On the computational complexity of upper distance fractional domination

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Abstract

Let $n \geq 1$ be an integer and let $G = (V, E)$ be a graph. In this paper we study a nondiscrete generalization of $\Gamma_n(G)$, the maximum cardinality of a minimal n-dominating set in G. A real-valued function $f : V \to [0,1]$ is n-dominating if for each $v \in V$, the sum of the values assigned to the vertices in the closed n-neighbourhood of v, $N_n[v]$, is at least one, i.e., $f(N_n[u]) \geq 1$. The weight of an n-dominating function f is $f(V)$, the sum of all values $f(v)$ for $v \in V$, and $\Gamma_{nf}(G)$ is the maximum weight over all minimal n-dominating functions. We show that the decision problems corresponding to the problems of computing $\Gamma_n(G)$ and $\Gamma_{nf}(G)$ are NP-complete, generalising the result of Cheston, Fricke, Hedetniemi and Jacobs for the case $n = 1$.

1 Introduction

Let $n \geq 1$ be an integer and $G = (V, E)$ a graph. A set $D \subseteq V$ is an *n-dominating set* if every vertex $v \in V - D$ is within distance n from some vertex of D. An n-dominating set

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is minimal if no proper subset is n-dominating. The *n-domination number of G*, denoted by $\gamma_n(G)$, is the minimum cardinality over all minimal n-dominating sets of G, while the *upper n-domination number of G*, denoted by $\Gamma_n(G)$, is the maximum cardinality over all minimal n-dominating sets of G. In this paper we consider a generalisation of $\Gamma_n(G)$.

Let $f : V \to [0,1]$. To simplify notation we will write $f(D)$ for $\sum_{v\in D} f(v)$ and we define the *weight of f* to be $\sum_{v\in V} f(v) = f(V)$. Given a vertex v its *closed n-neighbourhood*, denoted by $N_n[v]$, is the set containing v as well as all vertices within distance n from v. We say f is an *n-dominating function* if for each $v \in V$ we have that $f(N_n[v]) \geq 1$. Given an n-dominating function f, we say it is *minimal n-dominating* if it is minimal among all n-dominating functions under the usual partial ordering for real-valued functions (i.e., $f \leq g$ iff $f(v) \leq g(v)$ for all $v \in V$). The concepts introduced in this paper generalises those of Cheston, Fricke, Hedetniemi and Jacobs (see [1]).

The following result generalises a result of Cheston, Fricke, Hedetniemi and Jacobs (see [1]). It will prove to be very useful.

Lemma 1 *Let f be an n-dominating function for a graph $G = (V, E)$. Then f is minimal n-dominating if and only if whenever $f(v) > 0$ there exists some $u \in N_n[v]$ such that $f(N_n[u]) = 1$.*

Proof. Let $v \in V$ such that $f(v) > 0$. Then $f(N_n[v]) \geq 1$. Let $N_n[v] = \{w_1, \ldots, w_\ell\}$. If $f(N_n[w_i]) = 1$ for some i, we are done. Assume, therefore, that $f(N_n[w_i]) = 1 + \delta_i > 1$ for $i = 1, 2, \ldots, \ell$. Suppose $v = w_1$. If $f(v) = a$, then $f(N_n[v]) = a + f(N_n(v)) = 1 + \delta_1$. Let $\delta = \min\{\delta_1, \ldots, \delta_\ell\}$. Note that $\delta > 0$. Define $g : V \to [0,1]$ by $g(x) = f(x)$ if $x \neq v$ with $g(v) = \max\{0, a - \delta\}$, so that $g < f$. Note that $g(N_n[v]) = g(w_1) + \ldots + g(w_\ell) \geq a - \delta + g(w_2) + \ldots + g(w_\ell) = 1 + \delta_1 - \delta \geq 1$, while $g(N_n[w_i]) = f(N_n[w_i] - \{v\}) + g(v) \geq f(N_n[w_i] - \{v\}) + a - \delta = f(N_n[w_i]) - \delta \geq f(N_n[w_i]) - \delta_i = 1$ so that g is an n-dominating function of G with $g < f$, which contradicts the minimality of f.

For the converse, suppose there exists a g such that $g < f$ - let $v \in V$ such that $g(v) < f(v)$. Since $f(v) > 0$, there exists $u \in N_n[v]$ such that $f(N_n[u]) = 1$. But $g(N_n[u]) = g(N_n[u] - \{v\}) + g(v) < f(N_n[u] - \{v\}) + f(v) = 1$, which contradicts the fact that g is n-dominating. ∎

For a graph G with vertex set $V = \{v_1, \ldots, v_m\}$ we can identify functions from V into R as n-tuples $(x_1, \ldots, x_m) \in \mathrm{R}^m$. Such a function is n-dominating if and only if $0 \leq x_i \leq 1$ and

$\sum_{v_j \in N_n[v_i]} x_j \geq 1$ for $i = 1, \ldots, m$. If the aforementioned two conditions hold, by Lemma 1, the notion of minimality is equivalent to $x_i \prod_{v_j \in N_n[v_i]} (1 - \sum_{v_k \in N_n[v_j]} x_k) = 0$ for $i = 1, \ldots, m$.

For any graph G, the points $(x_1, \ldots, x_m) \in \mathbf{R}^m$ satisfying the aforementioned three conditions are precisely the set of all minimal n-dominating functions. Since this set is compact and the function $(x_1, \ldots, x_m) \rightarrow \sum x_i$ is continuous on $\mathbf{R}^m$, there exists a minimal n-dominating function of maximum weight. We denote the weight of such a function by $\Gamma_{nf}(G)$. Note that Γ, in this setting, is merely the weight obtained when the x_i are additionally constrained to be 0 or 1. Clearly $\Gamma_n(G) \leq \Gamma_{nf}(G)$.

In Section 2 we give an example of a graph G for which $\Gamma_n(G) < \Gamma_{nf}(G)$. Section 3 considers the complexity of the decision problems corresponding to the problems of computing $\Gamma_n(G)$ and $\Gamma_{nf}(G)$. The construction used in the latter, gives a new proof of the NP-completeness of upper fractional domination, originally settled by Cheston, Fricke, Hedetniemi and Jacobs in [1].

2 An example of $\Gamma_n(G) < \Gamma_{nf}(G)$

In this section we give an example of a graph such $\Gamma_n(G) < \Gamma_{nf}(G)$. We start by proving a useful lemma.

Let n and ℓ be positive integers and consider $P_{n+1} \times K_\ell$. Let $\{v \in V(P_{n+1} \times K_\ell) | deg(v) = \ell\} = A \cup B$ where $< A > \cong < B > \cong K_\ell$. Let $A = \{a_1, \ldots, a_\ell\}$, $B = \{b_1, \ldots, b_\ell\}$ and let $P : b_0 - a_0$ be a path of length $n - 1$. If $b_{\ell+1} \notin V(P_{n+1} \times K_\ell) \cup V(P)$, construct the graph $H(n, \ell) = (V', E')$ as follows:
(a) $V' = V(P_{n+1} \times K_\ell) \cup V(P) \cup \{b_{\ell+1}\}$
(b) $E' = E(P_{n+1} \times K_\ell) \cup E(P) \cup \{a_0 a_i | i = 1, \ldots, \ell\} \cup \{b_{\ell+1} b_i | i = 1, \ldots, \ell\}$. The graph $H(n, \ell)$ is depicted in Figure 1.

Lemma 2 *Let $G' = (V', E') = H(n, \ell)$ for some positive integers n and $\ell \geq 2$. Let $G = (V, E)$ be a graph such that $G' \subseteq G$ and $deg_{G'}(x) = deg_G(x)$ for all $x \in V' - \{b_0\}$. Let f be a minimal n-dominating function of G.*
(a) Suppose $f(B) > 1$. Then $f(V') \leq \ell$. Moreover, if equality holds, then $f(b_i) = 1$ for all $i = 1, \ldots, \ell$, while $f(V' - B) = 0$. Furthermore, if f is constrained to be a $0 - 1$ function, then $f(V' - B) = 0$.

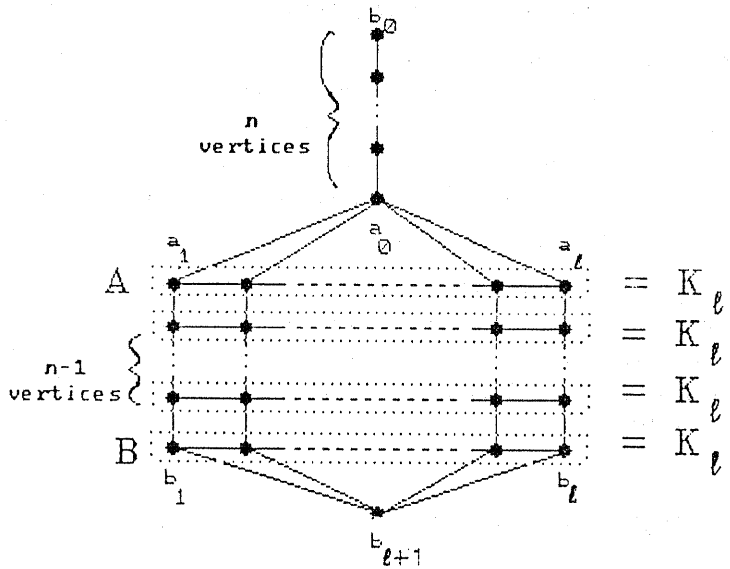

Figure 1: The graph $H(n, \ell)$

(b) If $f(B) \leq 1$, then $f(V') \leq 2$.

Proof. Let $\beta_i = f(b_i)$ for $i = 1, \ldots, \ell+1$, $I = N_n[b_{\ell+1}]$ and $J = V' - I$. Note that $f(I) \geq 1$ since f must n-dominate $b_{\ell+1}$.

(a) Suppose $f(B) > 1$. Note that this implies that $\beta_{\ell+1} = 0$, by the minimality of f. Assume that $\beta_i > 0$. Then, by Lemma 1, there exists $u \in N_n[b_i]$ such that $f(N_n[u]) = 1$. Since $\sum_{j=1}^{\ell} \beta_j > 1$, it follows that $u \notin I$, so that $u = a_i$. This implies that $f(V' - (B - \{b_i\})) = 1$. We now have that $f(V') = f(V' - (B - \{b_i\})) + \sum_{j \in \{1,\ldots,\ell\} - \{i\}} \beta_j \leq \ell$ with equality occurring only if $\beta_j = 1$ for $j \in \{1, \ldots, \ell\} - \{i\}$. If $\beta_j = 1$ for some $j \in \{1, \ldots, \ell\} - \{i\}$, then, by symmetry, we have that $f(V' - (B - \{b_j\})) = 1$. Hence $1 = f(V' - B) + \beta_i = f(V' - B) + \beta_j$, so that $\beta_i = \beta_j$. Also $f(V' - B) = 0$.

Now let D be a minimal n-dominating set of G' such that $D \cap B = \{b_i, b_j\}$. Since D is a *minimal* n-dominating set, it follows that $b_{\ell+1} \notin D$. Since $I \subseteq N_n[b_i] \cap N_n[b_j]$, it follows that $a_i \in N_n[b_i] - N_n[D - \{b_i\}]$ and $a_j \in N_n[b_j] - N_n[D - \{b_j\}]$. This implies that $(V' - B) \cap D = \emptyset$. Hence, if f is a minimal n-dominating function such that $f(B) > 1$, we see that $f(V' - B) = 0$.

(b) Suppose that $f(B) \leq 1$. We distinguish between two cases:

Case 1 $f(I) > 1$.
Since $f(I) > 1$, the minimality of f implies that $\beta_{\ell+1} = 0$.

Subcase 1.1 $\beta_i > 0$ for some $i \in \{1, \ldots, \ell\}$.
By the minimality of f, there exists $u \in N_n[b_i]$ such that $f(N_n[u]) = 1$. Since $f(I) > 1$, it follows that $u \notin I$, so that $u = a_i$. This implies that $1 = f(V' - (B - \{b_i\})) = f(V' - B) + \beta_i$ and so $f(V') = f(V' - B) + f(B) \leq f(V' - B) + \beta_i + f(B) \leq 1 + 1 = 2$.

Subcase 1.2 $f(B) = 0$.
Let $x \in V' - B$ such that $f(x) > 0$ and $d(x, B)$ is a minimum. By the minimality of f, there exists $u \in N_n[x]$ such that $f(N_n[u]) = 1$. Since $f(I) > 1$, it follows that $u \in J$. In this case $f(V') \leq f(N_n[u]) = 1$.

Case 2 $f(I) \leq 1$.
Since $b_{\ell+1}$ must be n-dominated by f, we have that $f(I) \geq 1$, so that $f(I) = 1$. We show that $f(J) \leq 1$: Suppose that $f(J) > 1$. If $f(a_i) > 0$ for some i, there exists $u \in N_n[a_i]$ such that $f(N_n[u]) = 1$. Since $f(J) > 1$, it follows that $u \in I$, so that $f(N_n[u]) \geq f(I) + f(a_i) = 1 + f(a_i) > 1$, which is a contradiction. Hence $f(A) = 0$. Now let $x \in J - A$ such that $f(x) > 0$ and $d(x, a_0)$ is a minimum. Then there exists $u \in N_n[x]$ such that $f(N_n[u]) = 1$. Since $f(J) > 1$, it follows that $u \notin J$. If $u \in I$, then $f(N_n[u]) \geq f(I) + f(x) > 1$, which is a contradiction, whence $u \notin V'$. If S is the vertex set of the $b_0 - x$ subpath of $\langle J \rangle$, then, since, $S \subseteq N_n[u]$, we have that $f(S) \leq 1$. Note that $f(J - S) = 0$, so that $f(J) \leq 1$, which is a contradiction. We conclude that $f(J) \leq 1$ so that $f(V') = f(I) + f(J) \leq 2$. ∎

We now show that we can construct a graph G for which $\Gamma_n(G) < \Gamma_{nf}(G)$.

Let $n \geq 1$ be an integer. Take four copies $H^1(n,5), H^2(n,5), H^3(n,5), H^4(n,5)$ of $H(n,5)$ and superscript each vertex according to the copy it appears in. Add the edges $b_0^1 b_0^2$, $b_0^2 b_0^3$, $b_0^3 b_0^4$ and $b_0^4 b_0^1$ to obtain the graph G. The graph G is depicted in Figure 2.

Lemma 3 *If G is the aforementioned graph, then $\Gamma_n(G) = 14$.*

Proof. Let $B^i = \{b_j^i | j = 1, \ldots, 5\}$ for $i = 1, \ldots, 4$ and let D be a minimal n-dominating set

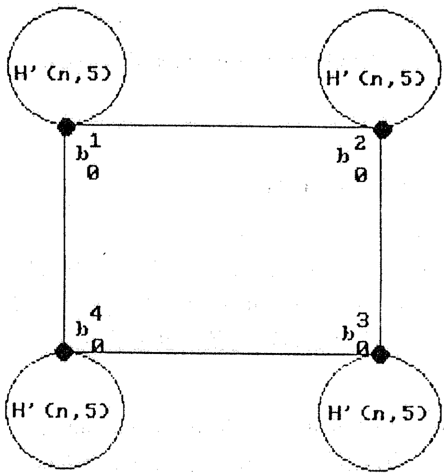

Figure 2: A graph for which $\Gamma_n(G) < \Gamma_{nf}(G)$

of G. Suppose $|B^i \cap D| > 1$ for $i = 1, 2, 3$. By Lemma 2(a), it follows that $(V(H^i(n,5) - B^i) \cap D = \emptyset$ for $i = 1, 2, 3$, so that vertex a_0^2 is not n-dominated by D. This shows that $|B^i \cap D| \leq 1$ for at least two of the graphs $H^i(n, 5)$. Lemma 2 now implies that $|D| \leq 2.2 + 2.5 = 14$. Figure 3 shows that $\Gamma_n(G) = 14$ with the square vertices forming a minimal n-dominating set of cardinality 14. ∎

Figure 4 shows that $\Gamma_{nf}(G) \geq 14\frac{2}{3}$. (Vertices not labelled are assumed to be labelled by 0.)

3 Complexity issues

In this section we show that the decision problems corresponding to the problems of computing $\Gamma_n(G)$ and $\Gamma_{nf}(G)$ are NP-complete. More specifically, these problems are:
UPPER DISTANCE DOMINATION (UDD)
Instance: A graph G and integers k and n.
Question: Is $\Gamma_n(G) \geq k$?

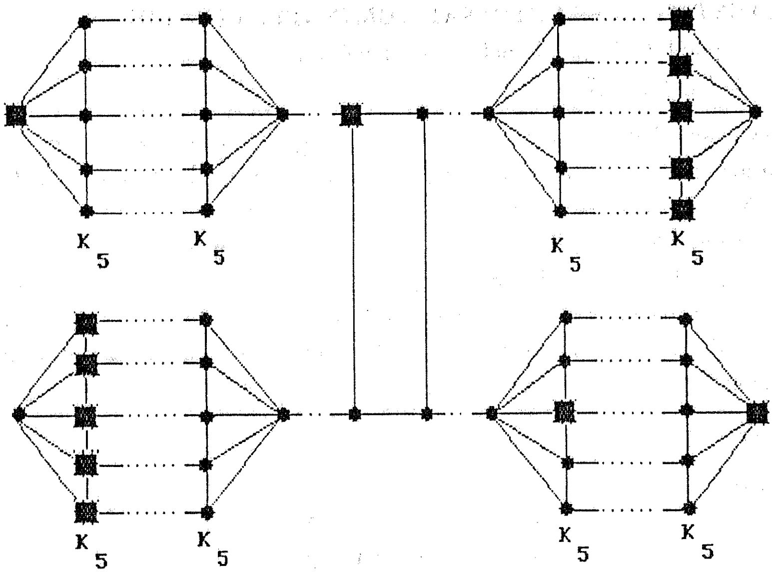

Figure 3: $\Gamma_n(G) = 14$

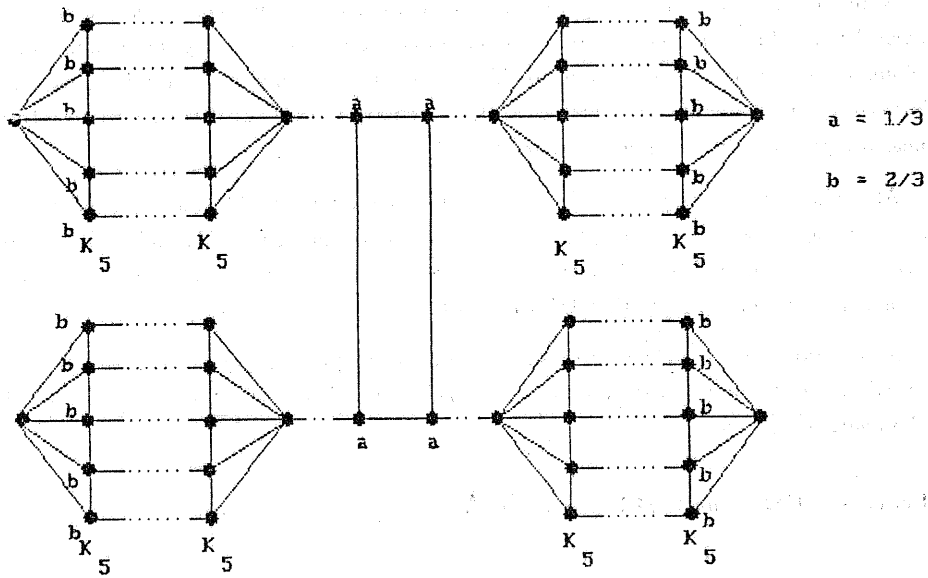

Figure 4: $\Gamma_{nf}(G) \geq 14\frac{2}{3}$

UPPER DISTANCE FRACTIONAL DOMINATION (UDFD)

Instance: A graph G, integer n and rational number q.

Question: Is $\Gamma_{nf}(G) \geq q$?

We now show that $\Gamma_{nf}(G)$ is computable and is always a rational number. If f is a minimal n-dominating function, let $S_f = \{v \in V | f(N_n[v]) = 1\}$. By Lemma 1, if $f(v) > 0$, then $v \in N_n[S_f]$, where $N_n[S] = \cup_{x \in S} N_n[x]$. Since f is n-dominating, for every vertex v, there is some $u \in N_n[v]$ with $f(u) \neq 0$. The previous two comments imply that we must have $N_n[N_n[S_f]] = V$. Let $\mathbf{S} = \{S | S \subseteq V \wedge N_n[N_n[S]] = V\}$. For each $S \in \mathbf{S}$, we consider the problem of finding a minimal n-dominating function f of maximum weight with the additional constraint that $S_f \supseteq S$. This subproblem can be solved using linear programming:

maximize $\qquad \sum_{v_i \in V} x_i$

subject to

$$0 \leq x_i \leq 1 \qquad \forall v_i \in N_n[S]$$
$$x_i = 0 \qquad \forall v_i \in V - N_n[S]$$
$$\sum_{v_j \in N_n[v_i]} x_j \geq 1 \qquad \forall v_i \in V - S$$
$$\sum_{v_j \in N_n[v_i]} x_j = 1 \qquad \forall v_i \in S.$$

Note that the conditions guarantee that a solution to this problem is n-dominating. Given that a solution is n-dominating, the second and fourth conditions guarantee minimal n-domination. Hence every solution to this problem is a minimal n-dominating function. Conversely, any minimal n-dominating function f having weight Γ_{nf} is the solution to this linear programming problem for some set $S_f \in \mathbf{S}$.

Also, since each member of $\mathbf{S}$ defines a linear programming problem and $\Gamma_{nf}(G)$ is the largest among these subproblems, $\Gamma_{nf}(G)$ is a computable function. This number is rational since each problem involves only rational numbers. Since linear programming can be solved in polynomial time, it follows that **UDFD** $\in NP$.

It is obvious that **UDD** $\in NP$, since we can, in polynomial time, guess at a subset of vertices, verify that it has cardinality at least k and then verify that it is a minimal n-dominating set. Thus we have:

Theorem 1 **UDD** *and* **UDFD** *are in* NP. ∎

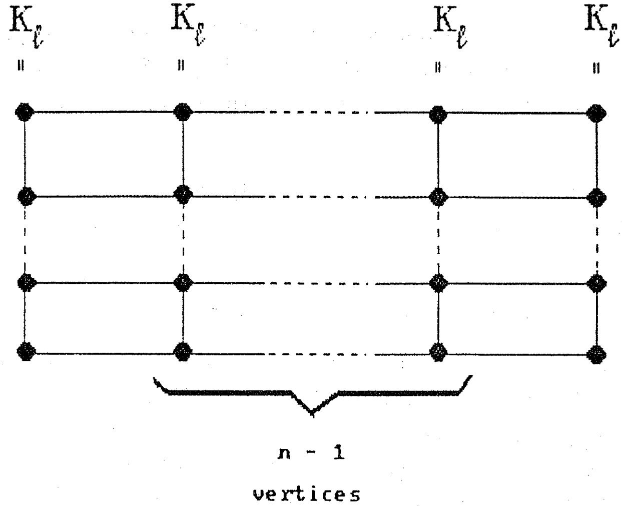

Figure 5: The graph $H'(n,\ell)$

Before proceeding further, we prove a lemma. Let n and ℓ be postive integers and consider $H'(n,\ell) = P_{n+1} \times K_\ell$. Let $\{v \in V(H'(n,\ell)) | deg(v) = \ell\} = A \cup B$ where $< A > \cong < B > \cong K_\ell$. Let $A = \{a_1, \ldots, a_\ell\}$ and $B = \{b_1, \ldots, b_\ell\}$. The graph $H'(n,\ell)$ is depicted in Figure 5.

Lemma 4 *Let $G' = (V', E') = H'(n,\ell)$ for some positive integers n and $\ell \geq 3$. Let $G = (V, E)$ be a graph such that $G' \subseteq G$ and $deg_{G'}(x) = deg_G(x)$ for all $x \in V' - \{a_1, b_1\}$. Let f be a minimal n-dominating function of G. Then $f(V') \leq \ell$. Moreover, if equality holds, then $f(a_i) = 1$ for all $i = 1, \ldots, \ell$ and $f(V' - A) = 0$ or $f(b_i) = 1$ for all $i = 1, \ldots, \ell$ and $f(V' - B) = 0$.*

Proof. Let $\alpha_i = f(a_i)$ and $\beta_i = f(b_i)$ for $i = 1, \ldots, \ell$. For $i = 1, \ldots, \ell$, let $X_i = \{v \in V - V' | d(v, a_i) \leq n\}$ and $Y_i = \{v \in V - V' | d(v, b_i) \leq n\}$. Note that $X_i = X_j$ and $Y_i = Y_j$ for $i, j \in \{2, \ldots, \ell\}$, while $X_i \subseteq X_1$ and $Y_i \subseteq Y_1$ for $i = 1, \ldots, \ell$.

Case 1 $f(A) > 1$.
In this case $\alpha_i > 0$ for some $i \in \{2 \ldots, \ell\}$, since otherwise $\alpha_1 > 1$, which is a contradiction. Then, by Lemma 1, there exists $u \in N_n[a_i] = V' - (B - \{b_i\}) \cup X_i$ such that $f(N_n[u]) = 1$.

Since $\sum_{j=1}^{\ell} \alpha_j > 1$, it follows that $u \notin (V' - B) \cup X_i$. Hence $u = b_i$, so that $f(V' - (A - \{a_i\}) \cup Y_i) = 1$. Hence $f(V') = f(V' - (A - \{a_i\}) + \sum_{j \in \{1,\ldots,\ell\}-\{i\}} \alpha_j \leq f(V' - (A - \{a_i\}) \cup Y_i) + \sum_{j \in \{1,\ldots,\ell\}-\{i\}} \alpha_j \leq 1 + (\ell - 1) = \ell$, with equality occurring only if $\alpha_j = 1$ for $j \in \{1, \ldots, \ell\} - \{i\}$.

Now let $\alpha_j = 1$ for some $j \in \{2, \ldots, \ell\} - \{i\}$. By symmetry, $f(V' - (A - \{a_j\}) \cup Y_j) = 1$. Hence $1 = f(V' - A \cup Y_i) + \alpha_j = f(V' - A \cup Y_i) + \alpha_i$, so that $\alpha_i = \alpha_j$. Also $f(V' - A \cup Y_i) = 0$, so that $f(V' - A) = 0$.

Case 2 $f(B) > 1$. This case is similar to case 1.

By cases 1 and 2, we may assume that $f(A) \leq 1$ and $f(B) \leq 1$. Let $x \in I = V' - A - B$ such that $f(x) > 0$. Then there exists $u \in N_n[x] \subseteq V' \cup X_2 \cup Y_2$ with $f(N_n[u]) = 1$. If $u \in I$, then, since $N_n[u] \supseteq V'$, it follows that $f(V') \leq 1$. If $u \in A \cup B$, say $u = a_i$, then $N_n[u] \supseteq (V' - B) \cup \{b_i\}$, whence $f(V') = f(V' - B) + f(B) = f((V' - B) \cup \{b_i\}) + \sum_{j \in \{1,\ldots,\ell\}-\{i\}} \beta_j \leq f(N_n[u]) + f(B) \leq 1 + 1 = 2$. Hence we may assume that if $x \in I$ such that $f(x) > 0$, there exists $u \in X_2 \cup Y_2$ such that $f(N_n[u]) = 1$. For $x \in I$ such that $f(x) > 0$, let P_x be a shortest path from x to $\{u \in X_2 \cup Y_2 | f(N_n[u]) = 1\}$; denote the other endpoint of P_x by $e(x)$. Let $S = \{x \in I | f(x) > 0 \wedge a_1 \in P_x\}$ and $T = \{x \in I | f(x) > 0 \wedge b_1 \in P_x\}$. Let $x' \in S$ such that $d(x', a_1) = \max\{d(x, a_1) | x \in S\}$ and let $x'' \in T$ such that $d(x'', b_1) = \max\{d(x, b_1) | x \in T\}$.

Case 1′ $S \neq \emptyset$ and $T = \emptyset$.
In this case, if $x \in I$ such that $d(x, a_1) > d(x', a_1)$, it follows that $f(x) = 0$. Hence $f(V') \leq f(N_n[e(x')]) + f(B) \leq 1 + 1 = 2$.

Case 2′ $S = \emptyset$ and $T \neq \emptyset$. This case is similar to Case 1′.

Case 3′ $S \neq \emptyset$ and $T \neq \emptyset$.
In this case, if $x \in I$ such that $d(x, a_1) > d(x', a_1)$ and $d(x, b_1) > d(x'', b_1)$, it follows that $f(x) = 0$. Hence $f(V') \leq f(N_n[e(x')]) + f(N_n[e(x'')]) \leq 1 + 1 = 2$. ∎

We now establish a polynomial transformation from the well-known 3-satisfiability problem (3-SAT) to UDD, thus proving it NP-hard. Let I be an instance of 3-SAT consisting of the set $\{C_1, C_2, \ldots, C_t\}$ of 3-literal clauses involving the literals $x_1, \overline{x}_1, x_2, \overline{x}_2, \ldots, x_m, \overline{x}_m$. Associate with each literal pair $x_i, \overline{x}_i$ the graph $H'(n, 3)$ depicted in Figure 5 – where the vertices a_1 and b_1 are renamed by $vx_i, v\overline{x}_i$ respectively. With each clause C_s associate the graph $H(n, 3)$ of Figure 1 – where the vertex b_0 is renamed by c_s. We insert an edge between literal vertex vx_i (or $v\overline{x}_i$) and clause vertex c_s if and only if x_i (or $\overline{x}_i$) is a literal in clause

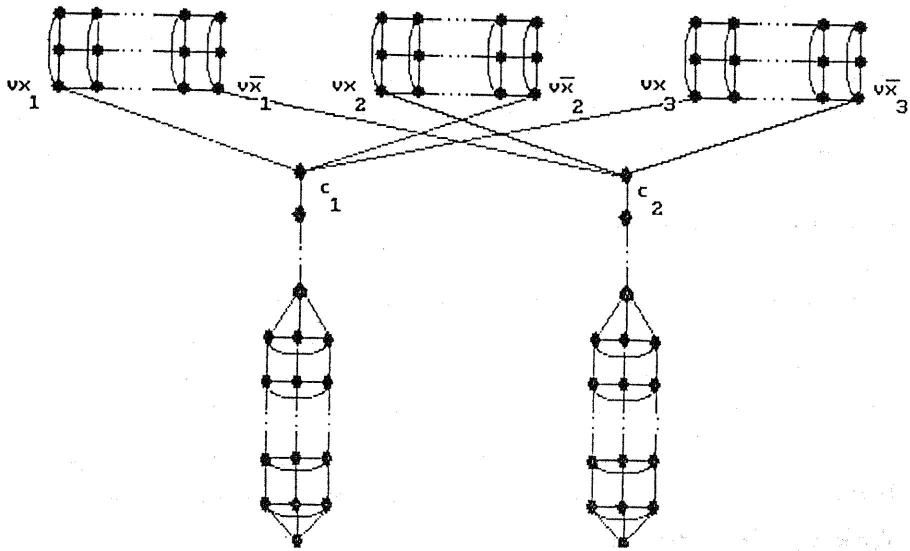

Figure 6: Graph for $(x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3)$

C_s – name the resulting graph G. The graph associated with $(x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3)$ is depicted in Figure 6. Clearly, this construction can be accomplished in polynomial time.

Theorem 2 **UDD** *is NP-complete.*

Proof. Given an instance I of 3-SAT, we construct the graph G as above and set $k = 3t+3m$. To show that this problem is NP-hard, it suffices to show that I is satisfiable if and only if G has a n-dominating set of cardinality at least k.

First, suppose g is a satifying truth assignment. We construct a minimal n-dominating set D of cardinality $3(t+m)$. For each $i = 1, \ldots, m$, do the following. If $g(x_i) = T$ ($g(\overline{x}_i) = T$ respectively) place in D the vertex vx_i ($v\overline{x}_i$ respectively) along with the other two vertices of the 3-clique containing vx_i ($v\overline{x}_i$ respectively). Next, for every clause associated subgraph, place the vertices b_1, b_2, b_3 in D. It is straightforward to verify that this is a minimal n-dominating set of cardinality $3(t+m)$.

Conversely, assume that D is a minimal n-dominating set of cardinality at least $3(t+m)$. We may think of D as a minimal n-dominating function. By Lemma 2 and Lemma 4, this function can be no more than 3 on each $H(n,3)$ and $H'(n,3)$ graph. Therefore, it must be exactly 3 on each such graph, since there are $t+m$ such subgraphs. By Lemma 4, for each $i = 1, \ldots, m$, exactly one of vx_i or $v\bar{x}_i$ is in D. We may define $g(x_i) = T$ iff $vx_i \in D$. By Lemma 2, each vertex c_s is not n-dominated by any vertex within a $H(n,3)$ graph. Hence it must be n-dominated by a vertex corresponding to one of its variables, so it follows that g is a satisfying truth assignment. ∎

Theorem 3 **UDFD** *is* NP*-complete.*

Proof. Given an instance I of 3-SAT, we map I to (G, n, q) with G the graph described before the statement of Theorem 2 and q is the rational number $3(t+m)$. We may then argue that I is satisfiable iff G has a minimal n-dominating function of weight at least $3(t+m)$. The argument is almost identical to the one given for Theorem 2. ∎

In closing, we note that our construction gives a new proof of the NP-completeness of **UDFD** for the case $n = 1$ established by Cheston, Fricke, Hedetniemi and Jacobs (see [1]).

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References

[1] G.A. Cheston, G. Fricke, S.T. Hedetniemi and D.P. Jacobs, On the computational complexity of upper fractional domination, *Discrete Applied Math.*, **27** (1990), 195-207.

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