Balanced Ternary Designs With Holes and Numbers of Common Triples

A. Khodkar
Centre for Combinatorics, Department of Mathematics,
The University of Queensland, Queensland 4072, Australia.

ABSTRACT

There exists a balanced ternary design with block size 3 and index 2 on $2v - \rho_2 + 4$ and $2v - \rho_2 + 1$ elements with a hole of size v, for all positive integers v and ρ_2, such that $v \geq 2\rho_2 + 1$. As an application of this result, we determine the numbers of common triples in two simple balanced ternary designs with block size 3 and index 2, for $\rho_2 = 3$ and 4.

1 Introduction

A *balanced ternary design* (BTD), based on a v-set X, has, as blocks, b multi-subsets of size k, so that each element occurs 0, 1 or 2 times in any block, and occurs altogether r times in the blocks, and each unordered pair of distinct elements occurs λ times altogether. A balanced ternary design is denoted by BTD$(v, b; \rho_1, \rho_2, r; k, \lambda)$, where ρ_i is the number of blocks in which an element occurs exactly i times $(i = 1, 2)$. A BTD is called *simple* if it contains no repeated blocks. It is clear that the parameters of a BTD satisfy

$$vr = bk, \qquad r = \rho_1 + 2\rho_2, \qquad \lambda(v-1) = r(k-1) - 2\rho_2.$$

(See also [1] for further details.) For brevity, in what follows we usually abbreviate the parameter set of BTD to $(v; \rho_2; k, \lambda)$, since from these four parameters we may obtain the remaining ones.

A $(w; \rho_2; k, \lambda)$ BTD with a *hole* of size v, or a frame-BTD $(w[v]; \rho_2; k, \lambda)$, is a collection of multi-sets (blocks) of size k chosen from a w-set W, so that the following conditions hold:

(i) $\{x_i \mid i = 1, 2, 3, \ldots, v\} = V$ is a subset of W called a hole;

(ii) any element in $W \setminus V$ occurs 0, 1 or 2 times per block, and precisely 2 times in ρ_2 blocks;

(iii) at most one (counting repetitions) element of each block is in V;

(iv) any pair ab, where a and b are distinct elements, not both in V, occurs λ times altogether in the blocks.

In this paper we shall concentrate on BTDs with $\lambda = 2$ and $k = 3$. Thus all blocks are of the form abc or aab. It is easy to check that such a design based on a v-set must satisfy the following:

(i) if $\rho_2 \equiv 0 \pmod 3$ then $v \equiv 0$ or $1 \pmod 3$;

(ii) if $\rho_2 \equiv 1 \pmod 3$ then $v \equiv 0 \pmod 3$;

(iii) if $\rho_2 \equiv 2 \pmod 3$ then $v \equiv 0$ or $2 \pmod 3$.

Moreover, since $k = 3$ we have $b \geq \rho_2 v$. So $v \geq 2\rho_2 + 1$.

In section 2, we shall prove the following result:

THEOREM 1. There exists a frame-BTD $((2v - \rho_2 + 1)[v]; \rho_2; 3, 2)$ and a frame-BTD $((2v-\rho_2+4)[v]; \rho_2; 3, 2)$, for all positive integers v and ρ_2 such that $v \geq 2\rho_2+1$.

In section 3, we deal with the numbers of triples common to two simple BTDs. Let $I_{\rho_2}(v)$ denote the set of integers s such that there exist two simple $(v, b; \rho_2; 3, 2)$ BTDs based on a common v-set and having s common blocks. As an application of Theorem 1 we shall determine $I_3(v)$ and $I_4(v)$. The problem for $\rho_2 = 1$ and 2, $I_1(v)$ and $I_2(v)$, has been settled in [2] and [3], respectively.

2 Construction

In this section we prove Theorem 1 and state two corollaries of this theorem. For this, we consider two cases: ρ_2 and v the same parity, and ρ_2 and v not the same parity.

2.1 ρ_2 and v are the same parity

2.1.1 Construction of a frame-BTD $((2v - \rho_2 + 1)[v]; \rho_2; 3, 2)$

Let $v - \rho_2 + 1 = 2m + 1$ and $2K_{2m+1}^{++\dots+}$ denote the complete graph on Z_{2m+1} with two edges joining each pair of vertices and with ρ_2 loops per vertex (denoted by the notation $++\dots+$, ρ_2 times). We can decompose this graph into v 2-factors. Each 2-factor arises from one of the differences $\{\overbrace{0, 0, \dots, 0}^{\rho_2 \text{ times}}, 1, 1, 2, 2, 3, 3, \dots, m, m\}$, where difference 0 corresponds to $2m + 1$ loops. None of these 2-factors contains a repeated edge. Now, let $V = \{x_1, x_2, \dots, x_v\}$. If each $x_i \in V$ forms triangles with the edges from one of the 2-factors, then we obtain a $(2v - \rho_2 + 1; \rho_2; 3, 2)$ BTD with a hole of size v.

2.1.2 Construction of a frame-BTD $((2v - \rho_2 + 4)[v]; \rho_2; 3, 2)$

Let $v - \rho_2 + 4 = 2m$. In this case we use the differences $\{\overbrace{0, 0, \dots, 0}^{\rho_2 \text{ times}}, 1, 2, 2, 3, 3, 4, 4, \dots,$

$m-2, m-2, m-1\}$ to obtain v 2-factors for $2K_{2m}^{++\cdots+}$. The remaining differences $\{1, m-1, m\}$ form $2m$ triangles either from the block $0\ 1\ m$ or the block $0\ 1\ m+1$ cyclically (mod $2m$).

2.2 ρ_2 and v are not the same parity

2.2.1 Construction of a frame-BTD $((2v-\rho_2+1)[v]; \rho_2; 3, 2)$

Let $v-\rho_2+1 = 2m$, and $\{F_1, F_2, F_3, ..., F_{2m-1}\}$ be a 1-factorization of K_{2m}. If we define $G_j = F_j \cup F_{j+1}$ for $1 \le j \le 2m-2$ and $G_{2m-1} = F_{2m-1} \cup F_1$, then $\{G_1, G_2, G_3, ..., G_{2m-1}\}$ is a 2-factorization of $2K_{2m}$ such that none of these G_j's has repeated edges. Now we may use the set $\{\overbrace{0, 0, ..., 0}^{\rho_2\ times}, G_1, G_2, G_3, ..., G_{2m-1}\}$, to obtain v 2-factors for $2K_{2m}^{++\cdots+}$, where as before difference 0 corresponds to $2m$ loops.

2.2.2 Construction of a frame-BTD $((2v-\rho_2+4)[v]; \rho_2; 3, 2)$

Let $v-\rho_2+4 = 2m+1$. In this case, we use the differences $\{\overbrace{0, 0, ..., 0}^{\rho_2\ times}, 1, 2, 3, 4, 4, 5, 5, ..., m, m\}$ to obtain v 2-factors for $2K_{2m+1}^{++\cdots+}$. The remaining differences $\{1, 2, 3\}$ form $2m+1$ triangle either from the block $0\ 1\ 3$ or from the block $0\ 2\ 3$ cyclically (mod $2m+1$). ∎

COROLLARY 1. If there exists a simple $(v; \rho_2; 3, 2)$ BTD, then there exist a simple $(2v-\rho_2+1; \rho_2; 3, 2)$ BTD and a simple $(2v-\rho_2+4; \rho_2; 3, 2)$ BTD.

PROOF. If we place a simple $(v; \rho_2; 3, 2)$ BTD in the holes of the frame-BTDs as described in Theorem 1, then we get the result. ∎

COROLLARY 2. (i) $(v-\rho_2+1)\{0, 1, 2, 3, ..., v-2, v\} + I_{\rho_2}(v) \subseteq I_{\rho_2}(2v-\rho_2+1)$; (ii) $(v-\rho_2+4)\{0, 1, 2, 3, ..., v-2, v\} + I_{\rho_2}(v) + \{0, v-\rho_2+4\} \subseteq I_{\rho_2}(2v-\rho_2+4)$ for every positive integers v and ρ_2, such that $v \ge 2\rho_2+1$.

PROOF. Suppose we have obtained $I_{\rho_2}(v)$ by induction. In order to obtain two designs, based on the same set of elements of size $2v-\rho_2+1$ or $2v-\rho_2+4$, we construct one design as described in Corollary 1, and then obtain a second design with an appropriate number of common blocks, where we may adjust the number of common blocks as follow:

(1) changing the allocation of the elements in the hole of size v;

(2) changing the embedded design in the hole of size v;

(3) in the cases which we described in 2.1.2 and 2.2.2, we may exchange the $v-\rho_2+4$ triples outside the hole, giving a further adjustment of 0 or $v-\rho_2+4$ added to the possible numbers of common blocks.

So we obtain (i) and (ii). ∎

3 The numbers of common blocks

In this section, we shall prove the following theorem by using Corollary 2 and induction.

THEOREM 2. (i) $I_3(v) = \{0, 1, 2, 3, ..., v(v+2)/3\} - \{v(v+2)/3-1, v(v+2)/3-2\}$ for $v \geq 9$ and $I_3(7) = \{0, 3, 4, 5, ..., 17, 18, 21\}$;
(ii) $I_4(v) = \{0, 1, 2, 3, ..., v(v+3))/3\} - \{v(v+3)/3 - 1, v(v+3)/3 - 2\}$ for $v \geq 12$ and $I_4(9) = \{0, 3, 4, 5, ..., 32, 33, 36\}$.

REMARK 1. It is impossible to have two $(v; \rho_2; 3, 2)$ BTDs based on the same v-set which have all but one block the same, or all but two blocks the same. So $\{v(v+2)/3-1, v(v+2)/3-2\} \not\subset I_3(v)$ and $\{v(v+3)/3-1, v(v+3)/3-2\} \not\subset I_4(v)$.

REMARK 2. Let D be a simple $(2\rho_2+1; \rho_2; 3, 2)$ BTD on the set X. Since D has $\rho_2(2\rho_2+1)$ blocks, all of them are of the form aab. Now if $B = \{xxy \mid x \neq y,\ x \text{ and } y \in X\}$, then B is a trivial $(2\rho_2+1; 2\rho_2; 3, 4)$ BTD. So $D' = B \setminus D$ is also a simple $(2\rho_2+1; \rho_2; 3, 2)$ BTD. Now let D_1 and D_2 be two simple $(2\rho_2+1; \rho_2; 3, 2)$ BTDs on the set X, such that $\mid D_1 \cap D_2 \mid = 1$ or 2, then $\mid D_1 \cap (B - D_2) \mid = \rho_2(2\rho_2+1) - 1$ or $\rho_2(2\rho_2+1) - 2$, respectively. And this is a contradiction by Remark 1. Thus 1 and $2 \notin I_{\rho_2}(2\rho_2+1)$.

Now we are ready to prove Theorem 2. This theorem is proved by induction. For the induction to work, we need v sufficiently large; in 3.1 and 3.2 we deal with $v = 7, 9, 10, 12, 13, 15$ when $\rho_2 = 3$, and $v = 9, 12, 15$ and 18 when $\rho_2 = 4$, and for large v we use Theorem 1. For brevity, we use the notation $T_{i_1 i_2 i_3 ... i_k}$ instead of $T_{i_1} \cup T_{i_2} \cup T_{i_3} \cup ... \cup T_{i_k}$ and we define $a = 10$, $b = 11$, $c = 12$, $d = 13$, $e = 14$, $f = 15$, $g = 16$, $h = 17$, $J_3(v) = \{0, 1, 2, 3, ..., v(v+2)/3\} - \{v(v+2)/3 - 1, v(v+2)/3 - 2\}$ and $J_4(v) = \{0, 1, 2, 3, ..., v(v+3)/3\} - \{v(v+3)/3 - 1, v(v+3)/3 - 2\}$.

3.1 The cases $v = 7, 9, 10, 12, 13, 15$ and $\rho_2 = 3$

3.1.1 The case $v = 7$

Let D and D' be two simple $(7; 3; 3, 2)$ BTDs which we obtain from initial blocks 001, 002, 003 and 006, 005, 004 cyclically (mod 7), respectively. We define $T_1 = \{001, 114, 440\}$, $T_2 = \{002, 225, 556, 660\}$, $T_3 = \{112, 223, 334, 445, 551\}$, $T_4 = \{224, 446, 662\}$ and $T_i' = \{aab \mid bba \in T_i\}$ for $1 \leq i \leq 4$. Now by Table 3.1 and Remark 2 we find $\{0, 3, 4, ..., 18, 21\} \subseteq I_3(7)$. Moreover, from Remark 1, we obtain 1, $2 \notin I_3(7)$. So $I_3(7) = J_3(7) \setminus \{1, 2\}$.

$\|D \cap D'\| = 0$	$\|[(D - T_{12}) \cup T'_{12}] \cap D'\| = 7$
$\|[(D - T_1) \cup T'_1] \cap D'\| = 3$	$\|[(D - T_{13}) \cup T'_{13}] \cap D'\| = 8$
$\|[(D - T_2) \cup T'_2] \cap D'\| = 4$	$\|[(D - T_{23}) \cup T'_{23}] \cap D'\| = 9$
$\|[(D - T_3) \cup T'_3] \cap D'\| = 5$	$\|[(D - T_{124}) \cup T'_{124}] \cap D'\| = 10$
$\|[(D - T_{14}) \cup T'_{14}] \cap D'\| = 6$	

Table 3.1

3.1.2 The case $v = 9$

Let D_1, D_2 and D_3 be the designs numbered 1, 2 and 3 in the Appendix, respectively. We define $T_1 = \{007, 775, 550\}$, $T_2 = \{004, 448, 880\}$, $T_3 = \{115, 553, 338, 881\}$, $T_4 = \{662, 227, 774, 446\}$, $T_5 = \{667, 778, 886\}$, $T_6 = \{007, 774, 446, 665, 550\}$, $S_1 = \{117{,}775{,}551\}$, $S_2 = \{113, 338, 881\}$, $S_3 = \{778, 880, 007\}$, $T'_i = \{aab|\ bba \in T_i\}$ for $1 \le i \le 6$ and $S'_j = \{aab|\ bba \in S_j\}$ for $1 \le j \le 3$. We now have the following common block numbers:

$\|D_1 \cap D_2\| = 0$	$\|[(D_3 - S_1) \cup S'_1] \cap D_1\| = 16$
$\|[(D_1 - T_1) \cup T'_1] \cap D_2\| = 1$	$\|D_3 \cap D_1\| = 17$
$\|[(D_1 - T_2) \cup T'_2] \cap D_2\| = 2$	$[(D_1 - T_{2356}) \cup T'_{2356}] \cap D_1\| = 18$
$\|[(D_1 - T_{12}) \cup T'_{12}] \cap D_2\| = 3$	$\|[(D_1 - T_{1234}) \cup T'_{1234}] \cap D_1\| = 19$
$\|[(D_1 - T_3) \cup T'_3] \cap D_2\| = 4$	$\|[(D_1 - T_{1235}) \cup T'_{1235}] \cap D_1\| = 20$
$\|[(D_1 - T_{13}) \cup T'_{13}] \cap D_2\| = 5$	$\|[(D_1 - T_{236}) \cup T'_{236}] \cap D_1\| = 21$
$\|[(D_1 - T_{23}) \cup T'_{23}] \cap D_2\| = 6$	$\|[(D_1 - T_{234}) \cup T'_{234}] \cap D_1\| = 22$
$\|[(D_1 - T_{123}) \cup T'_{123}] \cap D_2\| = 7$	$\|[(D_1 - T_{123}) \cup T'_{123}] \cap D_1\| = 23$
$\|[(D_1 - T_{34}) \cup T'_{34}] \cap D_2\| = 8$	$\|[(D_1 - T_{125}) \cup T'_{125}] \cap D_1\| = 24$
$\|[(D_1 - T_{134}) \cup T'_{134}] \cap D_2\| = 9$	$\|[(D_1 - T_{34}) \cup T'_{34}] \cap D_1\| = 25$
$\|[(D_1 - T_{234}) \cup T'_{234}] \cap D_2\| = 10$	$\|[(D_1 - T_{13}) \cup T'_{13}] \cap D_1\| = 26$
$\|[(D_1 - T_{1234}) \cup T'_{1234}] \cap D_2\| = 11$	$\|[(D_1 - T_{12}) \cup T'_{12}] \cap D_1\| = 27$
$\|[(D_3 - S_{23}) \cup S'_{23}] \cap D_1\| = 12$	$\|[(D_1 - T_6) \cup T'_6] \cap D_1\| = 28$
$\|[(D_3 - S_{13}) \cup S'_{13}] \cap D_1\| = 13$	$\|[(D_1 - T_3) \cup T'_3] \cap D_1\| = 29$
$\|[(D_3 - S_3) \cup S'_3] \cap D_1\| = 14$	$\|[(D_1 - T_1) \cup T'_1] \cap D_1\| = 30$
$\|[(D_3 - S_2) \cup S'_2] \cap D_1\| = 15$	$\|D_1 \cap D_1\| = 33$

Table 3.2

So we obtain $I_3(9) = J_3(9)$.

3.1.3 The case $v = 10$

Let D_1, D_2 and D_3 be the designs numbered 4, 5 and 6 in the Appendix, respectively. As before, we define $T_1 = \{114{,}440{,}001\}$, $T_2 = \{224{,}449{,}993{,}332\}$, $T_3 = \{559{,}996{,}662{,}221{,}115\}$, $T_4 = \{775{,}550{,}003{,}331{,}116{,}667\}$, $T_5 = \{448{,}883{,}337{,}774\}$, $T_6 = \{886{,}660{,}002{,}225{,}558\}$, $S_1 = \{007{,}772{,}220\}$, $S_2 = \{660{,}008{,}886\}$, $S_3 =\{773{,}331,$

119,992,221,118,889,993,330,009,997}, $S_4 = \{226,663,338,882\}$, $S_5 = \{774,440,003, 337\}$, $S_6 = \{448,883,332,224\}$, $S_7 = \{225,550,002\}$, $T_i' = \{aab |\ bba \in T_i\}$ for $1 \le i \le 6$ and $S_j' = \{aab |\ bba \in S_j\}$ for $1 \le j \le 7$. Now, by Table 3.3 we find $I_3(10) = J_3(10)$.

$\lvert D_3 \cap D_2\rvert = 0$	$\lvert[(D_1 - T_{2346}) \cup T'_{2346}] \cap D_1\rvert = 20$
$\lvert[(D_3 - S_1) \cup S'_1] \cap D_2\rvert = 1$	$\lvert[(D_1 - T_{1346}) \cup T'_{1346}] \cap D_1\rvert = 21$
$\lvert[(D_3 - S_2) \cup S'_2] \cap D_2\rvert = 2$	$\lvert[(D_1 - T_{1234}) \cup T'_{1234}] \cap D_1\rvert = 22$
$\lvert[(D_3 - S_{12}) \cup S'_{12}] \cap D_2\rvert = 3$	$\lvert[(D_1 - T_{1236}) \cup T'_{1236}] \cap D_1\rvert = 23$
$\lvert[(D_3 - S_4) \cup S'_4] \cap D_2\rvert = 4$	$\lvert[(D_1 - T_{346}) \cup T'_{346}] \cap D_1\rvert = 24$
$\lvert[(D_3 - S_{14}) \cup S'_{14}] \cap D_2\rvert = 5$	$\lvert[(D_1 - T_{234}) \cup T'_{234}] \cap D_1\rvert = 25$
$\lvert[(D_3 - S_{24}) \cup S'_{24}] \cap D_2\rvert = 6$	$\lvert[(D_1 - T_{134}) \cup T'_{134}] \cap D_1\rvert = 26$
$\lvert[(D_3 - S_{34} \cup S'_{34}] \cap D_2\rvert = 7$	$\lvert[(D_1 - T_{124}) \cup T'_{124}] \cap D_1\rvert = 27$
$\lvert[(D_3 - S_{134}) \cup S'_{134}] \cap D_2\rvert = 8$	$\lvert[(D_1 - T_{123}) \cup T'_{123}] \cap D_1\rvert = 28$
$\lvert[(D_3 - S_{234}) \cup S'_{234}] \cap D_2\rvert = 9$	$\lvert[(D_1 - T_{34}) \cup T'_{34}] \cap D_1\rvert = 29$
$\lvert[(D_3 - S_{1234}) \cup S'_{1234}] \cap D_2\rvert = 10$	$\lvert[(D_1 - T_{24}) \cup T'_{24}] \cap D_1\rvert = 30$
$\lvert[(D_1 - S_5) \cup S'_5] \cap D_2\rvert = 11$	$\lvert[(D_1 - T_{14}) \cup T'_{14}] \cap D_1\rvert = 31$
$\lvert D_1 \cap D_2\rvert = 12$	$\lvert[(D_1 - T_{13}) \cup T'_{13}] \cap D_1\rvert = 32$
$\lvert[(D_1 - S_6) \cup S'_6] \cap D_2\rvert = 13$	$\lvert[(D_1 - T_{12}) \cup T'_{12}] \cap D_1\rvert = 33$
$\lvert[(D_1 - S_7) \cup S'_7] \cap D_2\rvert = 14$	$\lvert[(D_1 - T_4) \cup T'_4] \cap D_1\rvert = 34$
$\lvert[(D_1 - S_{67}) \cup S'_{67}] \cap D_2\rvert = 15$	$\lvert[(D_1 - T_3) \cup T'_3] \cap D_1\rvert = 35$
$\lvert[(D_1 - T_{23456}) \cup T'_{23456}] \cap D_1\rvert = 16$	$\lvert[(D_1 - T_2) \cup T'_2] \cap D_1\rvert = 36$
$\lvert[(D_1 - T_{13456}) \cup T'_{13456}] \cap D_1\rvert = 17$	$\lvert[(D_1 - T_1) \cup T'_1] \cap D_1\rvert = 37$
$\lvert[(D_1 - T_{12456}) \cup T'_{12456}] \cap D_1\rvert = 18$	$\lvert D_1 \cap D_1\rvert = 40$
$\lvert[(D_1 - T_{12356}) \cup T'_{12356}] \cap D_1\rvert = 19$	

Table 3.3

3.1.4 The case $v = 12$

By Corollary 2(i) we see $5.\{0,1,2,3,4,5,7\} + \{0,3,4,5,\ldots,17,18,21\} \subseteq I_3(12)$. So it remains to show that $\{1,2\} \subset I_3(12)$. For this, if D_1 and D_2 are the designs number 7 and 8 in the Appendix, respectively, $T = \{11b, bb9, 996, 661\}$ and $T' = \{bb1, 99b, 669, 116\}$, then $|D_1 \cap D_2| = 2$ and $|[(D_1 - T) \cup T'] \cap D_2| = 1$. So we obtain $I_3(12) = J_3(12)$.

3.1.5 The case $v = 13$

First we decompose $2K_6^{+++}$ into eight 2-factors:

0: 00, 11, 22, 33, 44, 55;
0: 00, 11, 22, 33, 44, 55;
0: 00, 11, 22, 33, 44, 55;
F_1: (0,1,4,5,2,3);
F_2: (0,3,4,1,2,5);
F_3: (0,1,2,3,4,5);
F_4: (0,2,4),(1,3,5);
F_5: (0,2,4),(1,3,5).

Secondly, we take a copy of a design of order seven on the set $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. If we let $j \in I_3(7)$, then the following permutations give possible assignments of the seven elements $\{x_i\}$ to the first seven 2-factors of $2K_6^{+++}$. (Here, we use only seven 2-factors $\{0, 0, 0, F_1, F_2, F_3, F_4\}$, and add F_5 to the blocks obtained each time.)

							number of common blocks
0	0	0	F_1	F_2	F_3	F_4	$44 + j$
F_1	0	0	0	F_2	F_3	F_4	$32 + j$
F_4	0	0	F_1	F_2	0	F_3	$26 + j$
F_1	F_2	0	0	0	F_3	F_4	$20 + j$
F_4	F_1	0	0	F_2	0	F_3	$14 + j$
F_1	F_2	F_3	0	0	0	F_4	$8 + j$
F_4	F_1	F_2	0	0	0	F_3	$2 + j$

So we get $\{2, 8, 14, 20, 26, 32, 44\} + I_3(7) \subseteq I_3(13)$. Thus, it remains to show that $\{0, 1, 3, 4\} \subset I_3(13)$. For this, let D_1 and D_2 be the designs numbered 9 and 10 in the Appendix, respectively. Moreover, let $T_1 = \{11c, cc8, 881\}$, $T_2 = \{22c, cc5, 559, 991, 116, 662\}$ and $T_i' = \{aab|\ bba \in T_i\}$, for $i = 1$ and 2. We find that, $|D_1 \cap D_2| = 0$, $|[(D_1 - T_1) \cup T_1']) \cap D_2| = 1$, $|[(D_1 - T_2) \cup T_2'] \cap D_2| = 3$ and $|[(D_1 - T_{12}) \cup T_{12}'] \cap D_2| = 4$. So we obtain $I_3(13) = J_3(13)$.

3.1.6 The case $v = 15$

By Corollary 2(ii) we obtain $8.\{0, 1, 2, 3, 4, 5, 7\} + I_3(7) + \{0, 8\} \subseteq I_3(15)$. So it remains to show that $\{1, 2\} \subset I_3(15)$. For this, let D_1 and D_2 be the designs numbered 11 and 12 in the Appendix, respectively, $T = \{009, 99d, dd0\}$ and $T' = \{990, dd9, 00d\}$. Then $|D_1 \cap D_2| = 1$ and $|[(D_2 - T) \cup T'] \cap D_1| = 2$. Thus $I_3(15) = J_3(15)$.

3.2 The cases $v = 9$, 12, 15, 18 and $\rho_2 = 4$

3.2.1 The case $v = 9$

Let D and D' be the two simple (9;4;3,2) BTDs which we obtain from initial blocks 001, 002, 003, 004 and 008, 007, 006, 005 cyclically (mod 9) respectively. We define $T_1 = \{112, 226, 661\}$, $T_2 = \{002, 223, 336, 660\}$, $T_3 = \{003, 334, 445, 557, 770\}$, $T_4 = \{004, 447, 771, 113, 335, 550\}$, $T_5 = \{001, 114, 448, 883, 337, 778, 880\}$ and $T'_i = \{aab \mid bba \in T_i\}$ for $1 \le i \le 5$. From this information, we obtain the following common block numbers:

$\lvert D \cap D' \rvert = 0$	$\lvert[(D - T_{34}) \cup T'_{34}] \cap D'\rvert = 11$
$\lvert[(D - T_1) \cup T'_1] \cap D'\rvert = 3$	$\lvert[(D - T_{35}) \cup T'_{35}] \cap D'\rvert = 12$
$\lvert[(D - T_2) \cup T'_2] \cap D'\rvert = 4$	$\lvert[(D - T_{45}) \cup T'_{45}] \cap D'\rvert = 13$
$\lvert[(D - T_3) \cup T'_3] \cap D'\rvert = 5$	$\lvert[(D - T_{125}) \cup T'_{125}] \cap D'\rvert = 14$
$\lvert[(D - T_4) \cup T'_4] \cap D'\rvert = 6$	$\lvert[(D - T_{135}) \cup T'_{135}] \cap D'\rvert = 15$
$\lvert[(D - T_5) \cup T'_5] \cap D'\rvert = 7$	$\lvert[(D - T_{235}) \cup T'_{235}] \cap D'\rvert = 16$
$\lvert[(D - T_{13}) \cup T'_{13}] \cap D'\rvert = 8$	$\lvert[(D - T_{245}) \cup T'_{245}] \cap D'\rvert = 17$
$\lvert[(D - T_{23}) \cup T'_{23}] \cap D'\rvert = 9$	$\lvert[(D - T_{345}) \cup T'_{345}] \cap D'\rvert = 18$
$\lvert[(D - T_{24}) \cup T'_{24}] \cap D'\rvert = 10$	

Table 3.4

Also by Remark 2 we obtain $\{19, 20, 21, ..., 33, 36\} \subseteq I_4(9)$. And from Remark 1 we have 1 and $2 \notin I_4(9)$. So $I_4(9) = J_4(9) \setminus \{1, 2\}$.

3.2.2 The case $v = 12$

Let D_1 be a simple (12;4;3,2) BTD which we obtain from initial blocks 001, 002, 004, 005 and 039 cyclically (mod 12), and let D_2 be a design number 13 in the Appendix. We define $T_1 = \{002,227,770\}$, $T_2 = \{001,113,338,880\}$, $T_3 = \{004,445,559,99a, aa0\}$, $T_4 = \{005,556,667,779,99b, bb0\}$, $T_5 = \{112,223,334,446,668, 889,991\}$, $T_6 = \{115, 557, 77b, bb1, 448, 88a, aa2, 224\}$, $T_7 = \{335, 55a, aa3, 449, 992, 226, 66a, aab, bb4\}$, $S_1 = \{668, 88a, aa2, 224, 446\}$, $S_2 = \{aa2, 224, 445, 55a\}$, $S_3 = \{55a, aa3, 335\}$, $S_4 = \{559,991,115\}$, $S_5 = \{334,448,881,113\}$, $S_6 = \{116, 66a, aab, bb1\}$, $S_7 = \{002,226,667,77b, bb3, 338, 880\}$ and $S_8 = \{227,770,001,112\}$. Also, $T'_i = \{aab \mid bba \in T_i\}$, for $1 \le i \le 7$ and $S'_j = \{aab \mid bba \in S_j\}$, for $1 \le j \le 8$. Now, from Table 3.5 we obtain $I_4(12) = J_4(12)$.

3.2.3 The case $v = 15$

By Corollary 2(i) we obtain $6.\{0, 1, 2, ..., 7, 9\} + I_4(9) \subseteq I_4(15)$. So if we show that 1 and $2 \in I_4(15)$, then $I_4(15) = J_4(15)$. For this, let D_1 and D_2 be two simple (15;4;3,2) BTDs, which we obtain from initial blocks 004, 005, 006, 007, 013, 023 and 002, 00a, 009, 008, 014, 034 cyclically (mod 15), respectively. Now if $T_1 = \{007, 77b, bb0\}$, $T_2 = \{229, 991, 115, 559, 99d, dd2\}$ and $T'_i = \{aab \mid bba \in T_i\}$ for $i = 1$ and 2, then $\lvert[(D_1 - T_1) \cup T'_1] \cap D_2\rvert = 1$ and $\lvert[(D_1 - T_2) \cup T'_2] \cap D_2\rvert = 2$. So this case is also completed.

$\|[(D_1 - S_1) \cup S_1'] \cap D_2\| = 0$	$\|[(D_1 - T_{4567}) \cup T_{4567}'] \cap D_1\| = 30$
$\|[(D_1 - S_2) \cup S_2'] \cap D_2\| = 1$	$\|[(D_1 - T_{3567}) \cup T_{3567}'] \cap D_1\| = 31$
$\|D_1 \cap D_2\| = 2$	$\|[(D_1 - T_{2567}) \cup T_{2567}'] \cap D_1\| = 32$
$\|[(D_1 - S_3) \cup S_3'] \cap D_2\| = 3$	$\|[(D_1 - T_{2467}) \cup T_{2467}'] \cap D_1\| = 33$
$\|[(D_1 - S_4) \cup S_4'] \cap D_2\| = 4$	$\|[(D_1 - T_{2367}) \cup T_{2367}'] \cap D_1\| = 34$
$\|[(D_1 - S_5) \cup S_5'] \cap D_2\| = 5$	$\|[(D_1 - T_{1367}) \cup T_{1367}'] \cap D_1\| = 35$
$\|[(D_1 - S_6) \cup S_6'] \cap D_2\| = 6$	$\|[(D_1 - T_{567}) \cup T_{567}'] \cap D_1\| = 36$
$\|[(D_1 - S_{36}) \cup S_{36}'] \cap D_2\| = 7$	$\|[(D_1 - T_{467}) \cup T_{467}'] \cap D_1\| = 37$
$\|[(D_1 - S_{46}) \cup S_{46}'] \cap D_2\| = 8$	$\|[(D_1 - T_{367}) \cup T_{367}'] \cap D_1\| = 38$
$\|[(D_1 - S_7) \cup S_7'] \cap D_2\| = 9$	$\|[(D_1 - T_{357}) \cup T_{357}'] \cap D_1\| = 39$
$\|[(D_1 - S_{37}) \cup S_{37}'] \cap D_2\| = 10$	$\|[(D_1 - T_{347}) \cup T_{347}'] \cap D_1\| = 40$
$\|[(D_1 - S_{47}) \cup S_{47}'] \cap D_2\| = 11$	$\|[(D_1 - T_{247}) \cup T_{247}'] \cap D_1\| = 41$
$\|[(D_1 - S_{57}) \cup S_{57}'] \cap D_2\| = 12$	$\|[(D_1 - T_{147}) \cup T_{147}'] \cap D_1\| = 42$
$\|[(D_1 - S_{67}) \cup S_{67}'] \cap D_2\| = 13$	$\|[(D_1 - T_{67}) \cup T_{67}'] \cap D_1\| = 43$
$\|[(D_1 - S_{457}) \cup S_{457}'] \cap D_2\| = 14$	$\|[(D_1 - T_{57}) \cup T_{57}'] \cap D_1\| = 44$
$\|[(D_1 - S_{467}) \cup S_{467}'] \cap D_2\| = 15$	$\|[(D_1 - T_{47}) \cup T_{47}'] \cap D_1\| = 45$
$\|[(D_1 - S_{567}) \cup S_{567}'] \cap D_2\| = 16$	$\|[(D_1 - T_{46}) \cup T_{46}'] \cap D_1\| = 46$
$\|[(D_1 - S_{678}) \cup S_{678}'] \cap D_2\| = 17$	$\|[(D_1 - T_{45}) \cup T_{45}'] \cap D_1\| = 47$
$\|[(D_1 - S_{4567}) \cup S_{4567}'] \cap D_2\| = 18$	$\|[(D_1 - T_{35}) \cup T_{35}'] \cap D_1\| = 48$
$\|[(D_1 - S_{4678}) \cup S_{4678}'] \cap D_2\| = 19$	$\|[(D_1 - T_{25}) \cup T_{25}'] \cap D_1\| = 49$
$\|[(D_1 - S_{5678}) \cup S_{5678}'] \cap D_2\| = 20$	$\|[(D_1 - T_{15}) \cup T_{15}'] \cap D_1\| = 50$
$\|[(D_1 - T_{234567}) \cup T_{234567}'] \cap D_1\| = 21$	$\|[(D_1 - T_7) \cup T_7'] \cap D_1\| = 51$
$\|[(D_1 - T_{134567}) \cup T_{134567}'] \cap D_1\| = 22$	$\|[(D_1 - T_6) \cup T_6'] \cap D_1\| = 52$
$\|[(D_1 - T_{124567}) \cup T_{124567}'] \cap D_1\| = 23$	$\|[(D_1 - T_5) \cup T_5'] \cap D_1\| = 53$
$\|[(D_1 - T_{123567}) \cup T_{123567}'] \cap D_1\| = 24$	$\|[(D_1 - T_4) \cup T_4'] \cap D_1\| = 54$
$\|[(D_1 - T_{34567}) \cup T_{34567}'] \cap D_1\| = 25$	$\|[(D_1 - T_3) \cup T_3'] \cap D_1\| = 55$
$\|[(D_1 - T_{24567}) \cup T_{24567}'] \cap D_1\| = 26$	$\|[(D_1 - T_2) \cup T_2'] \cap D_1\| = 56$
$\|[(D_1 - T_{14567}) \cup T_{14567}'] \cap D_1\| = 27$	$\|[(D_1 - T_1) \cup T_1'] \cap D_1\| = 57$
$\|[(D_1 - T_{13567}) \cup T_{13567}'] \cap D_1\| = 28$	$\|D_1 \cap D_1\| = 60$
$\|[(D_1 - T_{13467}) \cup T_{13467}'] \cap D_1\| = 29$	

Table 3.5

3.2.4 The case $v = 18$

Let D_1 and D_2 be designs numbered 14 and 15 in the Appendix, respectively. If $T = \{ddg, gg3, 33h, hh1, 11d\}$ and $T' = \{ggd, 33g, hh3, 11h, dd1\}$, then $|D_1 \cap D_2| = 2$ and $|[(D_1 - T) \cup T'] \cap D_2| = 1$. Moreover, by Corollary 2(ii) we have 9.{0, 1, 2,...,7, 9} + $I_4(9)$ + {0, 9} $\subseteq I_4(18)$. So $I_4(18) = J_4(18)$.

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5 References

1. Elizabeth J. Billington, *Designs with repeated elements in blocks: a survey and some recent results*, Congressus Numerantium **68** (1989), 123–146.

2. Elizabeth J. Billington and D.G. Hoffman, *Pairs of simple balanced ternary designs with prescribed numbers of triples in common*, Australas. J. Combin. **5** (1992), 59–71.

3. Elizabeth J. Billington and E.S. Mahmoodian, *Multi-set designs and numbers of common triples* (submitted)

No.	Designs											
1	004	115	225	336	446	553	662	775	880	007	116	227
	337	445	550	665	774	881	006	117	228	338	448	558
	667	778	886	012	023	013	412	423	413			
2	001	112	223	331	440	552	661	771	882	005	114	220
	334	442	551	663	772	883	003	118	226	335	447	554
	664	773	884	067	078	068	567	578	568			
3	003	110	221	337	441	553	663	775	880	006	117	227
	338	443	551	667	774	881	007	113	228	332	448	558
	661	778	886	025	045	024	625	645	624			
4	001	114	221	331	440	550	660	771	881	991	002	115
	224	332	449	559	662	774	883	993	003	116	225	337
	448	558	667	775	886	996	078	079	089	278	279	289
	345	356	346	456								
5	004	114	227	339	442	552	662	774	884	994	005	115
	228	336	443	554	665	775	885	995	006	116	229	337
	446	553	668	776	883	996	078	079	089	178	179	189
	012	013	023	123								
6	007	110	220	330	440	550	660	772	882	992	008	118
	221	338	448	558	663	773	886	993	009	119	226	331
	449	559	669	778	889	997	416	417	467	516	517	567
	234	245	235	345								
7	009	114	225	334	448	551	661	770	880	992	aa5	bb2
	00a	11a	226	339	449	557	663	773	886	996	aa6	bb6
	00b	11b	228	33a	44b	559	667	77b	88a	99a	aab	bb9
	012	023	013	123	045	056	046	456	178	189	179	789
	247	27a	24a	47a	358	38b	35b	58b				
8	003	11b	224	332	445	559	665	775	882	990	aa1	bb0
	006	115	227	331	443	558	664	778	884	994	aa9	bba
	008	118	22b	335	44a	55b	66b	776	88b	998	aa7	bb4
	014	047	017	147	025	05a	02a	25a	126	169	129	269
	368	38a	36a	68a	379	39b	37b	79b				
9	00a	115	224	33a	443	553	662	772	883	991	aa2	bb1
	cc8	00b	116	229	33b	44b	557	668	776	884	994	aa5
	bba	cc5	00c	11c	22c	337	44c	559	66b	77b	881	99b
	aa8	bbc	cc7	012	023	013	123	045	056	046	456	078
	089	079	789	147	17a	14a	47a	258	28b	25b	58b	369
	36c	39c	a69	a6c	a9c							
10	001	11b	221	334	440	556	667	778	889	995	aab	bb8
	cc0	003	114	225	33c	447	554	663	77a	88c	99c	aa0
	bb7	cc2	009	11a	226	332	448	551	66a	773	885	996
	aa9	bb2	ccb	028	139	24a	35b	46c	570	681	792	8a3
	9b4	ac5	b06	c17	027	138	249	35a	46b	57c	680	791
	8a2	9b3	ac4	b05	c16							

No.	Designs											
11	00c	11c	228	334	442	552	661	773	884	991	aa0	bb0
	cc5	dd8	ee3	00d	11d	22a	33b	449	557	66d	776	88c
	99b	aa9	bb6	cc7	dda	ee5	00e	11e	22e	33c	44d	55a
	66e	77e	88e	99e	aac	bba	ccd	dde	eec	012	013	023
	123	045	046	056	456	078	079	089	789	147	14a	17a
	47a	158	15b	18b	58b	269	26c	29c	69c	27b	27d	2bd
	7bd	368	36a	38a	68a	359	35d	39d	59d	4bc	4be	4ce
	bce											
12	009	11e	22b	337	446	558	665	771	882	996	aa4	bb4
	cc0	dd0	ee8	00a	119	22c	33a	447	559	66b	77a	889
	99a	aa5	bbd	cc9	dd1	ee2	00b	11b	22d	33d	44c	55b
	66c	77c	883	99d	aad	bb3	cce	ddc	eea	014	018	048
	148	025	027	057	257	036	03e	06e	36e	126	12a	16a
	26a	135	13c	15c	35c	234	239	249	349	45d	45e	4de
	5de	678	67d	68d	78d	79b	79e	7be	9be	8ab	8ac	8bc
	abc											
13	007	110	220	330	440	550	660	771	882	990	aa1	bba
	008	118	221	331	441	551	661	772	883	993	aa4	bb6
	00a	119	229	33a	447	558	662	773	884	994	aa6	bb7
	00b	11b	22b	33b	44b	55b	668	776	88b	996	aa2	bb9
	234	235	245	634	635	645	579	57a	59a	879	87a	89a
14	00e	11d	22d	337	443	55f	665	774	884	995	aa3	bb6
	cc3	dd0	ee2	ffd	gg7	hh1	00f	11e	22f	335	44f	55e
	667	77e	88g	99f	aa5	bbe	cc4	ddc	ee4	ffb	gg3	hh8
	00g	11f	22g	33f	44g	55g	668	77f	88e	99g	aa8	bbg
	cc7	dd8	eea	ffc	ggc	hhd	00h	11g	22h	33h	44h	55h
	66e	77h	88f	99h	aa9	bbh	cch	ddg	eec	ffe	gge	hhe
	014	015	045	145	023	026	036	236	078	079	089	789
	0ab	0ac	0bc	abc	127	12a	17a	27a	138	13b	18b	38b
	169	16c	19c	69c	249	24b	29b	49b	258	25c	28c	58c
	39d	39e	3de	9de	46a	46d	4ad	6ad	57b	57d	5bd	7bd
	6fg	6fh	6gh	afg	afh	agh						
15	009	119	229	339	449	559	669	779	889	99a	aab	bbc
	ccd	dde	eef	ffg	ggh	hh9	00a	11a	22a	33a	44a	55a
	66a	77a	88a	99b	aac	bbd	cce	ddf	eeg	ffh	gg9	hha
	00b	11b	22b	33b	44b	55b	66b	77b	88b	99c	aad	bbe
	ccf	ddg	eeh	ff9	gga	hhb	00c	11c	22c	33c	44c	55c
	66c	77c	88c	99d	aae	bbf	ccg	ddh	ee9	ffa	ggb	hhc
	01d	12d	23d	34d	45d	56d	67d	78d	80d	02e	24e	46e
	68e	81e	13e	35e	57e	70e	03f	36f	60f	14f	47f	71f
	25f	58f	82f	04g	48g	83g	37g	72g	26g	61g	15g	50g
	04h	48h	83h	37h	72h	26h	61h	15h	50h	013	124	235
	346	457	568	670	781	802						

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