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On extremal graphs with exactly one Steiner tree connecting any k vertices*

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Abstract

The problem of determining the largest number $f(n;\overline{\kappa} \leq \ell)$ of edges for graphs with n vertices and maximal local connectivity at most ℓ was considered by Bollobás. Li et al. studied the largest number $f(n;\overline{\kappa}_3 \leq 2)$ of edges for graphs with n vertices and at most two internally disjoint Steiner trees connecting any three vertices. In this paper, we further study the largest number $f(n;\overline{\kappa}_k = 1)$ of edges for graphs with n vertices and exactly one Steiner tree connecting any k vertices with $k \geq 3$. It turns out that this is not an easy task to finish, unlike the same problem for the classical connectivity parameter. We determine the exact values of $f(n;\overline{\kappa}_k = 1)$ for $k = 3, 4, n$, and characterize the graphs which attain each of these values.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [3]. We refer to the number of vertices in a graph as the *order* of the graph and the number of its edges as its *size*. We use the basic notations $e(G)$, $\delta(G)$ and $d(v)$ to denote the size of G, the minimum degree of G and the degree of a vertex v, respectively. We say that two paths are *internally disjoint* if they have no common vertex except the end vertices. For any two distinct vertices u and v in a graph G, the *local connectivity* $\kappa_G(u,v)$ is the maximum number of internally disjoint paths connecting u and v. Then the connectivity of G is defined as $\kappa(G) = \min\{\kappa_G(u,v) : u,v \in V(G), u \neq v\}$; whereas $\overline{\kappa}(G) = \max\{\kappa_G(u,v) : u,v \in V(G), u \neq v\}$ is called the *maximal local connectivity* of G, introduced by Bollobás.

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Bollobás [1] considered the problem of determining the largest number $f(n;\overline{\kappa} \leq \ell)$ of edges for graphs with n vertices and maximal local connectivity at most ℓ. In other words, $f(n;\overline{\kappa} \leq \ell) = \max\{e(G) : |V(G)| = n \text{ and } \overline{\kappa}(G) \leq \ell\}$. Determining the exact value of $f(n;\overline{\kappa} \leq \ell)$ has got a great attention and many results have been worked out, see [1–2, 5–7, 15–16, 18].

For a graph $G(V,E)$ and a subset S of V where $|S| \geq 2$, an *S-Steiner tree* or a *Steiner tree connecting* S is a subgraph $T(V',E')$ of G which is a tree such that $S \subseteq V'$. Two S-Steiner trees T_1 and T_2 are called *internally disjoint* if $E(T_1) \cap E(T_2) = \varnothing$ and $V(T_1) \cap V(T_2) = S$. Note that T_1 and T_2 are vertex-disjoint in $G \setminus S$. For $S \subseteq V$, the *generalized local connectivity* $\kappa(S)$ is the maximum number of internally disjoint trees connecting S in G. The *generalized k-connectivity* is defined as $\kappa_k(G) = \min\{\kappa(S) : S \subseteq V(G), |S| = k\}$. These concepts can be found in [4]. Many results have been worked out on the generalized connectivity; we refer the reader to [9–12, 14] for details.

In analogue to the classical maximal local connectivity, another parameter $\overline{\kappa}_k(G) = \max\{\kappa(S) : S \subseteq V(G), |S| = k\}$, called the *maximal generalized local connectivity* of G, was introduced in [8]. The authors studied the largest number $f(n;\overline{\kappa}_3 \leq 2)$ of edges for graphs with n vertices and at most two internally disjoint Steiner trees connecting any three vertices. Later, Li and Mao [13] determined the exact value of $f(n;\overline{\kappa}_k \leq \ell)$ for $k = n$ and $n-1$, and for a general k they construct a graph to obtain a sharp lower bound.

In this paper, we will study the problem of determining the largest number $f(n;\overline{\kappa}_k = 1)$ of edges for graphs with n vertices and maximal generalized local connectivity exactly equal to 1, that is, $f(n;\overline{\kappa}_k = 1) = \max\{e(G) : |V(G)| = n \text{ and } \overline{\kappa}_k(G) = 1\}$. It is easy to see that for $k = 2$, $f(n;\overline{\kappa} = 1) = n-1$, and if a graph G satisfies $\overline{\kappa}(G) = 1$, then G must be a tree. It turns out that for $k \geq 3$, the problem is not easy to attack.

This paper is organized as follows. In Section 2, we introduce a graph operation to describe three graph classes. In Section 3, we first estimate the exact value of $f(n;\overline{\kappa}_3 = 1)$, that is, $f(n;\overline{\kappa}_3 = 1) = \frac{4n-3-r}{3}$ for $n = 3r+q$, $0 \leq q \leq 2$. Then, in Section 4, we determine $f(n;\overline{\kappa}_4 = 1)$ for $n = 4r+q$, $0 \leq q \leq 3$. Finally, in Section 5, $f(n;\overline{\kappa}_n = 1)$ is determined to be $\binom{n-1}{2}+1$. Furthermore, we characterize extremal graphs attaining each of these values. For general k, we get the lower bound of $f(n;\overline{\kappa}_k = 1)$ by constructing extremal graphs for $n = r(k-1)+q$, $0 \leq q \leq k-2$.

2 Preliminaries

In this section, we first give some definitions frequently used in the sequel, and then introduce a graph operation to describe three graph classes.

For a graph G, we say a path $P = u_1u_2\ldots u_q$ is an *ear* of G if $V(G) \cap V(P) = \{u_1, u_q\}$. If $u_1 \neq u_q$, P is an *open ear*; otherwise P is a *closed ear*. By $\ell(P)$ we denote the length of P and C_q a cycle on q vertices.

Let H_1 and H_2 be two disjoint graphs. The *adding operation* $H_1 + H_2$ of H_1

and H_2 is defined from the disjoint union of H_1 and H_2 by adding exactly one edge between a vertex of H_1 and a vertex of H_2, arbitrarily. Since the added edge is arbitrarily chosen, the adding operation defines a class of graphs rather than a single graph. Sometimes the adding operation contains exactly one graph, for example, $K_2 + K_1 = \{P_3\}$. In this case, we will use the notation $H_1 + H_2$ to mean the graph in the class $H_1 + H_2$ for brevity. As we will see, this does not violate the correctness of our proofs. Also note that for a graph $G \in H_1 + H_2$, $|V(G)| = |V(H_1)| + |V(H_2)|$ and $e(G) = e(H_1) + e(H_2) + 1$.

$\{C_3\}^i + \{C_4\}^j + \{C_5\}^k + \{K_1\}^\ell$ is a class of connected graphs obtained from i copies of C_3, j copies of C_4, k copies of C_5 and ℓ copies of K_1 by the adding operations such that $0 \le i \le \lfloor \frac{n}{3} \rfloor$, $0 \le j \le 2$, $0 \le k \le 1$, $0 \le \ell \le 2$ and $3i + 4j + 5k + \ell = n$. Note that these operations are taken in an arbitrary order.

Let $n = 3r + q$, $0 \le q \le 2$. If $q = 0$, $\mathcal{G}_n^0 = \{C_3\}^r$. If $q = 1$, $\mathcal{G}_n^1 = \{C_3\}^r + K_1$ or $\{C_3\}^{r-1} + C_4$. If $q = 2$, $\mathcal{G}_n^2 = \{C_3\}^r + \{K_1\}^2$ or $\{C_3\}^{r-1} + C_4 + K_1$ or $\{C_3\}^{r-1} + C_5$ or $\{C_3\}^{r-2} + \{C_4\}^2$.

Let $A, B, D_1, D_2, D_3, F_1, F_2, F_3, F_4$ be the graphs shown in Figure 1.

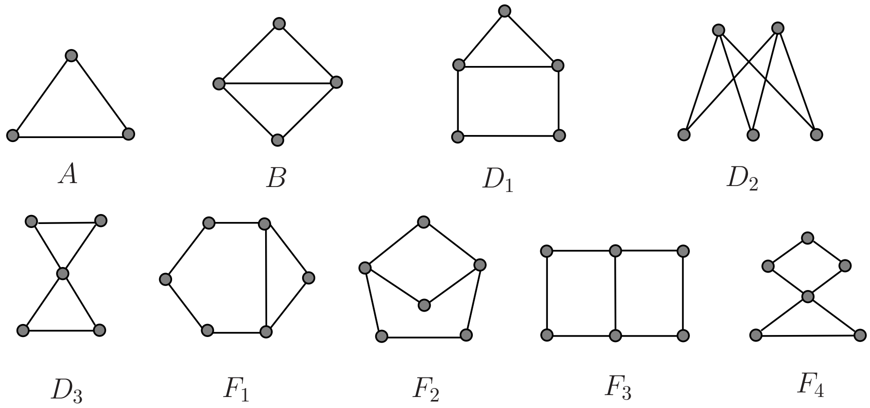

Figure 1. The graphs used for the second graph class

$\{A\}^{i_0} + \{B\}^{i_1} + \{D_1\}^{i_2} + \{D_2\}^{i_3} + \{D_3\}^{i_4} + \{F_1\}^{i_5} + \{F_2\}^{i_6} + \{F_3\}^{i_7} + \{F_4\}^{i_8} + \{K_1\}^{i_9}$ is composed of another connected graph class by the adding operations such that (1) $0 \le i_0 \le 2$, $0 \le i_1 \le \lfloor \frac{n}{4} \rfloor$, $0 \le i_2 + i_3 + i_4 \le 2$, $0 \le i_5 + i_6 + i_7 + i_8 \le 1$, $0 \le i_9 \le 2$; (2) D_i and F_j are not simultaneously in a graph belonging to this graph class where $1 \le i \le 3$, $1 \le j \le 4$; (3) $3i_0 + 4i_1 + 5(i_2 + i_3 + i_4) + 6(i_5 + i_6 + i_7 + i_8) + i_9 = n$.

Let $n = 4r + q$, $0 \le q \le 3$. If $q = 0$, $\mathcal{H}_n^0 = \{B\}^r$. If $q = 1$, $\mathcal{H}_n^1 = \{B\}^r + K_1$ or $\{B\}^{r-1} + D_i$ $(1 \le i \le 3)$. If $q = 2$, $\mathcal{H}_n^2 = \{B\}^r + \{K_1\}^2$ or $\{B\}^{r-1} + \{A\}^2$ or $\{B\}^{r-1} + D_i + K_1$ or $\{B\}^{r-2} + D_i + D_j$ $(1 \le i, j \le 3)$ or $\{B\}^{r-1} + F_i$ $(1 \le i \le 4)$. If $q = 3$, $\mathcal{H}_n^3 = \{B\}^r + A$.

Define the third graph class as follows: for $n = 5$, $\mathcal{K}_5 = \{G : |V(G)| = 5, e(G) = 7\}$; for $n \ge 6$, $\mathcal{K}_n = K_{n-1} + K_1$.

The following observation is obvious.

Observation 2.1. *Let G and G' be two connected graphs. If G' is a subgraph of G and $\overline{\kappa}_k(G') \geq 2$, then $\overline{\kappa}_k(G) \geq 2$.*

Next we state a famous theorem which is fundamental for calculating the number of edge-disjoint spanning trees and getting from it a useful lemma for our following results.

Theorem 2.2. *(Nash-Williams [17], Tutte [19]) A multigraph contains k edge-disjoint spanning trees if and only if for every partition $\mathcal{P}$ of its vertex sets it has at least $k(|\mathcal{P}|-1)$ cross-edges, whose ends lie in different partition sets.*

Lemma 2.3. *Let M be a subset of edges of K_n ($n \geq 5$) where $0 \leq |M| \leq n-3$, and G be a graph obtained from K_n by deleting M. Then G contains two edge-disjoint spanning trees.*

Proof. Let $\mathcal{P}$ be a partition of $V(G)$ into p sets $V_1, V_2, \ldots, V_p$ where $1 \leq p \leq n$, and let $\mathcal{E}$ represent the cross-edges. Set $|V_i| = n_i$, $1 \leq i \leq p$. If $p = 1$ then this case is trivial, so we suppose next that $2 \leq p \leq n$. By Theorem 2.2, in order to obtain two edge-disjoint spanning trees, we only need to prove that the inequality $|\mathcal{E}| \geq \binom{n}{2} - \sum_{i=1}^{p} \binom{n_i}{2} - |M| \geq 2(p-1)$, that is equivalent to saying that $\binom{n}{2} - |M| - 2(p-1) \geq \sum_{i=1}^{p} \binom{n_i}{2}$, holds. As $|M| \leq n-3$, and $\sum_{i=1}^{p} \binom{n_i}{2}$ attains the maximum value $\binom{n-p+1}{2}$ by $n_i = n-(p-1)$ and $n_j = 1$ where $j \neq i$, we only need to prove that $\binom{n}{2} - (n-3) - 2(p-1) \geq \binom{n-p+1}{2}$ holds. Let $f(n,p) = \binom{n}{2} - (n-3) - 2(p-1) - \binom{n-p+1}{2}$. Our aim is to prove that $f(n,p) \geq 0$. Now $f(n,p) = \binom{n-1}{2} - 2(p-2) - \binom{n-p+1}{2} = \frac{1}{2}(n-1)(n-2) - 2(p-2) - \frac{1}{2}[(n-1)-(p-2)](n-p) = \frac{1}{2}[(n-1)(p-2) + (p-2)(n-p-4)] = \frac{1}{2}(p-2)(2n-p-5)$. Since $2 \leq p \leq n$ and $n \geq 5$, it follows immediately that $f(n,p) \geq 0$. □

3 The case $k = 3$

We consider the case $k = 3$ in this section. At first, we begin with a necessary and sufficient condition for $\overline{\kappa}_3(G) = 1$.

Proposition 3.1. *Let G be a connected graph. Then $\overline{\kappa}_3(G) = 1$ if and only if every cycle in G has no ear.*

Proof. To settle the "only if" part, assume, to the contrary, that C is a cycle in G and P is an ear of C. Set $V(C) \cap V(P) = \{u, v\}$ where u and v may be the same vertex. If $\ell(P) = 1$, then P is an open ear; pick a vertex from uCv and vCu respectively, say u_1 and u_2. Then $T_1 = u_2Cu_1$ and $T_2 = u_1Cu_2 \cup uv$ are two internally disjoint trees connecting $\{u, u_1, u_2\}$, a contradiction to $\overline{\kappa}_3(G) = 1$. If $\ell(P) \geq 2$, pick a vertex in $C \setminus \{u, v\}$ and $P \setminus \{u, v\}$, respectively, say u_1 and u_2. Then there are also two internally disjoint trees connecting $\{u, u_1, u_2\}$, another contradiction.

To prove the "if" part, let S be a set of any three vertices. We need to prove that $\kappa_3(S) = 1$. Since every cycle in G has no ear, then every maximal bridgeless subgraph of G is a cycle and each edge incident with it is a cut edge. If two vertices in S belong to different cycles C_1 and C_2, then it is immediate to check that only one tree connects S, since the cut edge in the path from C_1 to C_2 can be used only once. If three vertices in S belong to a cycle, then it is immediate to see that only one tree connects S. Thus $\overline{\kappa}_3(G) = 1$. □

Lemma 3.2. *Let G be a connected graph of order 5 and size at least 6. Then $\overline{\kappa}_3(G) \geq 2$.*

Proof. Let H be a connected spanning subgraph of G and suppose H has size exactly 6. Since the possible connected graphs of order 5 and size 6 are D_1, D_2, D_3 and $B + K_1$, it is easy to see that each of these graphs has a cycle with an ear. Then by Proposition 3.1, it follows that $\overline{\kappa}_3(H) \geq 2$. By Observation 2.1, it follows that $\overline{\kappa}_3(G) \geq 2$. □

Theorem 3.3. *Let $n = 3r + q$, where $0 \leq q \leq 2$, and let G be a connected graph of order n such that $\overline{\kappa}_3(G) = 1$. Then $e(G) \leq \frac{4n-3-q}{3}$, with equality if and only if $G \in \mathcal{G}_n^q$.*

Proof. We apply induction on n. For $n = 3$, $e(G) \leq 3$, and let $G = C_3 \in \mathcal{G}_n^0$. For $n = 4$, if $G = K_4 \setminus e$, then there exists a cycle C_3 with an open ear of length 2, which contradicts to Proposition 3.1. Similarly, $G \neq K_4$. So G is obtained from K_4 by deleting two edges arbitrarily, that is, $G = C_3 + K_1$ or C_4, and then $G \in \mathcal{G}_n^1$. For $n = 5$, by Lemma 3.2, $e(G) \leq 5$ and if $e(G) = 5$, then $G = C_3 + \{K_1\}^2$ or $C_4 + K_1$ or C_5, and then $G \in \mathcal{G}_n^2$. Let $n \geq 6$. Assume that the assertion holds for graphs of order less than n. We will show that the assertion holds for graphs of order n. We distinguish two cases according to whether or not G has cut edges.

If G has no cut edge, then G is bridgeless, and combining with Proposition 3.1, G is a cycle. Then $e(G) = n < \frac{4n-5}{3}$, since $n \geq 6$.

Suppose that there exists at least one cut edge in G. Pick one, say e. Let G_1 and G_2 be two connected components of $G \setminus e$. Set $V(G_1) = n_1$, $V(G_2) = n_2$ where $n_1 + n_2 = n$. Clearly, $e(G) = e(G_1) + e(G_2) + 1$. Furthermore, set $n_1 \equiv q_1 \pmod 3$, $n_2 \equiv q_2 \pmod 3$ where $q_1, q_2 \in \{0, 1, 2\}$.

If $q_1 = 0$ or $q_2 = 0$, without loss of generality, say $q_1 = 0$. By the induction hypothesis, $e(G_1) \leq \frac{4n_1-3}{3}$, $e(G_2) \leq \frac{4n_2-3-q_2}{3}$. If $e(G_1) < \frac{4n_1-3}{3}$ or $e(G_2) < \frac{4n_2-3-q_2}{3}$, then $e(G) < \frac{4n-3-q_2}{3}$. If $e(G_1) = \frac{4n_1-3}{3}$ and $e(G_2) = \frac{4n_2-3-q_2}{3}$, then by the induction hypothesis, $G_1 \in \mathcal{G}_{n_1}^0$, $G_2 \in \mathcal{G}_{n_2}^{q_2}$. It follows that $G = G_1 + G_2 \in \mathcal{G}_n^{q_2}$ and $e(G) = \frac{4n-3-q_2}{3}$.

If $q_1 = 1$ and $q_2 = 1$, by the hypothesis induction, $e(G_1) \leq \frac{4n_1-4}{3}$, $e(G_2) \leq \frac{4n_2-4}{3}$. If $e(G_1) < \frac{4n_1-4}{3}$ or $e(G_2) < \frac{4n_2-4}{3}$, then $e(G) < \frac{4n-5}{3}$. If $e(G_1) = \frac{4n_1-4}{3}$ and $e(G_2) = \frac{4n_2-4}{3}$, then by the induction hypothesis, $G_1 \in \mathcal{G}_{n_1}^1$, $G_2 \in \mathcal{G}_{n_2}^1$. It follows that $G \in \mathcal{G}_n^2$ and $e(G) = \frac{4n-5}{3}$.

If $q_1 \in \{1, 2\}$ and $q_2 = 2$, then $e(G_1) \le \frac{4n_1-3-q_1}{3}$ and $e(G_2) \le \frac{4n_2-5}{3}$. Thus $e(G) \le \frac{4n-5-q_1}{3} < \frac{4n-2-q_1}{3}$. □

So we get the following result for $k = 3$.

Theorem 3.4. *$f(n; \overline{\kappa}_3 = 1) = \frac{4n-3-q}{3}$, where $n = 3r + q$ and $0 \le q \le 2$.*

4 The case $k = 4$

In this section, we turn our consideration to the case $k = 4$. Similarly, we will give a necessary and sufficient condition for $\overline{\kappa}_4(G) = 1$. First of all, we begin with a claim useful for simplifying our argument. Let $P_1 = u_1w_1w_2 \dots w_kv_1$ be an ear of a cycle C. Set $H = C \cup P_1$ and add another ear $P_2 = u_2w_1'w_2' \dots w_l'v_2$ to H. We claim that there is always a cycle C' in $H \cup P_2$ which has two ears in the following cases: if $u_2, v_2 \in V(C)$, then $C' = C_1^*$; if $u_2, v_2 \in V(P_1)$, then $C' = C_2^*$; if $u_2 \in v_1Cu_1$, $v_2 \in V(P_1)$ and P_1 is an open ear, then $C' = C_3^*$; if $u_2 \in v_1Cu_1$, $v_2 \in V(P_1)$ and P_1 is a closed ear, then $C' = C_4^*$. See Figure 2 for an illustration.

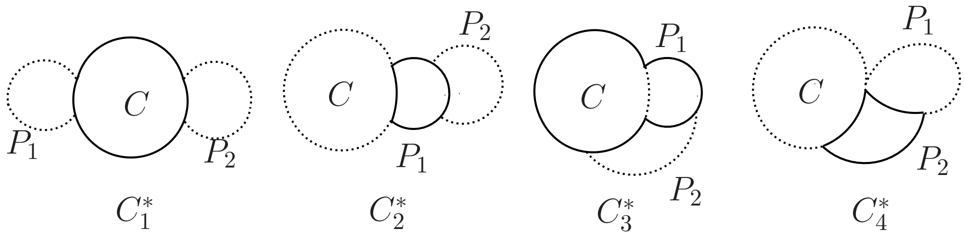

Figure 2. $C_i^*(1 \le i \le 4)$

Proposition 4.1. *Let G be a connected graph. Then $\overline{\kappa}_4(G) = 1$ if and only if every cycle in G has at most one ear.*

Proof. To settle the "only if" part, let C be a cycle in G. Assume, to the contrary, that C has two ears P_1 and P_2. In Figure 3, we list all cases that C has two ears. The marked dots are the chosen four vertices, and different trees are marked with different lines. Note that an ear P of the cycle C divides this cycle into two segments, say C_1 and C_2. If an ear P of C has length 1, then both C_1 and C_2 have length at least 2. In this case, we replace P with C_1 such that $P \cup C_2$ forms a new cycle and C_1 is an ear of this cycle, which has length at least 2. From Figure 3, we can find two internally disjoint trees connecting four vertices in G, a contradiction.

To prove the "if" part, since every maximal bridgeless subgraph of G is a cycle C or $C \cup P$, where P is an ear of C, then every edge incident to a maximal bridgeless subgraph of G is a cut edge of G. Similar to Proposition 3.1, it is easy to check that only one tree connects every four vertices in G, and so $\overline{\kappa}_4(G) = 1$. □

Lemma 4.2. *Let G be a connected graph of order 5 and size 6. Then $G \in \{B + K_1, D_1, D_2, D_3\}$ and $\overline{\kappa}_4(G) = 1$.*

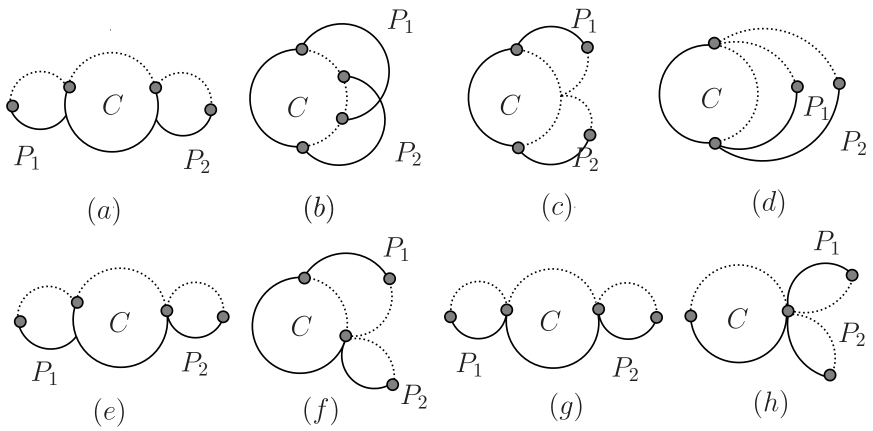

Figure 3. Graphs for Proposition 4.1

Proof. We can easily obtain $\delta(G) \leq 2$; otherwise $e(G) \geq \frac{3n}{2} = \frac{15}{2}$. If $\delta(G) = 1$, by deleting a vertex of degree 1, say v, we obtain a graph $G^* = K_4 \setminus e$. Observe that $G^* + K_1$ has no cycle with two ears. Thus by Proposition 4.1, $\overline{\kappa}_4(G) = 1$.

Suppose that $\delta(G) = 2$, without loss of generality, let $d(v) = 2$. Then $G \setminus v$ is C_4 or $C_3 + K_1$. Adding v back, there are four graphs D_1, D_2, D_3 or $B + K_1$, and for each of the graphs, $\overline{\kappa}_4(G) = 1$. □

Lemma 4.3. *Let G be a connected graph of order 5 and size at least 7. Then $\overline{\kappa}_4(G) \geq 2$.*

Proof. By Observation 2.1, we need to check the case that G has order 5 and size exactly 7. First, similar to Lemma 4.2, $\delta(G) \leq 2$. Suppose that $\delta(G) = 1$, without loss of generality, let $d(v) = 1$. Then $|V(G \setminus v)| = 4$ and $e(G \setminus v) = 6$, which implies that $G \setminus v$ is K_4. Then there are two internally disjoint trees connecting the four vertices of the clique K_4. It follows that $\overline{\kappa}_4(G \setminus v) \geq 2$, and hence $\overline{\kappa}_4(G) \geq 2$.

If $\delta(G) = 2$, suppose that v has degree 2, then $|V(G \setminus v)| = 4$ and $e(G \setminus v) = 5$, giving that $G \setminus v$ is $K_4 \setminus e$. Adding v again, the graph G has a cycle with two ears, and by Proposition 4.1, $\overline{\kappa}_4(G) \geq 2$. □

Lemma 4.4. *Let G be a connected graph of order 6 and size 7. Then $G \in \{B + \{K_1\}^2, \{C_3\}^2, D_1 + K_1, D_2 + K_1, D_3 + K_1, F_1, F_2, F_3, F_4\}$ and $\overline{\kappa}_4(G) = 1$.*

Proof. Obviously, $\delta(G) \leq 2$. If $\delta(G) = 1$, by deleting a vertex of degree 1 we get the graphs in Lemma 4.2. Adding v again, it is easy to check that $\overline{\kappa}_4(G) = 1$.

If $\delta(G) = 2$, without loss of generality, let $d(v) = 2$, then $|V(G \setminus v)| = 5$ and $e(G \setminus v) = 5$. Then $G \setminus v$ is C_5 or $C_4 + K_1$ or $K_3 + \{K_1\}^2$. Adding v again, the graph G belongs to $\{B + \{K_1\}^2, F_1, F_2, F_3, F_4\}$, and for each of the graphs, it is easy to check that $\overline{\kappa}_4(G) = 1$. □

Lemma 4.5. *Let G be a connected graph of order 6 and size at least 8. Then $\overline{\kappa}_4(G) \geq 2$.*

Proof. By Observation 2.1, we need to check the case that G has order 6 and size exactly 8. We can easily obtain $\delta(G) \leq 2$; otherwise $e(G) \geq \frac{3n}{2} = 9$. If $\delta(G) = 1$, we delete a vertex of degree one to get a graph of order 5 and size 7. Then by Lemma 4.3, it follows that $\overline{\kappa}_4(G) \geq 2$.

If $\delta(G) = 2$, without loss of generality, let $d(v) = 2$, then $|V(G \setminus v)| = 5$ and $e(G \setminus v) = 6$. It follows that $G \setminus v$ is one of the graphs in Lemma 4.2. Adding v again, there is a cycle with two ears, and by Proposition 4.1, $\overline{\kappa}_4(G) \geq 2$. □

Theorem 4.6. *Let $n = 4r + q$, where $0 \leq q \leq 3$, and let G be a connected graph of order n such that $\overline{\kappa}_4(G) = 1$. Then*

$$e(G) \leq \begin{cases} \frac{3n-2}{2} & if\ q = 0, \\ \frac{3n-3}{2} & if\ q = 1, \\ \frac{3n-4}{2} & if\ q = 2, \\ \frac{3n-3}{2} & if\ q = 3. \end{cases}$$

with equality if and only if $G \in \mathcal{H}_n^q$.

Proof. We apply induction on n. For $n = 4$, it is easy to see that $e(G) \leq 5$ and if $e(G) = 5$, and then $G = B \in \mathcal{H}_n^0$. For $n = 5$, if G is a connected graph of order 5 and size at least 7, then $\overline{\kappa}_4(G) \geq 2$ by Lemma 4.3. In other cases, either $e(G) \leq 5$ or $G \in \mathcal{H}_n^1$ by Lemma 4.2. For $n = 6$, if G is a connected graph of order 6 and size at least 8, then $\overline{\kappa}_4(G) \geq 2$ by Lemma 4.5. In other cases, either $e(G) \leq 6$ or $G \in \mathcal{H}_n^2$ by Lemma 4.4. Let $n \geq 7$, and suppose that the assertion holds for graphs of order less than n. We show that the assertion holds for graphs of order n. We consider two cases according to whether or not G has cut edges.

If G has no cut edge, then G is bridgeless, and combining with Proposition 4.1, G is a cycle or a cycle with an ear. If G is a cycle, then $e(G) = n < \frac{3n-4}{2}$, since $n \geq 7$. If G is a cycle with an ear, then $e(G) = n + 1 < \frac{3n-4}{2}$, since $n \geq 7$.

Suppose that G has cut edges. Without loss of generality, let e be a cut edge. Let G_1 and G_2 be two connected components of $G \backslash e$. Set $V(G_1) = n_1$, $V(G_2) = n_2$ where $n_1 + n_2 = n$. Clearly, $e(G) = e(G_1) + e(G_2) + 1$. Furthermore, set $n_1 \equiv q_1 \pmod 4$, $n_2 \equiv q_2 \pmod 4$ where $q_1, q_2 \in \{0, 1, 2, 3\}$.

If $q_1 = 0$, $q_2 \in \{0, 1, 2\}$ or $q_1 = 1$, $q_2 = 1$, by the induction hypothesis, $e(G_1) \leq \frac{3n_1-2-q_1}{2}$, $e(G_2) \leq \frac{3n_2-2-q_2}{2}$. If $e(G_1) < \frac{3n_1-2-q_1}{2}$ or $e(G_2) < \frac{3n_2-2-q_2}{2}$, then $e(G) < \frac{3n-2-q_1-q_2}{2}$. If $e(G_1) = \frac{3n_1-2-q_1}{2}$ and $e(G_2) = \frac{3n_2-2-q_2}{2}$, then by the induction hypothesis, $G_1 \in \mathcal{H}_{n_1}^{q_1}$, $G_2 \in \mathcal{H}_{n_2}^{q_2}$, and it follows that $G = G_1 + G_2 \in \mathcal{H}_n^{q_1+q_2}$ and $e(G) = \frac{3n-2-q_1-q_2}{2}$.

If $q_1 = 0$, $q_2 = 3$, by the induction hypothesis, $e(G_1) \leq \frac{3n_1-2}{2}$, $e(G_2) \leq \frac{3n_2-3}{2}$. If $e(G_1) < \frac{3n_1-2}{2}$ or $e(G_2) < \frac{3n_2-3}{2}$, then $e(G) < \frac{3n-3}{2}$. If $e(G_1) = \frac{3n_1-2}{2}$ and $e(G_2) = \frac{3n_2-3}{2}$, then by the induction hypothesis, $G_1 \in \mathcal{H}_{n_1}^0$, $G_2 \in \mathcal{H}_{n_2}^3$, and it follows that $G = G_1 + G_2 \in \mathcal{H}_n^3$ and $e(G) = \frac{3n-3}{2}$.

If $q_1 = 1$, $q_2 = 2$, then $e(G_1) \leq \frac{3n_1-3}{2}$ and $e(G_2) \leq \frac{3n_2-4}{2}$, and thus $e(G) \leq \frac{3n-5}{2} < \frac{3n-3}{2}$.

If $q_1 = 1$, $q_2 = 3$, then $e(G_1) \le \frac{3n_1-3}{2}$, $e(G_2) \le \frac{3n_2-3}{2}$, and so $e(G) \le \frac{3n-4}{2} < \frac{3n-2}{2}$.

If $q_1 = 2$, $q_2 = 2$, then $e(G_1) \le \frac{3n_1-4}{2}$, $e(G_2) \le \frac{3n_2-4}{2}$, and it follows that $e(G) \le \frac{3n-6}{2} < \frac{3n-3}{2}$.

If $q_1 = 2$, $q_2 = 3$, then $e(G_1) \le \frac{3n_1-4}{2}$, $e(G_2) \le \frac{3n_2-3}{2}$, and so $e(G) \le \frac{3n-5}{2} < \frac{3n-3}{2}$.

If $q_1 = 3$, $q_2 = 3$, by the induction hypothesis, $e(G_1) \le \frac{3n_1-3}{2}$, $e(G_2) \le \frac{3n_2-3}{2}$. If $e(G_1) < \frac{3n_1-3}{2}$ or $e(G_2) < \frac{3n_2-3}{2}$, then $e(G) < \frac{3n-4}{2}$. If $e(G_1) = \frac{3n_1-3}{2}$ and $e(G_2) = \frac{3n_2-3}{2}$, then by the induction hypothesis, $G_1 \in \mathcal{H}^3_{n_1}$, $G_2 \in \mathcal{H}^3_{n_2}$, and it follows that $G = G_1 + G_2 \in \mathcal{H}^2_n$ and $e(G) = \frac{3n-4}{2}$. □

So we get the following result for $k = 4$.

Theorem 4.7.

$$f(n; \overline{\kappa}_4 = 1) = \begin{cases} \frac{3n-2}{2} & if\ q = 0, \\ \frac{3n-3}{2} & if\ q = 1, \\ \frac{3n-4}{2} & if\ q = 2, \\ \frac{3n-3}{2} & if\ q = 3, \end{cases}$$

where $n = 4r + q$ and $0 \le q \le 3$.

5 The case $k = n$

Let us turn now to the case $k = n$. Let $n \ge 5$, since $k = 3$ and $k = 4$ have been considered before. Observe that in this case the edge disjoint trees are the same as the internally disjoint trees.

Theorem 5.1. *Let G be a connected graph of order n such that $\overline{\kappa}_n(G) = 1$ where $n \ge 5$. Then $e(G) \le \binom{n-1}{2} + 1$, with equality if and only if $G \in \mathcal{K}_n$.*

Proof. Let $G = K_5 \setminus M$, where M is a subset of the edges of K_5. On one hand, to make $\overline{\kappa}_5(G) = 1$, M should contain at least 3 edges by Lemma 2.3, and then $e(G) \le 7$. On the other hand, to form two edge-disjoint spanning trees, G should contain at least 8 edges, since each tree consists of at least 4 edges. Thus, G must have order 5 and size 7, meaning that it belongs to $\mathcal{K}_5$. It suffices to verify the case $n \ge 6$. By Lemma 2.3 again, to make $\overline{\kappa}_n(G) = 1$, $e(G) \le \binom{n}{2} - (n-2) = \binom{n-1}{2} + 1$.

Now we show that $\mathcal{K}_n$ is equal to $K_{n-1} + K_1$. Suppose H is a graph with order n, size $\binom{n-1}{2} + 1$ and $\overline{\kappa}_n(H) = 1$ but different from $K_{n-1} + K_1$.

We claim that $2 \le \delta(H) \le n-3$. Otherwise, if $\delta(H) = 1$, then $H = K_{n-1} + K_1$. If $\delta(H) \ge n-2$, then $e(H) = \frac{\Sigma_{v\in V(H)} d(v)}{2} \ge \frac{(n-2)n}{2}$, H is obtained from K_n by deleting at most $\frac{n}{2}$ edges. Since $n \ge 6$, then $\frac{n}{2} \le n-3$. By Lemma 2.3, H has two edge-disjoint spanning trees, a contradiction.

Let v be a vertex of H with degree equal to $\delta(H)$, and let $H^* = H \setminus v$. Since there are $n - 1 - d(v)$ vertices not adjacent to v in H and H is obtained from K_n by deleting $n-2$ edges, H^* is obtained from K_{n-1} by deleting $n - 2 - (n - 1 - d(v)) = d(v) - 1 \le (n-1) - 3$ edges. By Lemma 2.3, H^* has two edge-disjoint spanning trees T_1^* and T_2^*. By adding an edge incident with v to T_1^* and T_2^* respectively, we will obtain two edge-disjoint spanning trees of H, a contradiction. Thus $\mathcal{K}_n$ is equal to $K_{n-1} + K_1$. □

So we get the following result for $k = n$.

Theorem 5.2. *$f(n; \overline{\kappa}_n = 1) = \binom{n-1}{2} + 1$ where $n \geq 5$.*

Remark: Let G be a connected graph. For $k = 3$ and $k = 4$, we get necessary and sufficient conditions for $\overline{\kappa}_k(G) = 1$ by means of the number of ears of cycles. Naturally, one might think that this method can always be applied for $k = 5$, i.e., every cycle in G has at most two ears, but unfortunately we found a counterexample: Let G be a graph which contains a cycle with three independent closed ears. Set $C = u_1u_2u_3$, $P_1 = u_1v_1w_1u_1$, $P_2 = u_2v_2w_2u_2$, and $P_3 = u_3v_3w_3u_3$. Then, $\overline{\kappa}_5(G) = 1$. In fact, let S be the set of chosen five vertices. Obviously, for each i, if v_i and w_i are in S, then $\overline{\kappa}_5(S) = 1$. So only one vertex in $P_i \setminus u_i$ can be chosen. Suppose that v_1, v_2, v_3 have been chosen. By symmetry, u_1, u_2 are chosen. It is easy to check that there is only one tree connecting $\{u_1, u_2, v_1, v_2, v_3\}$. The remaining case is that all u_1, u_2 and u_3 are chosen. Then, no matter which are the other two vertices, only one tree can be found.

For general k with $5 \leq k \leq n-1$, we can only get the following lower bound of $f(n; \overline{\kappa}_k = 1)$. The exact value is not easy to obtain.

Theorem 5.3.

$$f(n; \overline{\kappa}_k = 1) \geq \begin{cases} r\binom{k-1}{2} + r - 1, & \text{if } q = 0; \\ r\binom{k-1}{2} + \binom{q}{2} + r, & \text{if } 1 \leq q \leq k-2. \end{cases}$$

for $n = r(k-1) + q$, $0 \leq q \leq k-2$.

Proof. If $q = 0$, let $G = \{K_{k-1}\}^r$, then $e(G) = r\binom{k-1}{2} + r - 1$. If $1 \leq q \leq k-2$, let $G = \{K_{k-1}\}^r + K_q$, and then $e(G) = r\binom{k-1}{2} + \binom{q}{2} + r$. In every case, it is easy to verify that $\overline{\kappa}_k(G) = 1$. □

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