of the degree diameter problem (DDP) in which we aim to maximise the number of vertices in a graph with given degree and diameter. In the DDP we can add any number of vertices and edges to a graph as long as we satisfy the degree and diameter constraints. In MaxDDBS our selection of edges is restricted by the requirement that the resulting graph is a subgraph of some given host architecture. In this way DDP may be thought of as MaxDDBS with the host architecture being a complete graph. For more detail information regarding the degree-diameter problem, see [7].

Let $G = (V, E)$ be a host graph, an undirected graph without loops and multiple edges. The *degree* of a vertex v in G is the number of edges incident to v. The largest degree among all vertices in G is denoted by Δ. The *distance* between any two vertices u and v in G is defined to be the length of the shortest path connecting them. The *diameter* of G is the largest distance between any two vertices in G.

The maximum degree-diameter bounded subgraph problem is stated as follows: *Given a connected undirected host graph G and positive integers Δ, D, find the largest connected subgraph S with maximum degree at most Δ and diameter at most D.*

Let $N_G(\Delta, D)$ denote the order of a largest subgraph of G with given maximum degree Δ and given diameter D. The aim of this problem is to determine $N_G(\Delta, D)$.

This problem was first introduced by Dekker et al. [1] in 2012. In their paper, they discuss various practical applications; for example, in order to perform an efficient computation in parallel and distributed processing, the existence of a sub-network of bounded degree and diameter within the parallel architecture might optimise the communication time. They also gave a heuristic approximation algorithm to solve MaxDDBS since it is computationally hard. In fact, this problem is known to be NP-hard since it contains other well-known NP-hard problem as subproblems.

In the same paper, Dekker et al. gave bounds for the order of the largest subgraphs in some host graphs of practical interest, such as the mesh and the hypercube. This approach to MaxDDBS problem leads to more results considering other host graphs G. The result in [1] is later improved in [6] and [2]. Besides the mesh, MaxDDBS has also been studied on honeycomb networks [3] and triangular networks [4].

Another interesting network is the bounded-degree derivative of hypercube networks, called a butterfly network. In an interconnection network, modelled as a graph, vertices represent processing units and edges signify a direct line of communication between two processors. With the rise of parallel processing, architectures such as the butterfly network took on importance in terms of communication between processors and between processors and memory. The butterfly network displays efficiency with Fast Fourier Transforms (FFT) by employing a "divide and conquer" approach. This structure is ideal for breaking up discrete Fourier transforms (DFTs) into subtransforms as well as integrating the smaller structures into a larger DFT. Butterfly networks have been used in parallel computing systems such as IBM, SP1/SP2, MIT Transit Project, and NEC Cenju-3 [8].

Formally, a set of vertices V of an r-dimensional butterfly $BF(r)$ [5] corresponds to a set of pairs $\langle w, i\rangle$ where w is an r-bit binary number, written as $w_r \dots w_1, w_j \in \{0,1\}, j = 1,\dots,r$ and i is the *level* of a vertex $(0 \le i \le r)$. Two vertices $w = \langle w_r w_{r-1} \dots w_1, i\rangle$ and $w' = \langle w'_r w'_{r-1} \dots w'_1, i'\rangle$ are connected by an edge if and only if $i' = i + 1$ and one of these two conditions holds:

Type 1. w and w' are identical; or

Type 2. w and w' differ at precisely the i'th bit.

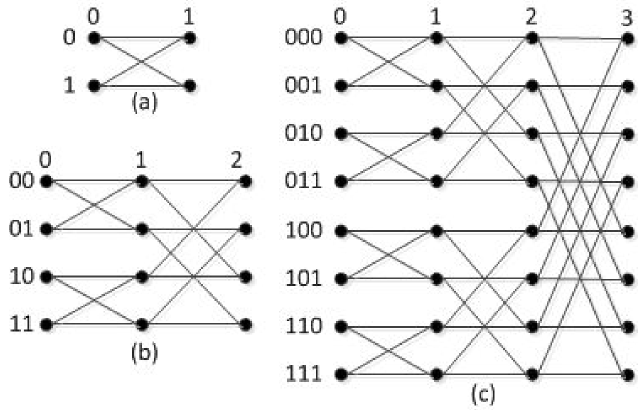

Figure 1: Butterfly network of dimension (a) 1; (b) 2; and (c) 3.

Let $N_{BF(r)}(\Delta, D)$ denote the order of a largest subgraph of $BF(r)$ with given maximum degree Δ and given diameter D. Since the maximum degree of a butterfly network is 4, we can consider subgraphs of maximum degree $\Delta = 2, 3, 4$. The case $\Delta = 1$ is trivial, being K_2.

In the following section we further consider butterfly subgraphs when the degree condition is also constrained.

2 Subgraphs of maximum degree $\Delta = 4$

An r-dimensional butterfly network is a graph with maximum degree 4 (except when $r = 1$) on $(r + 1) \times 2^r$ vertices and diameter $2r$. It is not difficult to check that combining the vertex sets $\{\langle w, i\rangle | i \in [r]\}$ into a single vertex results in a hypercube.

Lemma 2.1. *Let $BF(r)$ be an r-dimensional butterfly network; then there exists a path of length r from any vertex at level 0 to any vertex at level r.*

Proof. Take any $w = \langle w_r w_{r-1} \dots w_1, 0\rangle$ and $w' = \langle w'_r w'_{r-1} \dots w'_1, r\rangle$, vertices of levels 0 and r, respectively. Note that $w_i, w'_i \in \{0,1\}, i = 1,\dots,r$. To prove the lemma, we need to find a path of length r between w and w'. From the definition of a butterfly network, there are two types of edges connecting vertices of level i to level

$i+1$, defined as type 1 and type 2. We construct a path starting from w with the following pattern, for $i = 1, \ldots, r$:

- if $w_i = w'_i$ take edge of type 1 ; or
- if $w_i \neq w'_i$ take the edge of type 2.

The existence of those edges is guaranteed by the definition and clearly this path has length r. □

Subgraphs of maximum degree $\Delta = 4$ for even and odd diameter will be treated differently. Theorem 2.2 gives the lower bound for the number of vertices in a subgraph with maximum degree 4 and even diameter.

Theorem 2.2. *Let $D = 2t$. For any $r \geq t$, $N_{BF(r)}(4, D) \geq (t+1) \times 2^t$.*

Proof. $BF(r)$ has diameter $2r$, so when $r = t$ then $N_{BF(r)}(4, D) = |BF(t)| = (t+1) \times 2^t$. When $r > t$, clearly $BF(t) \subset BF(r)$, so $N_{BF(r)}(4, D) \geq |BF(t)| = (t+1) \times 2^t$. □

Theorem 2.3 gives the lower bound for the number of vertices in a subgraph with maximum degree 4 and odd diameter.

Theorem 2.3. *Let $D = 2t - 1$. In an r-dimensional butterfly network,*

1. *$N_{BF(r)}(4, D) \geq (t+1) \times 2^{t-1}$ if $r = t$;*
2. *$N_{BF(r)}(4, D) \geq (t+2) \times 2^{t-1}$ if $r > t$.*

Proof. We need to construct a subgraph with diameter $D = 2t - 1$.

Case 1. $r = t$

Define

$$\begin{aligned} V_1 &= \{\langle w, 0\rangle : w \text{ is } r-\text{bit binary number starting with } 0\} \\ &= \{\langle 0w_{r-1} \ldots w_1, 0\rangle : w_j \in \{0,1\}, j = 1, \ldots r-1\}; \\ V_2 &= \{\langle w, i\rangle : w \text{ is } r-\text{bit binary number ending with } 0, i = 1, \ldots, t\} \\ &= \{\langle w_r \ldots w_2 0, i\rangle : w_j \in \{0,1\}, j = 2, \ldots r, i = 1, \ldots, t\}. \end{aligned}$$

By simple counting, we have $|V_1| = 2^{t-1}$ and $|V_2| = t \times 2^{t-1}$. Define H to be a subgraph induced by the vertex set $V_1 \cup V_2$. The subgraph H has $|V_1| + |V_2| = (t+1) \times 2^{t-1}$ vertices.

Claim: Graph H has diameter $2t - 1$.

To check the diameter of the graph H, we need to check the pairwise distance among all vertices. However, this can be simplified into three cases:

- Vertices in V_2.
  The subgraph induced by V_2 is isomorphic to $BF(t-1)$; thus it has diameter $2t - 2$.

- Vertices in V_1 and V_2.
  Each vertex in V_1 has distance 1 to the set V_2 (each vertex in V_1 is adjacent to one vertex in V_2), so the distances between vertices from V_1 to any vertex in V_2 is at most $2t-1$.

- Vertices in V_1.
  Let $V_2' \subset V_2$ be defined as follows:

$$\begin{aligned} V_2' &:= \{\langle w, i\rangle : w \text{ is an } r\text{-bit binary number that starts and ends with } 0, \\ &\qquad i = 1, \dots, r-1\} \\ &= \{\langle 0w_{r-1} \dots w_2 0, i\rangle : i = 1, \dots, r-1\}. \end{aligned}$$

  The subgraph induced by V_2' is isomorphic to $BF(t-2)$ and all vertices in V_1 have distance 1 to V_2' by definition. Take $u, v \in V_1$. Vertices u and v have distance 1 to the set V_2'. Suppose u is adjacent to x and v is adjacent to y, where $x, y \in V_2'$ (note that x, y can be the same vertex). The distance $d(x, y) \le 2(t-2)$ since $\langle V_2' \rangle$ is isomorphic to $BF(t-2)$. Therefore, $d(u, v) \le 1+2t-4+1 = 2t-2$.

From these three cases, the greatest distance between any two vertices, and hence the diameter, is $2t-1$.

Case 2. $r > t$
We will consider the case $r = t+1$ and construct a subgraph of diameter $2t-1$ in $BF(t+1)$. For $r > t+1$, we simply embed this subgraph in $BF(r)$.

Define

$$\begin{aligned} V_1 &= \{\langle w, 0\rangle : w \text{ is an } r\text{-bit binary number that starts with } 00\} \\ &= \{\langle 00w_{r-2} \dots w_1, 0\rangle : w_j \in \{0,1\}, j = 1, \dots r-2\}; \\ V_2 &= \{\langle w, i\rangle : w \text{ is an } r\text{-bit binary number that starts and ends with } 0, \\ &\qquad i = 1, \dots, t\} \\ &= \{\langle 0w_{r-1} \dots w_2 0, i\rangle : w_j \in \{0,1\}, j = 2, \dots r-1, i = 1, \dots, t\}; \\ V_3 &= \{\langle w, t+1\rangle : w \text{ is a } (t+1)\text{-bit binary number that ends with } 00\} \\ &= \{\langle w_r \dots w_3 00, t+1\rangle : w_j \in \{0,1\}, j = 3, \dots r\}. \end{aligned}$$

Define H' to be a subgraph induced by the vertex set $V_1 \cup V_2 \cup V_3$ in $BF(t+1)$. Each set has cardinality $|V_1| = 2^{t-1}$, $|V_2| = t \times 2^{t-1}$, and $|V_3| = 2^{t-1}$, so the subgraph H' has cardinality $|V_1| + |V_2| + |V_3| = (t+2) \times 2^{t-1}$.
Claim: The graph H' has diameter $2t-1$.

To check the diameter, we need to check the pairwise distance among all vertices. However, the subgraph induced by the set $V_1 \cup V_2$ is isomorphic to the graph constructed in Case 1, so the distance among vertices in $V_1 \cup V_2$ is at most $2t-1$. It is sufficient to check distances in the following three cases:

- Vertices in V_1 and V_3.
  From the definition of the sets, each vertex of V_1 has distance 1 to set V_2, or

more precisely, it is adjacent to $\langle w, 1\rangle \in V_2$ for some w. Similarly, each vertex of V_3 also has distance 1 to set V_2, i.e. it is adjacent to $\langle w', t\rangle \in V_2$ for some w'. Since the graph induced by V_2 is isomorphic to $BF(t-1)$, by Lemma 2.1, the distance between $\langle w, 1\rangle$ to $\langle w', t\rangle$ is $t-1$. Hence the distance between vertices in V_1 and V_3 is $t+1$.

- Vertices in V_2 and V_3.
  Each vertex in V_3 has distance 1 to the set V_2 (each vertex in V_3 is adjacent to one vertex in V_2), so the distances between vertices from V_3 to any vertex in V_2 is at most $2t-1$.

- Vertices in V_3.
  Define $V_2' \subset V_2$ as follows:

$$\begin{aligned} V_2' &:= \{\langle w, i\rangle : w \text{ is } r\text{−bit binary number starts with 0 and ends with 00}\} \\ &= \{\langle 0w_{r-1}\dots w_3 00, i\rangle : w_j \in \{0,1\}, j = 3, \dots, r-1, i = 2, \dots, t\}. \end{aligned}$$

  The subgraph induced by V_2' has $(t-2) \times 2^{t-2}$ vertices and is isomorphic to $BF(t-2)$ and all vertices in V_1 have distance 1 to V_2' by definition. Take $u, v \in V_3$. Now u and v have distance 1 to the set V_2'. Suppose u is adjacent to x and v is adjacent to y, where $x, y \in V_2'$ (note that x, y can be the same vertex). The distance $d(x, y) \le 2(t-2)$ since $\langle V_2'\rangle$ is isomorphic to $BF(t-2)$. Therefore, $d(u, v) \le 1 + 2t - 4 + 1 = 2t - 2$.

From all cases, the greatest distance between any two vertices, and hence the diameter, is $2t-1$. □

Figure 2 shows an example of a subgraph of maximum degree 4 and diameter 3 in $BF(2)$ and $BF(3)$.

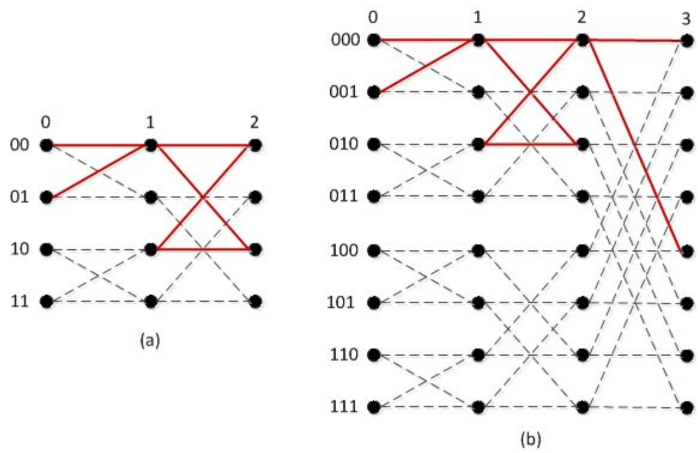

Figure 2: (a)$N_{BF(2)}(4, 3) \ge 6$ and (b) $N_{BF(3)}(4, 3) \ge 8$

Corollary 2.4 gives the relation between the lower bound obtained in Theorem 2.2 and Theorem 2.3.

Corollary 2.4. *Let t be a positive integer and let LB of $N_{BF(r)}(\Delta, D)$ denote the lower bound of $N_{BF(r)}$ obtained from Theorems 2.2 and 2.3 when the maximum degree is Δ and diameter is D. Then the following relations hold:*

$$LB \text{ of } N_{BF(r)}(4, 2t-1) = \begin{cases} \frac{1}{2} \times LB \text{ of } N_{BF(r)}(4, 2t) & \text{if } r = t; \\ \frac{1}{2} \times LB \text{ of } N_{BF(r)}(4, 2t) + 2^{t-1} & \text{if } r > t. \end{cases}$$

3 Subgraph of maximum degree $\Delta = 3$

While the maximum degree of butterfly networks is 4, trivalent networks are of interest and sometimes desired by network administrators. In this section, we focus on subgraphs of butterfly networks with maximum degree 3.

Figure 3 shows subgraphs of degree 3 with diameter $2t$ in $BF(t)$ for $t = 2, 3,$ and 4.

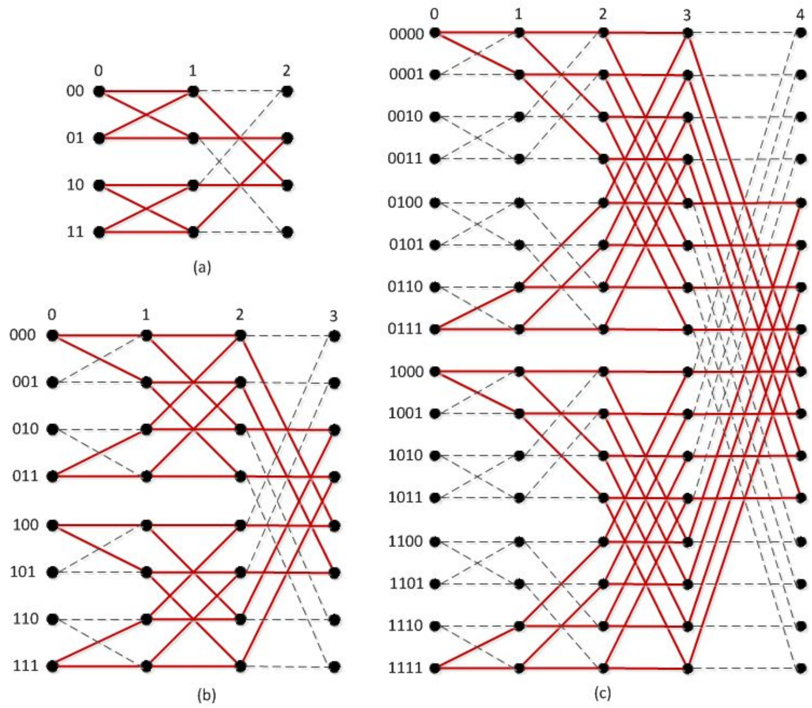

Figure 3: (a)$N_{BF(2)}(3, 4) \geq 10$; (b) $N_{BF(3)}(3, 6) \geq 24$; and (c)$N_{BF(4)}(3, 8) \geq 52$

Keeping the same diameter, the subgraphs shown in Figure 3 cannot be improved in higher dimension since all the vertices of degree 2 have eccentricity equal to the diameter.

Theorem 3.1. *Let $D = 2t$. For any $r \geq t, t > 1$, $N_{BF(r)}(3, 2t) \geq 7 \times 2^{t-1} - 4$ and when $t = 1$, $N_{BF(r)}(3, 2) = 4$.*

Proof. When $t = 1$, subgraph of $BF(r)$ that has maximum degree 3 and diameter 2 is either C_4 or $K_{1,3}$. When $t > 1$, we consider two cases:

Case 1. $r = t$.
The construction of a subgraph with diameter $D = 2t$ will be done inductively. Let G_t be a subgraph of diameter $2t$ contained in $BF(t)$. For $t = 2$, the subgraph G_2 is shown in Figure 3(a). This graph has maximum degree 3, 10 vertices with diameter 4. Note that $10 = 7 \times 2^{2-1} - 4$.

To construct G_3 ($t = 3$), place 2 copies of G_2 in $BF(3)$, one starting at $\langle 000, 1\rangle$ and another starting at $\langle 001, 1\rangle$. Moreover, add vertices at level 0, $\langle w, 0\rangle$ where $w \in \{000, 011, 100, 111\}$. These vertices link the two copies of G_2. Subgraph G_3 contains $2 \times 10 + 4 = 24 = 7 \times 2^{3-1} - 4$ vertices. From Figure 3(b), we can check that the diameter of G_3 is 6.

In general, assume we have constructed G_t, a subgraph of $BF(t)$ with diameter $2t$ on $7 \times 2^{t-1} - 4$ vertices. To construct G_{t+1}, we proceed as follows:

Let $w = \langle w_t w_{t-1} \dots w_1, i\rangle \in G_t$. Define:
$V_1 = \{\langle w''_{t+1} w''_t \dots w''_1, 0\rangle : w''_{t+1} = 0 \text{ or } 1, w''_t = \dots = w''_1 = 0 \text{ or } 1\}$.
$V_2 = \{\langle w_t w_{t-1} \dots w_1 0, i + 1\rangle : \langle w_t w_{t-1} \dots w_1, i\rangle \in G_t\}$.
$V_3 = \{\langle w_t w_{t-1} \dots w_1 1, i + 1\rangle : \langle w_t w_{t-1} \dots w_1, i\rangle \in G_t\}$.
Note that $|V_1| = 4$ and $|V_2| = |V_3| = |G_t|$.

Now $V(G_{t+1}) = V_1 \cup V_2 \cup V_3$. The subgraph induced by $V(G_{t+1})$, G_{t+1}, has cardinality $2 \times |G_t| + 4 = 2 \times (7 \times 2^{t-1} - 4) + 4 = 7 \times 2^t - 4$. The maximum degree is 3 because G_t has maximum degree 3 and the vertices in V_1 are adjacent to vertices of degree 2 in G_t.

We claim that the diameter of G_{t+1} is $2(t + 1)$. To check the diameter, it is sufficient to check the following distances:

1. Between V_1 and V_2 (similarly between V_1 and V_3).
   Each vertex in V_1 is adjacent to a vertex in V_2 (similarly to V_3). So, their distance is at most $1 + 2t$.

2. Among vertices in V_1.
   Each vertex in V_1 is adjacent to a vertex in V_2. Take $u, v \in V_1$ and $x, y \in V_2$, where u, v are adjacent to x, y, respectively. The distance $d(u, v) = d(u, x) + d(x, y) + d(y, v)$ is at most $1 + 2t + 1 = 2t + 2 = 2(t + 1)$ since the distance between any two vertices in V_2 is at most $2t$.

3. Between V_2 and V_3.
   Take any $w' = \langle w_{t+1} w_t \dots w_2 0, i\rangle \in V_2$ and $w'' = \langle w'_{t+1} w'_t \dots w'_2 1, j\rangle \in V_3$. We need to show the existence of a path of length at most $2t + 2$ joining them.

   (a) $i \le j$
   Consider the following vertices

      i. $v = \langle w_{t+1} w_t \dots w_t, 0\rangle \in V_1$;
      ii. $v' = \langle w_{t+1} w_t \dots w_t 0, 1\rangle \in V_2$;
      iii. $v'' = \langle w_{t+1} w_t \dots w_t 1, 1\rangle \in V_3$;
      iv. $u'' = \langle w'_{t+1} w'_t \dots w'_2 1, t + 1\rangle \in V_3$.

The path connecting vertices w' and w'' is

$$w' - v' - v - v'' - u'' - w''.$$

The length of each segment of the path:

i. $d(w', v') = (i - 1)$;
ii. $d(v', v) = 1$ (adjacent);
iii. $d(v, v'') = 1$ (adjacent);
iv. $d(v'', u'') = t$ (by Lemma 2.1);
v. $d(u'', w'') = (t + 1) - j$.

The total distance between $w' - w''$ is

$$d(w', w'') = (i - 1) + 1 + 1 + t + (t + 1) - j \leq i + 2t + 2 - i = 2t + 2.$$

(b) $i > j$

Consider the following vertices

i. $v' = \langle (1 - w_t)w_t w_{t-1} \dots w_2 0, t + 1\rangle \in V_2$;
ii. $u' = \langle w'_{t+1} w'_t \dots w'_t 0, 1\rangle \in V_2$;
iii. $v = \langle w'_{t+1} w'_t \dots w'_t, 0\rangle \in V_1$;
iv. $v'' = \langle w'_{t+1} w'_t \dots w'_t 1, 1\rangle \in V_3$.

The path connecting vertices w and w' is

$$w' - v' - u' - v - v'' - w''.$$

The length of each segment of the path:

i. $d(w', v') = (t + 1) - i$;
ii. $d(v', u') = t$ (by Lemma 2.1);
iii. $d(u', v) = 1$ (adjacent);
iv. $d(v, v'') = 1$ (adjacent);
v. $d(v'', w'') = j - 1$.

The total distance between $w' - w''$ is

$$d(w', w'') = (t + 1) - i + t + 1 + 1 + (j - 1) \leq 2t + 2 - j + j = 2t + 2.$$

Case 2. $r > t$

This subgraph is the subgraph constructed in Case 1. This subgraph cannot be improved since all the vertices with degree 2 have eccentricity equal to the diameter. In other words, adding any vertex adjacent to a vertex of degree 2 will increase the diameter. □

Theorem 3.2. *Let $D = 2t - 1$. In the r-dimensional butterfly network, $r \geq t$, we have*

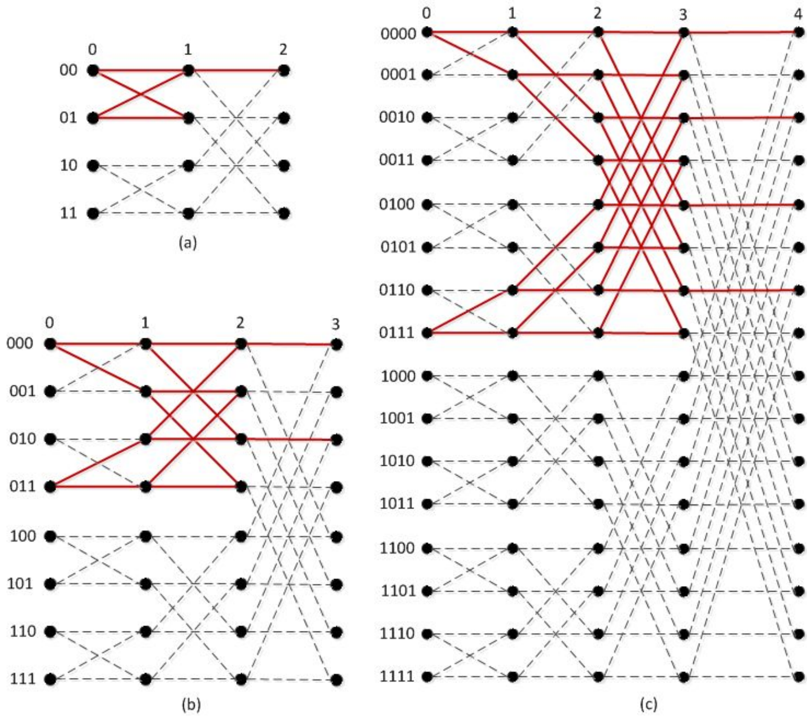

Figure 4: (a)$N_{BF(2)}(3,3) \geq 5$; (b) $N_{BF(3)}(3,5) \geq 12$; and (c)$N_{BF(4)}(3,7) \geq 26$

(i) $N_{BF(r)}(3,D) \geq 7 \times 2^{t-2} - 2$ *if* $r = t$;

(ii) $N_{BF(r)}(3,D) \geq 7 \times 2^{t-2} - 1$ *if* $t+1 \leq r \leq 2t-2$;

(iii) $N_{BF(r)}(3,D) \geq 2^{t+1} - 2$ *if* $r \geq 2t-1$.

Proof. Let $BF(r)$ be the r-dimensional butterfly network.

Case 1. $r = t$

Let G_t be a subgraph of diameter $2t$ contained in $BF(t)$. For $t = 2$, a subgraph of $BF(t)$ that has degree 3 and diameter 3 is given in Figure 4(a). It contains $5 = 7 \times 2^{2-2} - 2$ vertices. More explicitly, $G_2 = \{\langle 00,0\rangle, \langle 00,1\rangle, \langle 00,2\rangle, \langle 01,0\rangle, \langle 01,1\rangle\}$.

Similar to the proof of Theorem 3.1, we will construct the subgraph inductively.

To construct G_3 ($t = 3$), define:

$$\begin{aligned}
V_1 &= \{\langle 0w''_t \ldots w''_1, 0\rangle : w''_t = \ldots = w''_1 = 0 \text{ or } 1\} = \{\langle 000,0\rangle, \langle 011,0\rangle\}.\\
V_2 &= \{\langle w_2w_10, i+1\rangle : \langle w_2w_1, i\rangle \in G_2\}\\
&= \{\langle 000,1\rangle, \langle 000,2\rangle, \langle 000,3\rangle, \langle 010,1\rangle, \langle 010,2\rangle\}.\\
V_3 &= \{\langle w_2w_11, i+1\rangle : \langle w_2w_1, i\rangle \in G_2, i = 0,1\}\\
&= \{\langle 001,1\rangle, \langle 001,2\rangle, \langle 011,1\rangle, \langle 011,2\rangle\}.\\
U &= \{\langle 010,3\rangle\}.
\end{aligned}$$

G_3 contains $12 = 7 \times 2^{3-2} - 2$ vertices. From Figure 4(b), we can check that the diameter of G_3 is 5.

In general, assume we have constructed G_t, a subgraph of $BF(t)$ with diameter $2t-1$ on $7 \times 2^{t-2} - 2$ vertices. To construct G_{t+1}, we proceed as follows:
Let $w = \langle w_t w_{t-1} \dots w_1 w_0, i \rangle \in G_t$. Define:
$V_1 = \{\langle 0w''_t \dots w''_1, 0\rangle : w''_t = \dots = w''_1 = 0 \text{ or } 1\} = \{\langle 00\dots0,0\rangle, \langle 01\dots1,0\rangle\}$.
$V_2 = \{\langle w_t w_{t-1} \dots w_1 0, i+1\rangle : \langle w_t w_{t-1} \dots w_1, i\rangle \in G_t\}$.
$V_3 = \{\langle w_t w_{t-1} \dots w_1 1, i+1\rangle : \langle w_t w_{t-1} \dots w_1, i\rangle \in G_t, i = 0, \dots, t-1\}$.
$U = \{\langle 0u_t u_{t-1} \dots u_3 10, t+1\rangle\}$.

The cardinalities of those sets are
$|V_1| = 2$;
$|V_2| = |V(G_t)| = 7 \times 2^{t-2} - 2$;
$|V_3| = |V(G_t)| - 2^{t-2} = 7 \times 2^{t-2} - 2 - 2^{t-2} = 6 \times 2^{t-2} - 2$;
$|U| = 2^{t-2}$.
$V(G_{t+1}) = V_1 \cup V_2 \cup V_3 \cup U$, with cardinality

$$|V(G_{t+1})| = 2 + 7 \times 2^{t-2} - 2 + 6 \times 2^{t-2} - 2 + 2^{t-2} = 7 \times 2^{(t+1)-2} - 2.$$

The maximum degree is 3 because G_t has maximum degree 3 and vertices in V_1 are adjacent to vertices of degree 2 in G_t.

We claim that the diameter of G_{t+1} is $2(t+1)-1$. To check the diameter, it is sufficient to check the following distances:

1. Between V_1 and V_2 (similarly between V_1 and V_3).
   Each vertex in V_1 is adjacent to a vertex in V_2 (similarly to V_3). So, their distance is at most $1 + 2t - 1 = 2t$.

2. Between vertices in V_1.
   $V_1 = \{u = \langle 00\dots0,0\rangle, v = \langle 01\dots1,0\rangle\}$. Let $u' = \langle 01\dots1,t\rangle$. There is a path connecting u and v through u'. By Lemma 2.1, $d(u,u') = t$ and $d(u',v) = t$, thus, $d(u,v) = 2t$.

3. Between V_1 and U.
   By Lemma 2.1, the distance between a vertex in V_1 and a vertex in U is $t+1$.

4. Between V_2 and U.
   Each vertex in U is adjacent to a vertex in V_2, so their distance is at most $1 + 2t - 1 = 2t$.

5. Between V_2 and V_3.
   Take any $w' = \langle w_{t+1} w_t \dots w_2 0, i\rangle \in V_2, 1 \le i \le t+1$ and $w'' = \langle w'_{t+1} w'_t \dots w'_2 1, j\rangle \in V_3, 1 \le j \le t$. We need to show the existence of a path of length at most $2t+1$ joining them.

   (a) $i \le j$
   Consider the following vertices

      i. $v = \langle w_{t+1} w_t \dots w_t, 0\rangle \in V_1$;
      ii. $v' = \langle w_{t+1} w_t \dots w_t 0, 1\rangle \in V_2$;

iii. $v'' = \langle w_{t+1}w_t \dots w_t 1, 1\rangle \in V_3$;
iv. $u'' = \langle w'_{t+1}w'_t \dots w'_2 1, t\rangle \in V_3$

The path connecting vertices w' and w'' is

$$w' - v' - v - v'' - u'' - w''.$$

The length of each segment of the path:
i. $d(w', v') = i - 1$;
ii. $d(v', v) = 1$ (adjacent);
iii. $d(v, v'') = 1$ (adjacent);
iv. $d(v'', u'') = t - 1$ (by Lemma 2.1);
v. $d(u'', w'') = t - j$.

The total length of the path is

$$d(w', w'') = (i - 1) + 1 + 1 + (t - 1) + t - j \leq i + 2t - i = 2t.$$

(b) $i > j$

Consider the following vertices
i. $v' = \langle w_{t+1}w_t w_{t-1} \dots w_2 0, t\rangle \in V_2$;
ii. $u' = \langle w'_{t+1}w'_t \dots w'_t 0, 1\rangle \in V_2$;
iii. $v = \langle w'_{t+1}w'_t \dots w'_t, 0\rangle \in V_1$;
iv. $v'' = \langle w'_{t+1}w'_t \dots w'_t 1, 1\rangle \in V_3$

The path connecting vertices w and w' is

$$w' - v' - u' - v - v'' - w''.$$

The length of each segment of the path:
i. $d(w', v') = \begin{cases} t - i & \text{if } i \leq t \\ 1 & \text{if } i = t + 1 \end{cases}$
ii. $d(v', u') = t - 1$ (by Lemma 2.1);
iii. $d(u', v) = 1$ (adjacent);
iv. $d(v, v'') = 1$(adjacent);
v. $d(v'', w'') = j - 1$.

The total length of the path is

$$d(w', w'') = \begin{cases} (t - i) + (t - 1) + 1 + 1 + (j - 1) \leq 2t & \text{if } i \leq t \\ 1 + (t - 1) + 1 + 1 + (j - 1) \leq 2t + 1 & \text{if } i = t + 1 \end{cases}$$

6. Between V_3 and U.
Let $u = \langle 0u_t u_{t-1} \dots u_3 10, t + 1\rangle \in U$. From the definition of the butterfly network, u is adjacent to vertex $v' = \langle 0u_t u_{t-1} \dots u_3 10, t\rangle \in V_2$ by an edge of Type 1 (see Section 1). Hence, we can treat u similarly to vertex $v' \in V_2$ when $i = t+1$. Using the same path constructed in Point 5b for $i = t+1$, we obtain $d(u, v'') \leq 2t + 1, u \in U, v'' \in V_3$

7. Among vertices in U.
   Each vertex in U is adjacent to a vertex in V_2. Take $u, v \in U$ and $x, y \in V_2$, where u, v are adjacent to x, y, respectively. The distance $d(u, v) = d(u, x) + d(x, y) + d(y, v)$ is at most $1 + 2t - 1 + 1 = 2t + 1 = 2(t + 1) - 1$ since the distance among vertices in V_2 is at most $2t - 1$.

Case 2. $t + 1 \le r \le 2t - 2$
Let G_t be a subgraph of diameter $2t - 1$ contained in $BF(t)$, constructed in (i); we need to construct G'_t, a subgraph of diameter $2t - 1$ contained in $BF(r)$, where $r > t$. Define $V' = \{\langle 0 \ldots 0 w_t w_{t-1} \ldots w_1 0, i + 1\rangle : \langle w_t w_{t-1} \ldots w_1, i\rangle \in G_t\}$. Clearly $|V'| = |G_t| = 7 \times 2^{t-2} - 2$. Since the maximum degree is 3, we can only attach vertices to the vertices with degree 2 and eccentricity less than $2t - 1$, which are vertices in $V'_1 = \{u = \langle 0 \ldots 0, 0\rangle, v = \langle 0 \ldots 0 \underbrace{1 \ldots 1}_{t-1} 0, 0\rangle\} \subset V'$. From the proof of the previous case, we know that vertices in V'_1 have eccentricity $2t - 2$, so we can attach to any of them a vertex without affecting the diameter. However, since $d(u, v) = 2t - 2$, we can only add a vertex to exactly one of them; without loss of generality join vertex u to $\langle 0 \ldots 0, 0\rangle$. The graph G'_t is the graph induced by $V(G'_t) = V' \cup \{\langle 0 \ldots 0, 0\rangle\}$, with cardinality $7 \times 2^{t-2} - 1$.
Case 3. $r \ge 2t - 1$
The graph is a ternary tree of depth $t - 1$ rooted at $\langle 0 \ldots 0, t\rangle$, adding 1 depth extra from the first child of the root (2^{t-1} leaves). □

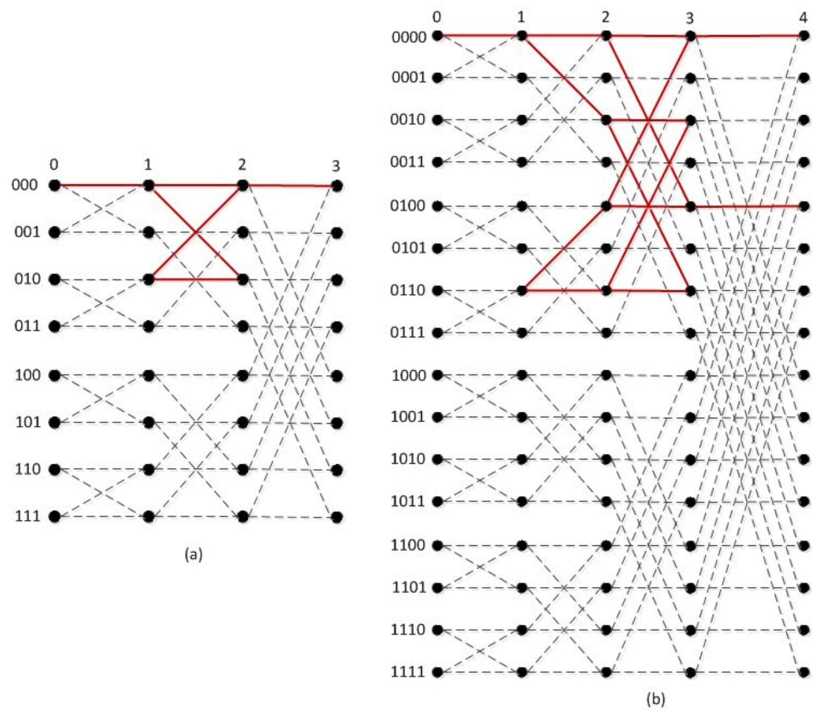

Figure 5: (a)$N_{BF(3)}(3, 3) \ge 6$ and (b) $N_{BF(4)}(3, 5) \ge 13$

By observation, we obtain the next corollary.

Corollary 3.3. *Let t be a positive integer. Let LB of $N_{BF(r)}(\Delta, D)$ denote the lower bound of $N_{BF(r)}$ obtained from Theorem 3.1 and Theorem 3.2 with maximum degree Δ and diameter D. Then the following relations hold.*

1. *Lower bound of $N_{BF(t)}(3, 2t-3) = \frac{1}{2}\times$ lower bound of $N_{BF(t)}(3, 2t-1)$;*
2. *Lower bound of $N_{BF(t)}(3, 2t-3) = \frac{1}{4}\times$ lower bound of $N_{BF(t)}(3, 2t)$.*

Corollary 3.4 shows the relation between the lower bound of the largest subgraphs of given diameter D with maximum degrees 4 and 3.

Corollary 3.4. *Let t be a positive integer. Let LB of $N_{BF(r)}(\Delta, D)$ denote the lower bound of the number of vertices in an r-dimensional butterfly network with maximum degree Δ and diameter D obtained in Theorems 2.2, 2.3, 3.1 and 3.2. Then the following relations hold:*

1. *LB of $N_{BF(r)}(3, 2t) = \frac{1}{2}\times$ LB of $N_{BF(r)}(4, 2t) + (6-t)2^{t-1} - 4$;*
2. *LB of $N_{BF(r)}(3, 2t-1) = \frac{1}{2}\times$ LB of $N_{BF(r)}(4, 2t-1) + (6-t)2^{t-2} - 2$ if $r = t$;*
3. *LB of $N_{BF(r)}(3, 2t-1) = \frac{1}{2}\times$ LB of $N_{BF(r)}(4, 2t-1) + (5-t)2^{t-2} - 1$ if $r > t$.*

4 Subgraph of maximum degree $\Delta = 2$

Since the butterfly network is a bipartite graph, the subgraph with maximum degree 2 of diameter D is a cycle C_{2D} (if it exists).

Figure 6 gives all possible cycle lengths that are contained in $BF(2)$ and $BF(3)$. As we see, the largest cycles in $BF(2)$ and $BF(3)$ are of sizes 8 and 24, respectively. In general, we prove in Theorem 4.1 that the largest cycle contained in $BF(r)$ is of size $r2^r$. Other observations are that there are no cycles C_6 and C_{10} in $BF(2)$ and $BF(3)$. Indeed, Theorem 4.2 shows that $BF(r)$ contains no cycle of size 6 and 10, for any r.

Theorem 4.1. *Let $BF(r)$ be a butterfly network of dimension r. The largest cycle contained in $BF(r)$ is of size $r2^r$.*

Proof. First, we prove that the size of cycles in $BF(r)$ is at most $r2^r$ and second, we give the explicit construction of a cycle achieving that size.

Consider vertices of degree 2 in $BF(r)$.

- $i = 0$
  Take two vertices $w = \langle w_r \dots w_2 0, 0\rangle$ and $w' = \langle w_r \dots w_2 1, 0\rangle$. Both w and w' have degree 2. Consider v, v' vertices of level 1 in $BF(r)$ where $v = \langle w_r \dots w_2 0, 1\rangle$ and $v' = \langle w_r \dots w_2 1, 1\rangle$. Vertex w is adjacent to v by an edge of type 1 and to v' by an edge of type 2. Vertex w' is adjacent to v by edge of type 2 and to v' by edge of type 1. If w is a vertex in a cycle, so are vertices

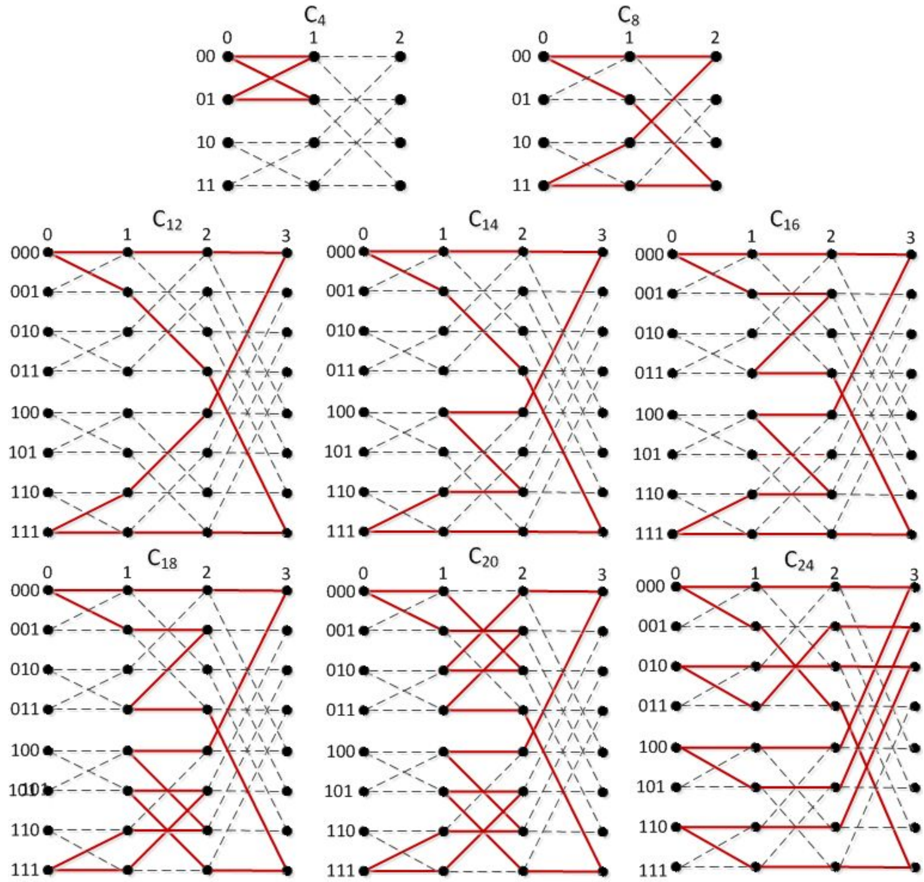

Figure 6: All cycle lengths in $BF(2)$ and $BF(3)$

v and v'. If w' is contained in the same cycle, then the cycle is of length 4. Therefore, a cycle of size greater than 4 contains at most one of w, w'. From the 2^r vertices of level 0, at most $\frac{1}{2} \times 2^r = 2^{r-1}$ vertices can be contained in the same cycle.

- $i = r$
  By a similar argument we have that, of the 2^r vertices in level r, just 2^{r-1} can be in the same cycle.

From the two cases above, at most 2^{r-1} vertices at level 0 and 2^{r-1} vertices at level r can be included in the same cycle. The cardinality of $BF(r)$ is $(r+1)2^r$, and thus the size of a largest cycle is at most $(r+1)2^r - 2^{r-1} - 2^{r-1} = r2^r$.

We now prove, by construction, the existence of a cycle of length $r2^r$ in $BF(r)$. In this construction, we are excluding vertices of degree 2 which have the form $\langle w_r \ldots w_2 1, 0\rangle$ and $\langle 1 w_{r-1} \ldots w_1, r\rangle$. We start from the vertex $\langle \underbrace{0 \ldots 0}_{r}, 0\rangle$.

The following algorithm gives a cycle construction.

1. $\langle 0 w_{r-1} \ldots w_2 0, j\rangle \to \langle 0 w_{r-1} \ldots w_2 0, j+1\rangle, j = 0, \ldots, r-1$

2. $\langle 0w_{r-1}\dots w_2 0, r\rangle \to \langle 1w_{r-1}\dots w_2 0, r-1\rangle$
3. $\langle 1w_{r-1}\dots w_2 0, j\rangle \to \langle 1w_{r-1}\dots w_2 0, j-1\rangle, j = 1,\dots,r-1$
4. $\langle 1w_{r-1}\dots w_2 0, 0\rangle \to \langle 1w_{r-1}\dots w_2 1, 1\rangle$
5. $\langle 1w_{r-1}\dots w_2 1, j\rangle \to \langle 1w_{r-1}\dots w_2 1, j+1\rangle, j = 1,\dots,r-2$
6. $\langle 1w_{r-1}\dots w_2 1, r-1\rangle \to \langle 0w_{r-1}\dots w_2 1, r\rangle$
7. $\langle 0w_{r-1}\dots w_{j+2} 0 \underbrace{1\dots1}_{j}, k\rangle \to \langle 0w_{r-1}\dots w_{j+2} 0 \underbrace{1\dots1}_{j}, k-1\rangle, j+2 \le k \le r, j \le r-2$
8. $\langle 0w_{r-1}\dots w_{j+2} 0 \underbrace{1\dots1}_{j}, j+1\rangle \to \langle 0w_{r-1}\dots w_{j+2} \underbrace{1\dots1}_{j+1}, j\rangle, j \le r-2$
9. $\langle 0w_{r-1}\dots w_{j+1} \underbrace{1\dots1}_{j}, j\rangle \to \langle 0w_{r-1}\dots w_{j+1} 0 \underbrace{1\dots1}_{j-1}, j-1\rangle, 1 \le j \le r-1$
10. $\langle 0\dots0 \underbrace{1\dots1}_{r-1}, r\rangle \to \langle 0\dots0 \underbrace{1\dots1}_{r-1}, r-1\rangle$

□

Theorem 4.2. *For any r, $BF(r)$ does not contain cycles C_6 and C_{10}.*

Proof. First we will prove that C_6 does not exist in $BF(r)$ and, using a similar argument, we show that C_{10} does not exist.

i. $BF(r)$ does not contain cycle C_6.
   In $BF(r)$, adjacent vertices have consecutive level, i.e. if u and v are adjacent and u lies in level i, then v lies either in level $i-1$ or $i+1$. Therefore, to check the existence of C_6 in $BF(r)$, we just need to check its existence in $BF(3)$, since if C_6 exists in $BF(r > 3)$, it should also exist in a subgraph of $BF(r > 3)$ isomorphic to $BF(3)$.

   For every cycle in $BF(3)$, due to the structure of a butterfly network, we can always find an isomorphic copy of the cycle that contains the vertex $\langle \underbrace{0\dots0}_{r}, 0\rangle$, which therefore contains vertices $\langle \underbrace{0\dots0}_{r}, 1\rangle$ and $\langle \underbrace{0\dots0}_{r-1} 1, 1\rangle$. Without loss of generality, to prove the non-existence of C_6 in $BF(r)$, we just need to show that there do not exist two paths of length 2, one beginning from $\langle \underbrace{0\dots0}_{r}, 1\rangle$ and the other from $\langle \underbrace{0\dots0}_{r-1} 1, 1\rangle$, that end at the same vertex distinct from the starting vertices.

If we label the vertices as $v_1 = \langle\underbrace{0\dots0}_{r},0\rangle$, $v_2 = \langle\underbrace{0\dots0}_{r-1}1,0\rangle$, …, $v_8 = \langle\underbrace{1\dots1}_{r},0\rangle$, $v_9 = \langle\underbrace{0\dots0}_{r},1\rangle$, $v_{10} = \langle\underbrace{0\dots0}_{r-1}1,1\rangle$, …, $v_{16} = \langle\underbrace{1\dots1}_{r},1\rangle$, …, $v_{25} = \langle\underbrace{0\dots0}_{r},3\rangle$, …, $v_{32} = \langle\underbrace{1\dots1}_{r},3\rangle$, then the adjacency matrix M of $BF(3)$ can be written as a block matrix, as follows

$$M = [m_{ij}] = \begin{bmatrix} 0 & A & 0 & 0 \\ A & 0 & B & 0 \\ 0 & B & 0 & C \\ 0 & 0 & C & 0 \end{bmatrix}$$

where each block $(0, A, B, C)$ is of size 8×8, and

$$A = \begin{bmatrix} 1&1&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0 \\ 0&0&1&1&0&0&0&0 \\ 0&0&0&0&1&1&0&0 \\ 0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&1&1 \end{bmatrix}, \quad B = \begin{bmatrix} 1&0&1&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0 \\ 1&0&1&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0 \\ 0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&1&0&1 \\ 0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&1&0&1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \\ 1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \end{bmatrix}.$$

Information regarding paths of length 2 is contained in the square of the adjacency matrix, M^2:

$$M^2 = [m'_{ij}] = \begin{bmatrix} A^2 & 0 & AB & 0 \\ 0 & A^2+B^2 & 0 & BC \\ AB & 0 & B^2+C^2 & 0 \\ 0 & BC & 0 & C^2 \end{bmatrix},$$

$$A^2+B^2=\begin{bmatrix}4&2&2&0&0&0&0&0\\2&4&0&2&0&0&0&0\\2&0&4&2&0&0&0&0\\0&2&2&4&0&0&0&0\\0&0&0&0&4&2&2&0\\0&0&0&0&2&4&0&2\\0&0&0&0&2&0&4&2\\0&0&0&0&0&2&2&4\end{bmatrix},\quad BC=\begin{bmatrix}1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\\1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\\1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\\1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\end{bmatrix}.$$

We need to find two paths of length 2, beginning at v_9 and v_{10} respectively, that end at the same vertex distinct from v_9 and v_{10}. Therefore, we focus on the entries m'_{9k} and m'_{10k}, where $k \neq 9, 10$ in the matrix M^2. For each k, either m'_{9k} or m'_{10k} is 0 indicating that these paths do not exist. We conclude that C_6 does not exist.

ii. $BF(r)$ does not contain cycle C_{10}.
Similar to the proof of Theorem 4.2(i), without loss of generality, to prove the non-existence of C_{10} in $BF(r)$, we just need to show that there are no two paths of length 4, beginning from $\langle\underbrace{0\ldots0}_{r},1\rangle$ and $\langle\underbrace{0\ldots0}_{r-1}1,1\rangle$ respectively, that end at the same vertex distinct from the starting vertices.
If we label the vertices as $v_1=\langle\underbrace{0\ldots0}_{r},0\rangle, v_2=\langle\underbrace{0\ldots0}_{r-1}1,0\rangle,\ldots, v_{32}=\langle\underbrace{1\ldots1}_{r},0\rangle$, $v_{33}=\langle\underbrace{0\ldots0}_{r},1\rangle$, $v_{34}=\langle\underbrace{0\ldots0}_{r-1}1,1\rangle,\ldots, v_{64}=\langle\underbrace{1\ldots1}_{r},1\rangle,\ldots, v_{161}=\langle\underbrace{0\ldots0}_{r},5\rangle$, $\ldots, v_{192}=\langle\underbrace{1\ldots1}_{r},5\rangle$, then the adjacency matrix of $BF(5)$ can be written as follows:

$$M=\begin{bmatrix}0&A_1&0&0&0&0\\A_1&0&B_1&0&0&0\\0&B_1&0&C_1&0&0\\0&0&C_1&0&D_1&0\\0&0&0&D_1&0&E_1\\0&0&0&0&E_1&0\end{bmatrix}$$

where each block $(0, A_1, B_1, C_1, D_1, E_1)$ is of size 32×32, and A_1 (respectively B_1, C_1) are themselves diagonal block matrices, where the block matrices in the main diagonal are all A (respectively B, C), defined in Part (i). Matrices D_1 and E_1 are defined as block matrices in terms of I_8, the identity matrix of size 8×8, as follows:

$$D_1=\begin{bmatrix}I_8&I_8&0&0\\I_8&I_8&0&0\\0&0&I_8&I_8\\0&0&I_8&I_8\end{bmatrix},\quad E_1=\begin{bmatrix}I_8&0&I_8&0\\0&I_8&0&I_8\\I_8&0&I_8&0\\0&I_8&0&I_8\end{bmatrix}.$$

To gain information regarding paths of length 4, we observe the 4th power of the adjacency matrix M, i.e. $M^4 = [m''_{ij}]$. Since we are observing the paths starting from vertices v_{33} and v_{34}, we only need the entries m''_{33k} and m''_{34k}, $k = 1, \dots, 192$. The second row of the block matrix M^4 is

$$\begin{bmatrix} 0 & (A_1^2 + B_1^2)^2 + (B_1C_1)^2 & 0 & B_1C_1(A_1^2 + B_1^2 + C_1^2 + D_1^2) & 0 & B_1C_1D_1E_1 \end{bmatrix}$$

Since the paths cannot end at v_{33}, v_{34}, we focus on the entries m''_{33k} and m''_{34k}, where $k \neq 33, 34$ in the matrix M^4. The path might exist when m''_{33k} or m''_{34k} are both non-zero. From the matrix, this happens when $k = 35, 36, 97, \dots, 104$. However, for these values of k, the pair of paths are not independent, i.e. they share some edges, so they cannot form a cycle. We conclude that C_{10} does not exist. □

The last observation regarding the cycles is that there is no C_6 in $BF(2)$ and no C_{22} in $BF(3)$. There is also no cycle C_{62} in $BF(4)$.

Conjecture 1. *There is no cycle of size $r2^r - 2$ in $BF(r)$ for any value of r.*

Conjecture 1 is a long standing problem for which here we are not able to provide a solution. However, if the conjecture is true then $N_{BF(r)}(2, r2^{(r-1)} - 1) = r2^r - 4$, otherwise $N_{BF(r)}(2, r2^{(r-1)} - 1) = r2^r - 2$.

5 Conclusion and Open Problems

In this paper, we have given the lower bounds for the $N_{BF(r)}(\Delta, D)$ for $\Delta = 2, 3, 4$ and the bounds are sharp. However, we have not proved the exact value of $N_{BF(r)}(\Delta, D)$. In the case $\Delta = 2$, we conjecture the non-existence of cycles of length $r2^r - 2$ in $BF(r)$. This is very interesting since a larger cycle (cycle of length $r2^r$) does exist in $BF(r)$.

In all real world architectures, optimisation of subnetworks subject to constraints on vertex degree and diameter plays an important role in network analysis. It would be valuable to calculate MaxDDBS for other real world architectures.

Acknowledgements

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References

[1] A. Dekker, H. Pérez-Rosés, G. Pineda-Villavicencio and P. Watters, The maximum degree & diameter-bounded subgraph and its applications, *J. Math. Model. Algorithms* 11(3) (2012), 249–268.

[2] S. Hashimoto, Sharper lower bounds in the maximum degree and diameter bounded subgraph problem in the mesh, *Discrete Applied Math.* in print, `http://arxiv.org/abs/1211.7128`.

[3] P. Holub, M. Miller, H. Perez-Roses and J. Ryan, Degree/diameter problem on honeycomb networks, *Discrete Applied Math.* 179 (2014), 139–151.

[4] P. Holub and J. Ryan, Degree diameter problem on triangular networks, *Australas. J. Combin.* 63 (2015), 333–345.

[5] T. F. Leighton, *Introduction to Parallel Algorithms and Architecture: Arrays, Trees, Hypercubes*, Morgan Kaufmann Publishers, 1992.

[6] M. Miller, H. Perez-Roses and J. Ryan, The maximum degree and diameter-bounded subgraph in the mesh, *Discrete Appl. Math.* 160 (12) (2012), 1782–1790.

[7] M. Miller and J. Siran, Moore graphs and beyond: A survey of the degree-diameter problem, *Elec. J. Combin.* (2013), 1–92. Dynamic Survey 14.

[8] A. William and A. Shanthakumari, Minimum cycle covers of butterfly and benes networks, *Int. J. Math. and Soft Computing* 2(1) (2012), 93–98.

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