AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **68(2)** (2017), Pages 304–316

The k-proper index of complete bipartite and complete multipartite graphs*

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Abstract

Let G be an edge-colored graph. A tree T in G is a *proper tree* if no two adjacent edges of it are assigned the same color. Let k be a fixed integer with $2 \leq k \leq n$. For a vertex subset $S \subseteq V(G)$ with $|S| \geq 2$, a tree is called an S-*tree* if it connects the vertices of S in G. A k-*proper coloring* of G is an edge-coloring of G having the property that for every set S of k vertices of G, there exists a proper S-tree T in G. The minimum number of colors that are required in a k-proper coloring of G is defined as the k-*proper index* of G, denoted by $px_k(G)$. In this paper, we determine the 3-proper index of all complete bipartite and complete multipartite graphs and partially determine the k-proper index of them for $k \geq 4$.

1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty in [2] for those not defined here. Let G be a graph, we use $V(G), E(G), |G|, \Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, order (number of vertices), maximum degree and minimum degree of G, respectively. For $D \subseteq V(G)$, let $\overline{D} = V(G) \backslash D$, and let $G[D]$ denote the subgraph of G induced by D.

Let G be a nontrivial connected graph with an *edge-coloring* $c: E(G) \rightarrow \{1, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with the same color. If adjacent edges of G receive different colors by c, then c is called a *proper coloring*. The minimum

* Supported by NSFC No.11371205 and 11531011, and PCSIRT.

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number of colors required in a proper coloring of G is referred as the *chromatic index* of G and denoted by $\chi'(G)$. Meanwhile, a path in G is called *a rainbow path* if no two edges of the path are colored with the same color. The graph G is called *rainbow connected* if for any two distinct vertices of G, there is a rainbow path connecting them. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is defined as the minimum number of colors that are required to make G rainbow connected. These concepts were first introduced by Chartrand et al. in [6] and have been well-studied since then. For further details, we refer the reader to a book [10].

Motivated by rainbow coloring and proper coloring in graphs, Andrews et al. [1] and, independently, Borozan et al. [3] introduced the concept of proper-path coloring. Let G be a nontrivial connected graph with an edge-coloring. A path in G is called *a proper path* if no two adjacent edges of the path are colored with the same color. The graph G is called *proper connected* if for any two distinct vertices of G, there is a proper path connecting them. The *proper connection number* of G, denoted by $pc(G)$, is defined as the minimum number of colors that are required to make G proper connected. For more details, we refer to a dynamic survey [9].

Chen et al. [7] recently generalized the concept of proper-path to proper tree. A tree T in an edge-colored graph is a *proper tree* if no two adjacent edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an *S-tree* if it connects S in G. Let G be a connected graph of order n with an edge-coloring and let k be a fixed integer with $2 \leq k \leq n$. A *k-proper coloring* of G is an edge-coloring of G having the property that for every set S of k vertices of G, there exists a proper S-tree T in G. The minimum number of colors that are required in a k-proper coloring of G is the *k-proper index* of G, denoted by $px_k(G)$. Clearly, $px_2(G)$ is precisely the proper connection number $pc(G)$ of G. For a connected graph G, it is easy to see that $px_2(G) \leq px_3(G) \leq \cdots \leq px_n(G)$. The following results are not difficult to obtain.

Proposition 1.1. *[7] If G is a nontrivial connected graph of order $n \geq 3$, and H is a connected spanning subgraph of G, then $px_k(G) \leq px_k(H)$ for any k with $3 \leq k \leq n$. In particular, $px_k(G) \leq px_k(T)$ for every spanning tree T of G.*

Proposition 1.2. *[7] For an arbitrary connected graph G with order $n \geq 3$, we have $px_k(G) \geq 2$ for any integer k with $3 \leq k \leq n$.*

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a graph having a Hamiltonian path is a *traceable graph*.

Proposition 1.3. *[7] If G is a traceable graph with $n \geq 3$ vertices, then $px_k(G) = 2$ for each integer k with $3 \leq k \leq n$.*

Armed with Proposition 1.3, we can easily obtain

$$px_k(K_n) = px_k(P_n) = px_k(C_n) = px_k(W_n) = px_k(K_{s,s}) = 2$$

for each integer k with $3 \leq k \leq n$, where K_n, P_n, C_n and W_n are respectively a complete graph, a path, a cycle and a wheel on $n \geq 3$ vertices and $K_{s,s}$ is a regular complete bipartite graph with $s \geq 2$.

A vertex set $D \subseteq G$ is called an *s-dominating set* of G if every vertex in $\overline{D}$ is adjacent to at least s distinct vertices of D. If, in addition, $G[D]$ is connected, then we call D a *connected s-dominating set*. Recently, Chang et al. [4] gave an upper bound for the 3-proper index of graphs with respect to the connected 3-dominating set.

Theorem 1.1. *[4] If D is a connected 3-dominating set of a connected graph G with minimum degree $\delta(G) \geq 3$, then $px_3(G) \leq px_3(G[D]) + 1$.*

Using this, we can easily obtain the following.

Theorem 1.2. *For any complete bipartite graph $K_{s,t}$ with $t \geq s \geq 3$, we have $2 \leq px_3(K_{s,t}) \leq 3$.*

Proof. Let U and W be the two partite sets of $K_{s,t}$, where $U = \{u_1, u_2, u_3, \ldots, u_s\}$ and $W = \{w_1, w_2, w_3, \ldots, w_t\}$. Obviously, $D = \{u_1, u_2, u_3, w_1, w_2, w_3\}$ is a connected 3-dominating set of $K_{s,t}$ and $\delta(K_{s,t}) \geq 3$. It follows from Theorem 1.1 that $px_3(K_{s,t}) \leq px_3(K_{s,t}[D]) + 1 = 3$. By Proposition 1.2, we have $px_3(K_{s,t}) \geq 2$. □

Naturally, we wonder among these complete bipartite graphs, whose 3-proper index is 2. Moreover, what are the exact values of $px_3(K_{s,t})$ with $s+t \geq 3, t \geq s \geq 1$ and $px_3(K_{n_1,n_2,\ldots,n_r})$ with $r \geq 3$? Moreover, what happens when $k \geq 4$? So our paper is organised as follows: In Section 2, we concentrate on all complete bipartite graphs and determine the value of the 3-proper index of each of them. In Section 3, we go on investigating all complete multipartite graphs and obtain the 3-proper index of each of them. In the final section, we turn to the case that $k \geq 4$, and give a partial answer. In the sequel, we use $c(uw)$ to denote the color of the edge uw.

2 The 3-proper index of a complete bipartite graph

In this section, we concentrate on all complete bipartite graphs $K_{s,t}$ with $s + t \geq 3, t \geq s \geq 1$ and obtain a complete answer of the value of $px_3(K_{s,t})$. From [7], we know $px_3(K_{1,t}) = t$. Hence, in the following we assume that $t \geq s \geq 2$. Our result will be divided into three separate theorems depending upon the value of s.

Theorem 2.1. *For any integer $t \geq 2$, we have*

$$px_3(K_{2,t}) = \begin{cases} 2 & \text{if } 2 \leq t \leq 4; \\ 3 & \text{if } 5 \leq t \leq 18; \\ \left\lceil \sqrt{\frac{t}{2}} \right\rceil & \text{if } t \geq 19. \end{cases}$$

Proof. Let U, W be the two partite sets of $K_{2,t}$, where $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. Suppose that there exists a 3-proper coloring $c : E(K_{2,t}) \to \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$. Corresponding to the 3-proper coloring, there is a color code(w) assigned to every vertex $w \in W$, consisting of an ordered 2-tuple (a_1, a_2), where $a_i = c(u_i w) \in \{1, 2, \ldots, k\}$ for $i = 1, 2$. In turn, if we give each vertex of W a code, then we can induce the corresponding edge-coloring of $K_{2,t}$.

Claim 1: $px_3(K_{2,t}) = 2$ if $2 \leq t \leq 4$.

Proof. Give the codes $(1, 2), (2, 1), (1, 1), (2, 2)$ to w_1, w_2, w_3, w_4 (if each of these vertices exists). Then it is easy to check that for every 3-subset S of $K_{2,t}$, the edge-colored $K_{2,t}$ has a proper path P connecting S. □

Claim 2: $px_3(K_{2,t}) > 2$ if $t > 4$.

Proof. Otherwise, give $K_{2,t}$ a 3-proper coloring with colors 1 and 2. Then for any 3-subset S of $K_{2,t}$, any proper tree connecting S must actually be a path. For $t > 4$, there are at least two vertices w_p, w_q in W such that $code(w_p) = code(w_q)$. We may assume that $code(w_1) = code(w_2)$. Then for an arbitrary integer i with $3 \leq i \leq t$, let $S = \{w_1, w_2, w_i\}$. There must be a proper path of length 4 connecting S. Suppose that the path is $w_a u_{a'} w_b u_{b'} w_c$, where $\{w_a, w_b, w_c\} = \{w_1, w_2, w_i\}$ and $\{u_{a'}, u_{b'}\} = \{u_1, u_2\}$. By symmetry, we can assume that $u_{a'} = u_1, u_{b'} = u_2$. Then $w_b = w_i$ for otherwise we have $c(w_a u_1) = c(u_1 w_b)$ or $c(w_b u_2) = c(u_2 w_c)$, a contradiction. By symmetry, let $w_a = w_1, w_c = w_2$. Thus $c(w_i u_1) \neq c(w_i u_2)$. Without loss of generality, we can suppose that $c(w_i u_1) = 1$ and $c(w_i u_2) = 2$. Hence, $code(w_i) = (1, 2)$ for each integer $3 \leq i \leq t$. Now let $S = \{w_3, w_4, w_5\}$. It is easy to verify that there is no proper path $w_a u_{a'} w_b u_{b'} w_c$ connecting S, for we always have $c(w_a u_{a'}) = c(u_{a'} w_b), c(w_b u_{b'}) = c(u_{b'} w_c)$. □

Claim 3: Let k be a integer where $k \geq 3$. Then $px_3(K_{2,t}) \leq k$ for $4 < t \leq 2k^2$.

Proof. Set $code(w_1) = (1, 1), code(w_2) = (1, 2), \ldots, code(w_k) = (1, k)$;

$code(w_{k+1}) = (2, 1), code(w_{k+2}) = (2, 2), \ldots, code(w_{2k}) = (2, k)$;

$\ldots$

$code(w_{k(k-1)+1}) = (k, 1), code(w_{k(k-1)+2}) = (k, 2), \ldots, code(w_{k^2}) = (k, k)$

(if each of these vertices exists). And let $code(w_{k^2+i}) = code(w_i)$ for $1 \leq i \leq k^2$ (if each of these vertices exists). Now, we prove that this induces a 3-proper coloring of $K_{2,t}$. First of all, we notice that each code appears at most twice. Let S be a 3-subset of $K_{2,t}$. We consider the following two cases.

Case 1: Let $S = \{w_l, w_m, w_n\}$, where $1 \leq l < m < n \leq t$.

Subcase 1.1: If there is a $j \in \{1, 2\}$ such that the colors of $u_j w_l, u_j w_m, u_j w_n$ are pairwise distinct, then the tree $T = \{u_j w_l, u_j w_m, u_j w_n\}$ is a proper S-tree.

Subcase 1.2: If there is no such j, that is, at least two of the edges $u_j w_l, u_j w_m, u_j w_n$ share the same color for both $j = 1$ and $j = 2$.

i) $code(w_l), code(w_m)$ and $code(w_n)$ are pairwise distinct. Without loss of generality, we suppose that $c(u_1w_l) = c(u_1w_m) = a, c(u_2w_l) = c(u_2v_n) = b$ $(1 \leq a, b \leq k^2)$. Then $c(u_1w_n) \neq c(u_1w_l), c(u_2w_l) \neq c(u_2w_m)$. If $a = b$, then we have $c(u_1w_n) \neq c(w_nu_2)$. So the path $P = w_lu_1w_nu_2w_m$ is a proper S-tree. Otherwise, the path $P = w_nu_1w_lu_2w_m$ is a proper S-tree.

ii) Two of the codes of the vertices in S are the same. Without loss of generality, we assume that $code(w_l) = code(w_m) = (a, b), code(w_n) = (x, y)$ $(1 \leq a, b, x, y \leq k^2)$. Notice that $(x, y) \neq (a, b)$, then suppose that $x \neq a$. Since $k \geq 3$, there are two positive integers $p, q \leq k$ such that $p \neq a, p \neq x$ and $q \neq b, q \neq p$. Pick a vertex w_r whose code is (p, q) (this vertex exists since all of the k^2 codes appear at least once). Then the tree $T = \{u_1w_m, u_1w_n, u_1w_r, w_ru_2, u_2w_l\}$ is a proper S-tree.

Case 2: $S = \{u_r, w_l, w_m\}$, where $1 \leq l < m \leq t$. By symmetry, let $r = 1$.

Suppose that $code(w_l) = (a, b), code(w_m) = (x, y)$ $(1 \leq a, b, x, y \leq k^2)$. If $a \neq x$ then the path $P = w_lu_1w_m$ is a proper S-tree. If $a = x$, then we consider whether $b = y$ or not. We discuss two subcases.

i) $b \neq y$, then at least one of them is not equal to a, assume that $b \neq a$. So the path $P = u_1w_lu_2w_m$ is a proper S-tree.

ii) $b = y$, that is $code(w_l) = code(w_m)$, so all of the k^2 codes appear at least at once. Since $k \geq 3$, there are two positive integers $p, q \leq k$ such that $p \neq a$ and $q \neq b, q \neq p$. Pick a vertex w_r whose code is (p, q). Then the path $P = w_lu_1w_ru_2w_m$ is a proper S-tree.

Case 3: $S = \{u_1, u_2, w_l\}$, where $1 \leq l \leq t$.

Suppose that $code(w_l) = (a, b)$ $(1 \leq a, b \leq k^2)$. If $a \neq b$, then the path $P = u_1w_lu_2$ is a proper S-tree. Otherwise, according to our edge-coloring, there exists a vertex w_r of W with the code (p, q) such that $q \neq a$ and $p \neq q$. Then the path $P = w_lu_2w_ru_1$ is a proper S-tree. □

Claim 4: $px_3(K_{2,t}) > k$ for $t > 2k^2$.

Proof. For any edge-coloring of $K_{2,t}$ with k colors, there must be a code which appears at least three times. Suppose that w_1, w_2, w_3 are the vertices with the same code and set $S = \{w_1, w_2, w_3\}$. Then for any tree T connecting S, there is a $j \in \{1, 2\}$ such that $\{u_jw_l, u_jw_m\} \subseteq E(T)$ for some $\{l, m\} \subseteq \{1, 2, 3\}$, $l \neq m$. But $c(u_jw_l) = c(u_jw_m)$, so T can not be a proper S-tree. Thus $px_3(K_{2,t}) > k$. □

By Claims 2–4, we have the following result: if $5 \leq t \leq 8$, $px_3(K_{2,t}) = 3$; if $t > 8$, let $k = \left\lceil \sqrt{\frac{t}{2}} \right\rceil$, then $3 \leq \sqrt{\frac{t}{2}} \leq k < \sqrt{\frac{t}{2}} + 1$, i.e., $2(k-1)^2 + 1 \leq t \leq 2k^2$, so we have $px_3(K_{2,t}) = k = \left\lceil \sqrt{\frac{t}{2}} \right\rceil$. Notice that $px_3(K_{2,t}) = 3$ for $5 \leq t \leq 18$. □

Theorem 2.2. *For any integer $t \geq 3$, we have*

$$px_3(K_{3,t}) = \begin{cases} 2 & \text{if } 3 \leq t \leq 12; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Let U, W be the two partite sets of $K_{3,t}$, where $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, \dots, w_t\}$. Suppose that there exists a 3-proper coloring $c : E(K_{3,t}) \to \{0, 1, 2, \dots, k-1\}$, $k \in \mathbb{N}$. Analogously to Theorem 2.1, corresponding to the 3-proper coloring, there is a color code(w) assigned to every vertex $w \in W$, consisting of an ordered 3-tuple (a_1, a_2, a_3), where $a_i = c(u_i w) \in \{0, 1, 2, \dots, k-1\}$ for $i = 1, 2, 3$. In turn, if we give each vertex of W a code, then we can induce the corresponding edge-coloring of $K_{3,t}$.

Case 1: $3 \le t \le 8$.

In this part, we give the vertices of W the codes which induce a 3-proper coloring of $K_{3,t}$ with colors 0 and 1. And by application of binary system, we can introduce the assignment of the codes in a clear way. Recall the Abelian group $\mathbb{Z}_2$. We build a bijection $f : \{w_1, w_2, \dots, w_8\} \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $f(w_{4a_1+2a_2+a_3+1}) = (a_1, a_2, a_3)$. For instance, $f(w_3) = (0, 1, 0)$. Under this condition, we use its restriction f_W on W. Now, we prove that f induces a 3-proper coloring of $K_{3,t}$. Let S be an arbitrary 3-subset.

Subcase 1.1: $S = \{w_l, w_m, w_n\}$ for some l, m, n.

Because there is no copy of any code, we can find a vertex in U, say u_1, such that u_1w_l, u_1w_m, u_1w_n are not all with the same color. We may assume that $c(u_1w_l) = c(u_1w_m) = 0$ and $c(u_1w_n) = 1$.

i) *code*$(w_l) = (0, 0, 0)$. Then there is a '1' in the code of w_m. By symmetry, assume that $c(u_2w_m) = 1$. Then there is a proper path $P = w_lu_2w_mu_1w_n$ connecting S.

ii) *code*$(w_l) = (0, 0, 1)$. If *code*$(w_m) = (0, 0, 0)$, then we return to *i)*. Otherwise, the code of w_m is neither $(0, 0, 0)$ nor $(0, 0, 1)$. So $c(u_2w_m) = 1$. Then the proper S-tree is the same as that in *i)*.

iii) *code*$(w_l) = (0, 1, 0)$. It is similar to *ii)*.

iv) *code*$(w_l) = (0, 1, 1)$. Then either $c(u_2w_m) = 0$ or $c(u_3w_m) = 0$. By symmetry, we suppose that $c(u_2w_m) = 0$. Then the path $P = w_mu_2w_lu_1w_n$ is a proper S-tree.

Subcase 1.2: $S = \{u_j, w_l, w_m\}$ for some j, l, m.

If $c(u_jw_l) \ne c(u_jw_m)$, then the path $P = w_lu_jw_m$ is a proper S-tree. Otherwise, by symmetry, we assume that $c(u_jw_l) = c(u_jw_m) = 0$, then there is a $j' \ne j$ such that $c(u_{j'}w_l) \ne c(u_{j'}w_m)$ (otherwise w_l, w_m will have the same code). So one of $c(u_{j'}w_l)$ and $c(u_{j'}w_m)$ equals 1, say $c(u_j'w_l) = 1$. Then the path $P = u_jw_lu_{j'}w_m$ is a proper S-tree.

Subcase 1.3: $S = \{u_{j_1}, u_{j_2}, w_l\}$ for some j_1, j_2, l.

If $c(u_{j_1}w_l) \ne c(u_{j_2}w_l)$, then the path $P = u_{j_1}w_lu_{i_2}$ is a proper S-tree. Otherwise, by symmetry, we assume that $c(u_{j_1}w_l) = c(u_{j_2}w_l) = 0$. By the sequence of the codes according to f and $t \ge 3$, we know that for any two vertices $u_{a'}, u_{b'}$ of U, there exists a vertex $w \in W$ such that $c(u_{a'}w) \ne c(u_{b'}w)$. Similar to Subcase 1.2, we can obtain a proper S-tree.

Subcase 1.4: $S = \{u_1, u_2, u_3\}$.

$P = u_1w_3u_2w_2u_3$ is a proper path connecting S.

Case 2: $9 \leq t \leq 12$.

Set $code(w_1) = (0, 0, 1), code(w_2) = (0, 1, 0), code(w_3) = (0, 1, 1)$,

$code(w_4) = (1, 0, 0), code(w_5) = (1, 0, 1), code(w_6) = (1, 1, 0)$.

And let $code(w_{6+i}) = code(w_i)$ for $1 \leq i \leq 6$ (if each of these vertices exist). For convenience, we denote $w_{6+i} = w'_i$. Now, we claim that this induces a 3-proper coloring of $K_{3,t}$. Let S be an arbitrary 3-subset of $K_{3,t}$. Based on Case 1, we only consider about the case that $\{w_i, w'_i\} \subseteq S$ for some $1 \leq i \leq 6$. By symmetry, we suppose that $i = 1$. First of all, we list three proper paths containing w_1, w'_1: $P_1 = w_1u_3w_2u_2w'_1$, $P_2 = w_1u_2w_3u_1w_4u_3w'_1$ and $P_3 = w_1u_1w_5u_2w_6u_3w'_1$, in which w_j can be replaced by w'_j for $2 \leq j \leq 6$. Then, we can always find a proper path from $\{P_1, P_2, P_3\}$ connecting S whichever the third vertex of S is.

Case 3: $t \geq 13$.

We claim that $px_3(K_{3,t}) = 3$. We prove it by contradiction. If there is a 3-proper coloring of $K_{3,t}$ with two colors 0 and 1, then any proper tree for an arbitrary 3-subset S is in fact a path. Consider the set $S \subseteq W$. As the graph is bipartite and we just care about the shortest proper path connecting S, there are only two possible types of such a path:

I: $w_au_{a'}w_bu_{b'}w_c$

II: $w_au_{a'}w_bu_{b'}w'u_{c'}w_c$

where $\{u_{a'}, u_{b'}, u_{c'}\} = U$ and $\{w_a, w_b, w_c\} = S, w' \in W \setminus S$.

Firstly, as $t \geq 13$, we know that some code appears more than once. But it can not appear more than twice. Otherwise, assume that w_i, w'_i, w''_i are the three vertices with the same code, and let $S = \{w_i, w'_i, w''_i\}$. Whether the proper path connecting S is type I or type II, it should be $c(w_au_{a'}) \neq c(w_bu_{a'})$, contradicting with the assumption that $code(w_i) = code(w'_i) = code(w''_i)$.

Secondly, we prove the following several claims by contradiction.

Claim 1: The repetitive code can not be $(0, 0, 0)$ or $(1, 1, 1)$.

Proof. Suppose that $code(w_1) = code(w_2) = (0, 0, 0)$. Let $S = \{w_1, w_2, w_3\}$ where $w_3 \in W\backslash\{w_1, w_2\}$, and let P be a proper path connecting S. Then w_1, w_2 are the two end vertices of P, and so the two end edges of it are assigned the same color. However, since the length of P is even, the colors of the end edges can not be the same, a contradiction. Analogously, the code $(1, 1, 1)$ cannot appear more than once. □

Claim 2: If the code $(0, 0, 1)$ is repeated, then there is no vertex in W with $(0, 0, 0)$ as its code.

Proof. Suppose that $code(w_1) = code(w_2) = (0, 0, 1), code(w_3) = (0, 0, 0)$. Let $S = \{w_1, w_2, w_3\}$, and let P be a proper path connecting S. Then w_3 is one of the end vertices of P. Moreover, the path P must be type II, for in type I, we need

$c(w_a u_{a'}) \neq c(w_b u_{a'})$ and $c(w_b u_{b'}) \neq c(w_c u_{b'})$, which is impossible for S. We can also deduce that $u_{a'} = u_3$ because $c(w_a u_{a'}) \neq c(w_b u_{a'})$. And $\{w_1, w_2\} \neq \{w_a, w_b\}$ since they are with the same code. So we have $w_a = w_3$. Thus, $\{w_b, w_c\} = \{w_1, w_2\}$ and $\{u_{b'}, u_{c'}\} = \{u_1, u_2\}$, contradicting with the fact that $c(w_b u_{b'}) \neq c(w_c u_{c'})$. □

Analogously, we have that the repetitive code $(0, 1, 0)$ or $(1, 0, 0)$ can not exist along with the code $(0, 0, 0)$, respectively. Symmetrically, the repetitive code $(0, 1, 1), (1, 0, 1)$ or $(1, 1, 0)$ can not exist along with the code $(1, 1, 1)$, respectively.

Finally, as $t \geq 13$ and no code could appear more than twice, there are at least 7 different codes in W and at least 5 codes repeated. But considering Claim 2 and its analogous results, it is a contradiction. So $px_3(K_{3,t}) = 3$ when $t \geq 13$. □

Theorem 2.3. *For a complete bipartite graph $K_{s,t}$ with $t \geq s \geq 4$, we have $px_3(K_{s,t}) = 2$.*

Proof. Let U, W be the two partite sets of $K_{s,t}$, where $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. And denote a cycle $C_s = u_1 w_1 u_2 w_2 \ldots u_s w_s u_1$. Moreover, if $u, v \in V(C_s)$, then we use $uC_s v$ to denote the segment of C_s from u to v in the clockwise direction, and we denote the opposite direction by $uC'_s v$. Then we demonstrate a 3-proper coloring of $K_{s,t}$ with two colors 0 and 1. Let $c(u_i w_i) = 0$ $(1 \leq i \leq s)$ and $c(u_i w_j) = 1$ $(1 \leq i \neq j \leq s)$. And assign $c(w_r u_i) = i \ (mod\ 2)$ $(1 \leq i \leq s, s < r \leq t)$. Now we prove that this coloring is a 3-proper coloring of $K_{s,t}$. Consider a 3-subset S.

i) $S \subseteq V(C_s)$. The proper path is a segment of C_s.

ii) $S = \{w_l, w_m, w_n\}$ where $l, m, n > s$.Then the path $P = w_l u_1 w_1 u_2 w_m u_3 w_3 u_4 w_n$ is a proper S-tree.

iii) $S = \{w_l, w_m, w_n\}$ where $l \leq s, m, n > s$. If $c(w_m u_l) = 1$, then the path $P = w_m u_l w_l C_s u_2 w_n$ is a proper S-tree. If $c(w_m u_l) = 0$, then the proper S-tree is the path $P = w_m u_l w_{l-1} u_{l-1} w_n u_{l-2} C'_s w_l$, where $u_0 = u_s, u_{-1} = u_{s-1}$ if $i_1 = 2$.

iv) $S = \{u_j, w_l, w_m\}$ where $l, m > s$. The way to find a proper S-tree is similar to that in *iii)*.

v) $S = \{u_j, w_l, w_m\}$ where $l \leq s, m > s$. If $c(w_m u_j) = 1$, then the proper S-tree is the path $P = w_m u_j w_j C_s w_l$. If $c(w_m u_j) = 0$, then the path $P = w_m u_j C'_s w_l$ is a proper S-tree.

vi) $S = \{u_{j_1}, u_{j_2}, w_i\}$ where $i > s$. The way to find a proper S-tree is similar to that in *v)*. □

Remarks. Here, we introduce a generalization of k-proper index which was recently proposed by Chang et al. in [5]. Let G be a nontrivial κ-connected graph of order n, and let k and ℓ be two integers with $2 \leq k \leq n$ and $1 \leq \ell \leq \kappa$. For $S \subseteq V(G)$, let $\{T_1, T_2, \ldots, T_\ell\}$ be a set of S-trees. They are *internally disjoint* if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for every pair of distinct integers i, j with $1 \leq i, j \leq \ell$. The (k, ℓ)-proper index of G, denoted by $px_{k,\ell}(G)$, is the minimum number of colors that are required in an edge-coloring of G such that for every k-subset S of $V(G)$, there exist ℓ internally disjoint proper S-trees connecting them. In their paper, they investigated the complete bipartite graphs and obtained the following.

Theorem 2.4. *[5] Let s and t be two positive integers with $t = O(s^r), r \in \mathbb{R}$ and $r \geq 1$. For every pair of integers k, ℓ with $k \geq 3$, there exists a positive integer $N_3 = N_3(k, \ell)$ such that $px_{k,\ell}(K_{s,t}) = 2$ for every integer $s \geq N_3$.*

Obviously, they did not give the exact value of $px_{k,\ell}(K_{s,t})$, even for $k = 3$ and $\ell = 1$. Our Theorem 2.3 completely determines the value of $px_{k,\ell}(K_{s,t})$ for $k = 3$ and $\ell = 1$, without using the condition that $t = O(s^r), r \in \mathbb{R}$ and $r \geq 1$.

3 The 3-proper index of a complete multipartite graph

With the aids of Theorems 2.1, 2.2 and 2.3, we are now able to determine the 3-proper index of all complete multipartite graphs. First of all, we give a useful theorem.

Theorem 3.1. *[8] Let G be a graph with n vertices. If $\delta(G) \geq \frac{n-1}{2}$, then G has a Hamiltonian path (i.e. G is traceable).*

Theorem 3.2. *Let $G = K_{n_1,n_2,\ldots,n_r}$ be a complete multipartite graph, where $r \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_r$. Set $s = \sum_{i=1}^{r-1} n_i$ and $t = n_r$. Then we have*

$$px_3(G) = \begin{cases} 3 & \text{if } G = K_{1,1,t}, 5 \leq t \leq 18 \\ & \quad \text{or } G = K_{1,2,t}, t \geq 13 \\ & \quad \text{or } G = K_{1,1,1,t}, t \geq 15; \\ \left\lceil \sqrt{\frac{t}{2}} \right\rceil & \text{if } G = K_{1,1,t}, t \geq 19; \\ 2 & \text{otherwise.} \end{cases}$$

Proof. The graph G has a $K_{s,t}$ as its spanning subgraph, so it follows from Propositions 1.1 and 1.2 that $2 \leq px_3(G) \leq px_3(K_{s,t})$. In the following, we discuss two cases according to the relationship between s and t.

Case 1: $s \leq t$. Let $U_1, U_2, \ldots, U_r$ denote the different r-partite sets of G, where $|U_i| = n_i$ for each integer $1 \leq i \leq r$.

When $s \geq 4$, then by Theorem 2.3, we have $px_3(G) = px_3(K_{s,t}) = 2$. When $s \leq 3$, there are only three possible values of $(n_1, n_2, \ldots, n_{r-1})$.

Subcase 1: $(n_1, n_2, \ldots, n_{r-1}) = (1, 1)$. Set $U_1 = \{u_1\}, U_2 = \{u_2\}$. Under this condition, giving the edge u_1u_2 an arbitrary color, the proof is exactly the same as that of Theorem 2.1. So it holds that $px_3(G) = px_3(K_{2,t})$.

Subcase 2: $(n_1, n_2, \ldots, n_{r-1}) = (1, 2)$. Set $U_1 = \{u_1\}, U_2 = \{u_2, u_3\}$ and $W = U_r$. By Theorem 2.2, we have $px_3(G) = px_3(K_{3,t}) = 2$ if $t \leq 12$; $px_3(G) \leq px_3(K_{3,t}) = 3$ if $t > 12$. We claim that $px_3(G) = 3$ if $t > 12$. Assume, to the contrary, that G has a 3-proper coloring with two colors 0 and 1. By symmetry, we assume that $c(u_1u_2) = 0$. With the similar reason in Case 3 of the proof of Theorem 2.2, no code can appear more than twice. And recall the bijection f defined in that proof. To label the vertices in W, we use its inverse $f^{-1} : (a_1, a_2, a_3) \mapsto w_{4a_1+2a_2+a_3+1}$, and denote by w'_i the copy of the vertex w_i with $1 \le i \le 8$. Then we prove the following results by contradiction.

Claim 1: $\{w_1, w'_1, w_2\} \nsubseteq W$ and $\{w_2, w'_2, w_1\} \nsubseteq W$.

Proof. Set $S = \{w_1, w'_1, w_2\}$. We know from the proof of Theorem 2.2 that there is no proper path of type I or II. So the proper path P connecting S is type III, defined as $w_a u_{a'} w_b u_{b'} u_{c'} w_c$. Then w_1, w'_1 must be the end vertices of P, and so $w_b = w_2$ and $u_{a'} = u_3$. Since $c(w_a u_{a'}) = 0$, $c(u_{b'} u_{c'}) = 1$, contradicting with $c(u_1 u_2) = 0$. Hence, we obtain $\{w_1, w'_1, w_2\} \nsubseteq W$. Similarly, we have $\{w_2, w'_2, w_1\} \nsubseteq W$. □

Claim 2: $\{w_4, w'_4, w_8\} \nsubseteq W$ and $\{w_8, w'_8, w_4\} \nsubseteq W$.

Proof. Set $S = \{w_4, w'_4, w_8\}$. Similar to Claim 1, any proper path P connecting S should be type III: $w_a u_{a'} w_b u_{b'} u_{c'} w_c$. Then w_8 must be an end vertex of P, and so both of the end edges of P are colored with 1. Thus $u_{a'} = u_1$. Then $\{u_{b'}, u_{c'}\} = \{u_2, u_3\}$ and $c(u_2u_3)$=0, contradicting with the fact that $u_2 u_3 \notin E(G)$. Similarly, we have $\{w_8, w'_8, w_4\} \nsubseteq W$. □

So there are four cases that some vertices can not exist in W at the same time, and each code appears at most twice. However, there are more than 12 vertices in W, a contradiction. So $px_3(G) = px_3(K_{3,t}) = 3$ when $t > 12$.

Subcase 3: $(n_1, n_2, \ldots, n_{r-1}) = (1, 1, 1)$. Set $U = \cup_{j=1}^{r-1} U_j = \{u_1, u_2, u_3\}$ and $W = U_r$.

Claim 3: $px_3(G) = 2$ if $t \le 14$.

Proof. By Theorem 2.2, we have $px_3(G) = px_3(K_{3,t}) = 2$ if $t \le 12$; $px_3(G) \le px_3(K_{3,t}) = 3$ if $t > 12$. When $t = 13$ or 14, we recall $code(w)$ defined in Case 2 of Theorem 2.2. Set

$$code(w_1) = (0, 0, 1), code(w_2) = (0, 1, 0), code(w_3) = (0, 1, 1), code(w_4) = (1, 0, 0),$$
$$code(w_5) = (1, 0, 1), code(w_6) = (1, 1, 0), code(w_7) = (1, 1, 1).$$

And let $code(w_{7+i}) = code(w_i)$ for $1 \le i \le 7$ (if each of these vertices exists) and $c(u_i u_j) = 0$ for $1 \le i \ne j \le 3$. For convenience, we denote $w_{7+i} = w'_i$. Now, we claim that this induces a 3-proper coloring of G. Let S be an arbitrary 3-subset of G. Based on Theorem 2.2, we only consider about the case that $w_7(w'_7) \in S$. When $S = \{w_1, w_7, w'_7\}$, then the path $P = w_7 u_1 w_1 u_3 u_2 w'_7$ is a proper path connecting S. Similarly, we can find a proper path in type III connecting S whichever the two other vertices of S are. □

Claim 4: $px_3(G) = 3$ if $t > 14$.

Proof. Assume, to the contrary, that G has a 3-proper coloring with two colors 0 and 1. If the edges of $G[U]$ are colored with two different colors, then we set u_2

the common vertex of two edges with two different colors. Moreover, without loss of generality, we suppose that $c(u_1u_2) = 0$. Similar to Subcase 2, we have $px_3(G) = 3$ if $t > 12$. If all the edges of $G[U]$ are colored with one color, say 0. Repeat the discussion in Subcase 2, then we know Claim 1 is also true under this condition. As $t \geq 15$ and no code could appear more than twice, there are at least 8 different codes in W and at least 7 codes repeated. But from Claim 1, we know $\{w_1, w_1', w_2\} \nsubseteq W$ and $\{w_2, w_2', w_1\} \nsubseteq W$. So $px_3(G) = 3$ when $t \geq 15$. ☐

Case 2: $s \geq t$. Under this condition, we have $\delta(G) \geq \frac{n-1}{2}$. By Theorem 3.1, we know G is traceable. Thus, it follows from Proposition 1.3 that $px_3(G) = 2$. ☐

4 The k-proper index

Now, we turn to the k-proper index of a complete bipartite graph and a complete multipartite graph for general k. Throughout this section, let k be a fixed integer with $k \geq 3$. Firstly, we generalize Theorem 1.1 to the k-proper index.

Theorem 4.1. *If D is a connected k-dominating set of a connected graph G with minimum degree $\delta(G) \geq k$, then $px_k(G) \leq px_k(G[D]) + 1$.*

Proof. Since D is a connected k-dominating set, every vertex v in $\overline{D}$ has at least k neighbors in D. Let $x = px_k(G[D])$. We first color the edges in $G[D]$ with x different colors from $\{2, 3, \ldots, x+1\}$ such that for every k vertices in D, there exists a proper tree in $G[D]$ connecting them. Then we color the remaining edges with color 1.

Next, we will show that this coloring makes G k-proper connected. Let $S = \{v_1, v_2, \ldots, v_k\}$ be any set of k vertices in G. Without loss of generality, we assume that $\{v_1, \ldots, v_p\} \subseteq D$ and $\{v_{p+1}, \ldots, v_k\} \subseteq \overline{D}$ for some p $(0 \leq p \leq k)$. For each $v_i \in \overline{D}$ $(p+1 \leq i \leq k)$, let u_i be the neighbour of v_i in D such that $\{u_{p+1}, \ldots, u_k\}$ is a $(k-p)$-set. It is possible since D is a k-dominating set. Then the edges $\{u_{p+1}v_{p+1}, \ldots, u_kv_k\}$ together with the proper tree connecting the vertices $\{v_1, \ldots, v_p, u_{p+1}, \ldots, u_k\}$ in $G[D]$ induces a proper S-tree. Thus, we have $px_k(G) \leq px_k(G[D]) + 1$. ☐

Based on this theorem, we can give a lower bound and a upper bound on the k-proper index of a complete bipartite graph, whose proof is similar to Theorem 1.2.

Theorem 4.2. *For a complete bipartite graph $K_{s,t}$ with $t \geq s \geq k$, we have $2 \leq px_k(K_{s,t}) \leq 3$.*

Let G be a complete bipartite graph. Using the techniques in Theorem 2.3, we can obtain the sufficient condition such that $px_k(G) = 2$.

Theorem 4.3. *For a complete bipartite graph $K_{s,t}$ with $t \geq s \geq 2(k-1)$, we have $px_k(K_{s,t}) = 2$.*

Proof. We demonstrate a k-proper coloring of $K_{s,t}$ with two colors 0 and 1, the same as Theorem 2.3. For completeness, we restate the coloring. Let U, W be the two partite sets of $K_{s,t}$, where $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$, $t \geq s \geq 2(k-1)$. Denote a cycle $C_s = u_1w_1u_2w_2 \ldots u_sw_su_1$. Let $c(u_iw_i) = 0$ $(1 \leq i \leq s)$ and $c(u_iw_j) = 1$ $(1 \leq i \neq j \leq s)$. And assign $c(w_ru_i) = i \ (mod\ 2)$ $(1 \leq i \leq s, s < r \leq t)$. Now, we show that for any k-subset $S \subseteq V(K_{s,t})$, there is a proper path P_S connecting all the vertices in S. Set $W_1 = \{w_1, w_2, \ldots, w_s\}$ and $W_2 = \{w_{s+1}, \ldots, w_t\}$ (if $t > s$). Then S can be divided into three parts, i.e., $S = S_1 \cup S_2 \cup S_3$, where $S_1 = S \cap W_1$, $S_2 = S \cap W_2$ and $S_3 = S \cap U$. Suppose $|S_1| = p$, $|S_2| = q$, then $p + q \leq k$. If $q = 0$, the path $P = u_1w_1u_2w_2 \ldots u_sw_s$ is a proper path connecting S. If $q \geq 1$, set $S_2 = \{w_{\alpha_1}, w_{\alpha_2}, \ldots, w_{\alpha_q}\}$, where $s < \alpha_1, \alpha_2, \ldots, \alpha_q \leq t$. Let $P = w_{\alpha_q}u_1w_1u_2w_2 \ldots u_sw_s$. Then consider the vertex set $W'_S = \{w_{2i} : w_{2i} \in W_1 \setminus S_1\}$. We have $|W'_S| \geq s/2 - p \geq k - p - 1 \geq q - 1$. So set $|W'_S| = \ell$ and $W'_S = \{w_{\beta_1}, w_{\beta_2}, \ldots, w_{\beta_{q-1}}, \ldots, w_{\beta_\ell}\}$, where $2 \leq \beta_1, \beta_2, \ldots, \beta_\ell \leq s$ are even. Then we construct a path P_S by replacing the subpath $u_{\beta_j}w_{\beta_j}u_{\beta_j+1}$ of P with $u_{\beta_j}w_{\alpha_j}u_{\beta_j+1}$ (and u_sw_s with $u_sw_{\alpha_j}$ if $\beta_j = s$) for $1 \leq j \leq q-1$. Hence, the new path P_S is a proper path contains all the vertices of U so that P_S connects S_3. By the replacement we know that P_S also connects S_1 as well as S_2. Thus we complete the proof. $\square$

With the aid of Theorems 4.3 and 3.1, we can easily obtain the following, the proof of which is similar to that of Theorem 3.2.

Theorem 4.4. *Let $G = K_{n_1,n_2,\ldots,n_r}$ be a complete multipartite graph, where $r \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_r$. Set $s = \sum_{i=1}^{r-1} n_i$ and $t = n_r$. If $t \geq s \geq 2(k-1)$ or $t \leq s$, then we have $px_k(G) = 2$.*

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(Received 21 Dec 2016)