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Approximating Vizing's independence number conjecture

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Abstract

In 1965, Vizing conjectured that the independence ratio of edge-chromatic critical graphs is at most $\frac{1}{2}$. We prove that for every $\epsilon > 0$ this conjecture is equivalent to its restriction on a specific set of edge-chromatic critical graphs with independence ratio smaller than $\frac{1}{2} + \epsilon$.

1 Introduction

All graphs in this article are simple. If G is a graph, then $V(G)$ denotes its vertex set and $E(G)$ denotes its edge set. If $e \in E(G)$ has end vertices v and w, then we also use the term vw to denote e. If v is a vertex of G, then $N_G(v)$ denotes the set of its neighbors, and $|N_G(v)|$ is the degree of v, which is denoted by $d_G(v)$. The maximum degree and the minimum degree of a vertex of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For $i \in \{1, \dots, \Delta(G)\}$ let $V_i(G) = \{v : d_G(v) = i\}$.

A k-edge-coloring of G is a function $\phi : E(G) \longrightarrow \{1, \dots, k\}$ such that $\phi(e) \neq \phi(f)$ for adjacent edges e and f. The chromatic index $\chi'(G)$ is the smallest number k such that there is k-coloring of G. In 1965 Vizing proved the fundamental result on the chromatic index of simple graphs.

Theorem 1.1 ([11]). *If G is a graph, then $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.*

Theorem 1.1 leads to a natural classification of simple graphs into two classes, namely Class 1 and Class 2 graphs depending upon whether their edge chromatic number is Δ and $\Delta+1$. For $k \geq 2$, a graph G is k-critical if $\Delta(G) = k$, $\chi'(G) = k+1$ and $\chi'(G - e) = k$ for every $e \in E(G)$. Let $\mathcal{C}(k)$ be the set of k-critical graphs, and $\mathcal{C} = \bigcup_{k=2}^{\infty} \mathcal{C}(k)$ be the set of critical graphs.

If G is a graph, then $\alpha(G)$ denotes the maximum cardinality of an independent set of vertices in G. The independence ratio of G is $\frac{\alpha(G)}{|V(G)|}$ and it is denoted by $\iota(G)$. In 1965, Vizing [10] conjectured that the independence ratio of edge-chromatic critical graphs is at most $\frac{1}{2}$.

Conjecture 1.2 ([10])**.** *If $G \in \mathcal{C}$, then $\iota(G) \leq \frac{1}{2}$.*

Clearly, Conjecture 1.2 can be reformulated as follows.

Conjecture 1.3 ([10])**.** *For all $k \geq 2$, if $G \in \mathcal{C}(k)$, then $\iota(G) \leq \frac{1}{2}$.*

Since the 2-critical graphs are the odd circuits, it follows that Conjecture 1.3 is true for $k = 2$. It is an open question whether it is true for $k \geq 3$. It is easy to see, that the bound $1/2$ cannot be replaced by a smaller one. The first results on this topic were obtained by Brinkmann et al. [1] who proved that the independence ratio of critical graphs is smaller than $\frac{2}{3}$. In [3] Conjecture 1.2 is verified for overfull graphs, i.e. graphs G with $|E(G)| > \Delta(G)\lfloor\frac{|V(G)|}{2}\rfloor$. In 2006, Luo and Zhao [4] proved that the conjecture is true for critical graphs whose order is at most twice the maximum degree of the graph. Later some improvements were achieved for specific values of Δ, see [4, 5, 6, 8, 9]. In 2011, Woodall [12] completed a major step in this research by proving that the independence ratio of critical graphs is bounded by $\frac{3}{5}$.

The main result of this article is that for each $\epsilon > 0$, Conjecture 1.2 is equivalent to its restriction on a specific set $\mathcal{C}_\epsilon$ of critical graphs and $\iota(G) < \frac{1}{2} + \epsilon$ for each $G \in \mathcal{C}_\epsilon$. For the proof of this statement we will deduce similar results for $\mathcal{C}(k)$, for each $k \geq 3$.

2 k-critical graphs and Meredith extension

This section first studies k-critical graphs and Conjecture 1.3. One of the fundamental statements in the theory of edge-coloring of graphs is Vizing's Adjacency Lemma.

Lemma 2.1 (Vizing's Adjacency Lemma [11])**.** *Let G be a critical graph. If $xy \in E(G)$, then at least $\Delta(G) - d_G(y) + 1$ vertices in $N_G(x) \setminus \{y\}$ have degree $\Delta(G)$.*

Lemma 2.1 implies that if v is a vertex of a k-critical graph, then it is adjacent to at least two vertices of degree k.

Definition 2.2. *For $k \geq 2$ and $t \geq 0$ let $\mathcal{C}(k,t)$ be the set of k-critical graphs G with the following properties:*

1. *$\delta(G) \geq k - 1$.*
2. *every $v \in V_{k-1}(G)$ is the initial vertex of $k-1$ distinguished paths $p_1^t(v), \dots, p_{k-1}^t(v)$ such that for all $i, j \in \{1, \dots, k-1\}$:*
   - *(a) $V(p_i^t(v)) \cap V_{k-1}(G) = \{v\}$,*
   - *(b) $|V(p_i^t(v))| \geq 2t(k-1) + 2$,*
   - *(c) if $i \neq j$, then $V(p_i^t(v)) \cap V(p_j^t(v)) = \{v\}$, and*
   - *(d) if $w \in V_{k-1}(G)$ and $w \neq v$, then $V(p_i^t(v)) \cap V(p_j^t(w)) = \emptyset$.*

For $k \geq 0$ and $t \geq 0$, let $\iota(k) = \sup\{\iota(G) : G \in \mathcal{C}(k)\}$ and $\iota(k,t) = \sup\{\iota(G) : G \in \mathcal{C}(k,t)\}$. We will prove that for any $k \geq 3$ and any $t \geq 0$, Conjecture 1.3 for $\mathcal{C}(k)$ is equivalent to its restriction on $\mathcal{C}(k,t)$. We prove upper bounds for $\iota(k,t)$ and $\lim_{t\to\infty} \iota(k,t) = \frac{1}{2}$. These statements are used to deduce the main result of this article.

The 2-critical graphs are the odd circuits and for any $k \geq 2$, there exists a k-critical graph G with $\delta(G) = 2$. Hence, the following lemma is an obvious consequence of Lemma 2.1 and Definition 2.2.

Proposition 2.3. *1. $\mathcal{C}(3,0) = \mathcal{C}(3)$ and $\mathcal{C}(2,t) = \mathcal{C}(2)$ for all $t \geq 0$.*

2. If $k \geq 2$ and $t \geq 0$, then $\mathcal{C}(k,t+1) \subseteq \mathcal{C}(k,t) \subseteq \mathcal{C}(k)$.

The following operation on graphs was first studied by Meredith [7].

Definition 2.4. *Let $k \geq 2$ and G be a graph with $\Delta(G) = k$, $v \in V(G)$ with $d_G(v) = d$, and let $v_1, \ldots, v_d$ be the neighbors of v. Let $u_1, \ldots, u_k$ be the vertices of degree $k-1$ in a complete bipartite graph $K_{k,k-1}$. The graph H is a Meredith extension of G (applied on v) if it is obtained from $G - v$ and $K_{k,k-1}$ by adding edges $v_i u_i$ for each $i \in \{1, ..., d\}$.*

The following theorem is Theorem 2.1 in [2].

Theorem 2.5 ([2])**.** *Let $k \geq 2$, G be a graph with $\Delta(G) = k$ and M be a Meredith extension of G. Then G is k-critical if and only if M is k-critical.*

Lemma 2.6. *Let $k \geq 2$, G be a graph with $\Delta(G) = k$ and H be a Meredith extension of G. Then $\iota(G) \leq \frac{1}{2}$ if and only if $\iota(H) \leq \frac{1}{2}$.*

Proof. We prove $\iota(G) > \frac{1}{2}$ if and only if $\iota(H) > \frac{1}{2}$.

Let $v \in V(G)$ and H be the Meredith extension of G applied on v. We have $|V(H)| = |V(G)| + 2k - 2$ and hence $|V(H)|$ and $|V(G)|$ have the same parity.

($\Rightarrow$) Let I_G be an independent set of G with more than $\frac{1}{2}|V(G)|$ vertices.

If $v \in I_G$, then all neighbors of v are not in I_G. Hence, H has an independent set I_H of cardinality $|I_G| - 1 + k$. Therefore, $|I_H| = |I_G| + k - 1 > \frac{1}{2}(|V(G)| + 2k - 2) = \frac{1}{2}|V(H)|$.

If $v \notin I_G$, then H has an independent set I_H of cardinality $|I_G| + (k-1)$, e.g. $I_G \cup V_k(K_{k,k-1})$. We deduce $|I_H| > \frac{1}{2}|V(H)|$ as above.

($\Leftarrow$) Let I_H be an independent set of H with $|I_H| > \frac{1}{2}|V(H)|$. We can assume that I_H is maximum. Let $K_{k,k-1}$ be the subgraph of H which was added to $G - v$ by applying Meredith extension on v.

If there is a vertex $w \in V_{k-1}(K_{k,k-1})$ which has a neighbor in $(V(H) - V(K_{k,k-1})) \cap I_H$, then $|V(K_{k,k-1}) \cap I_H| = k - 1$. Hence, if we contract $K_{k,k-1}$ to a single vertex v (to obtain G), then $I_G = I_H - V(K_{k,k-1})$ is an independent set in G which contains $|I_H| - (k-1)$ vertices. Hence $|I_G| = |I_H| - (k-1) > \frac{1}{2}(|V(H)| - (2k-2)) = \frac{1}{2}|V(G)|$.

If for every vertex $w \in V_{k-1}(K_{k,k-1})$ all neighbors in $H - V(K_{k,k-1})$ are not in I_H, then $|V(K_{k,k-1}) \cap I_H| = k$. If we contract $K_{k,k-1}$ to a single vertex v, then $I_G = (I_H - V(K_{k,k-1})) \cup \{v\}$ is an independent set in G. As above, we deduce that $|I_G| > \frac{1}{2}|V(G)|$. □

Lemma 2.7. *For every $k \geq 2$ and every $t \geq 0$: Every k-critical graph G can be extended to a graph $H \in \mathcal{C}(k,t)$ by a sequence of Meredith extensions.*

Proof. For $k = 2$ there is nothing to prove. Let $k \geq 3$. We first show that G can be extended to a graph of $\mathcal{C}(k,0)$. If $G \in \mathcal{C}(k,0)$, then we are done. Assume that $G \in \mathcal{C}(k) \setminus \mathcal{C}(k,0)$. We proceed in three steps. For an example see Figures 1, 2 and 3 (without step 2).

(1) Repeated application of Meredith extension on all vertices of degree smaller than $k-1$, yields a graph G_1 with $d_{G_1}(v) \in \{k-1,k\}$, for all $v \in V(G_1)$.

(2) Repeated application of Meredith extension on vertices of degree $k-1$ which are adjacent to another vertex of degree $k-1$, yields a graph G_2, with $d_{G_2}(v) \in \{k-1,k\}$, for all $v \in V(G_2)$, and $V_{k-1}(G_2)$ is an independent set.

(3) Repeated application of Meredith extension on vertices of degree $k-1$ which have a common neighbor yields a graph G_3 with $d_{G_3}(v) \in \{k-1,k\}$, $V_{k-1}(G_3)$ is an independent set, and $N_{G_3}(u) \cap N_{G_3}(w) = \emptyset$ for any two vertices $u,w \in V_{k-1}(G_3)$.

Let $H = G_3$. By Theorem 2.5, H is k-critical and it satisfies the conditions of Definition 2.2 for $t = 0$. Hence, $H \in \mathcal{C}(k,0)$.

Next we show that every graph G' of $\mathcal{C}(k,s)$ ($s \geq 0$) can be extended to a graph H' of $\mathcal{C}(k,s+1)$ by a sequence of Meredith extensions. Let $v \in V_{k-1}(G')$ and $p_j^s(v)$ be one of the $k-1$ distinguished paths which have v as initial vertex. Let z be the terminal vertex of $p_j^s(v)$. Apply Meredith extension on z and extend $p_j^s(v)-z$ to a path $p_j^{s+1}(v)$ that contains all vertices of the $K_{k,k-1}$ which is used in the Meredith extension. Then $|V(p_j^{s+1}(v))| = |V(p_j^s(v))| + 2k - 2 \geq 2s(k-1) + 2 + 2k - 2 = 2(s+1)(k-1) + 2$. If we repeat this procedure on all terminal vertices of the distinguished paths of G' we obtain a graph $H' \in \mathcal{C}(k,s+1)$. □

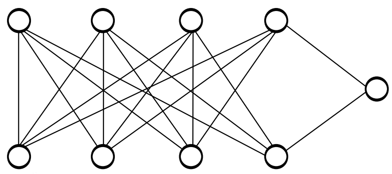

Figure 1: Graph $H \in \mathcal{C}(4)$

The notation in Figures 1, 2 and 3 are used in the proof of Theorem 2.11. For $i \in \{1,2,3\}$, the paths $p_i^0(v)$ and $p_i^0(w)$ are indicated by the bold edges. The following lemma is obvious.

Lemma 2.8. *Let $k \geq 2$, $t \geq 0$ and $G \in \mathcal{C}(k,t)$. If H is a Meredith extension of G, then $H \in \mathcal{C}(k,t)$.*

Theorem 2.9. *For every $k \geq 2$ and every $t \geq 0$: $\iota(k) \leq \frac{1}{2}$ if and only if $\iota(k,t) \leq \frac{1}{2}$.*

Proof. By Proposition 2.3, $\mathcal{C}(k,t) \subseteq \mathcal{C}(k)$ for all $k \geq 2$ and $t \geq 0$. Hence, if $\iota(k) \leq \frac{1}{2}$ then $\iota(k,t) \leq \frac{1}{2}$.

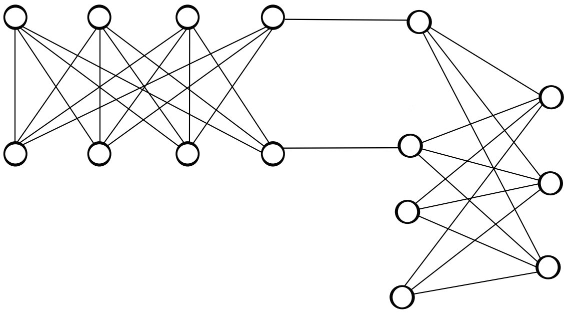

Figure 2: Graph $H' \in \mathcal{C}(4)$ (Step 1)

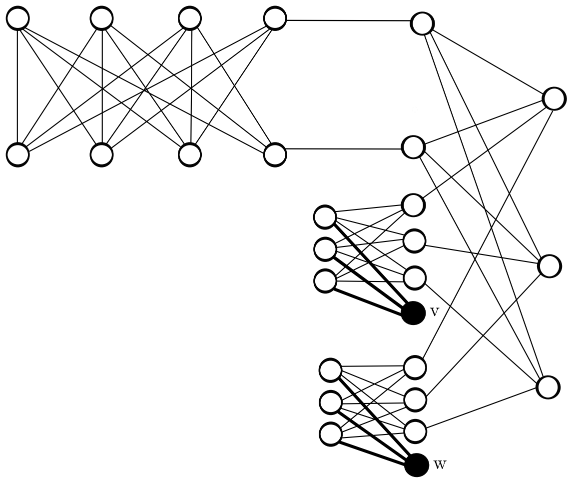

Figure 3: Graph $H_0 \in \mathcal{C}(4,0)$ (Step 3)

Let $G \in \mathcal{C}(k)$. If there is $t' \geq t$ such that $G \in \mathcal{C}(k,t')$, then we are done, since $\mathcal{C}(k,t') \subseteq \mathcal{C}(k,t)$ by Proposition 2.3. If $G \notin \mathcal{C}(k,t')$ for all $t' \geq t$, then it follows with Lemma 2.7 that there exists $H \in \mathcal{C}(k,t)$ which is obtained from G by a sequence of Meredith extensions. By our assumption, $\iota(H) \leq \frac{1}{2}$ and hence, $\iota(G) \leq \frac{1}{2}$ by Lemma 2.6. Therefore, $\iota(k) \leq \frac{1}{2}$. □

Theorem 2.10. *Let $k \geq 2$, $t \geq 0$ and $\varphi(k,t) = t(k-1)^2 + k - 1$. If $G \in \mathcal{C}(k,t)$, then $\iota(G) < \frac{1}{2} + \frac{1}{4k\varphi(k,t)+2}$.*

Proof. If $G \in \mathcal{C}(2)$, then $\iota(G) < \frac{1}{2}$. Let $G \in \mathcal{C}(k,t)$ ($k \geq 3$, $t \geq 0$) and I be an independent set of G and $Y = V(G) - I$. Let $I_k = I \cap V_k(G)$, $I_{k-1} = I \cap V_{k-1}(G)$, $Y_k = Y \cap V_k(G)$, $Y_{k-1} = Y \cap V_{k-1}(G)$.

Clearly, I contains vertices of $V_{k-1}(G)$. Let v be such a vertex. By definition, there are $k-1$ distinguished paths $p_1^t(v), \ldots, p_{k-1}^t(v)$ such that for all $i, j \in \{1, \ldots, k-1\}$

(a) $V(p_i^t(v)) \cap V_{k-1}(G) = \{v\}$,

(b) $|V(p_i^t(v))| \geq 2t(k-1) + 2$,

(c) if $i \neq j$, then $V(p_i^t(v)) \cap V(p_j^t(v)) = \{v\}$, and

(d) if $w \in V_{k-1}(G)$ and $w \neq v$, then $V(p_i^t(v)) \cap V(p_j^t(w)) = \emptyset$.

Consequently, $|Y \cap V(p_i^t(v))| \geq t(k-1)+1$ for each $i \in \{1,\dots,k-1\}$, and therefore $\varphi(k,t)|I_{k-1}| \leq |Y|$. Let $m_Y = |E(G[Y])|$. Since G is a critical graph it follows that $m_Y > 0$. With $|I_{k-1}| \leq \frac{1}{\varphi(k,t)}|Y|$ we deduce

$$k|I| - \frac{1}{\varphi(k,t)}|Y| \leq k|I| - |I_{k-1}| \leq k|Y| - 2m_Y < k|Y|.$$

Since $Y = V(G) - I$, it follows that

$$|I| < \frac{k + \frac{1}{\varphi(k,t)}}{2k + \frac{1}{\varphi(k,t)}}|V(G)|.$$

Therefore, $\iota(G) < \frac{1}{2} + \frac{1}{4k\varphi(k,t)+2}$ □

We now deduce our main results.

Theorem 2.11. *For each $k \geq 2$: $\lim_{t\to\infty} \iota(k,t) = \frac{1}{2}$.*

Proof. The statement is trivial for $k = 2$. We will first prove the following claim.

Claim 2.11.1. *For all $k \geq 3$ and $t \geq 0$: $\iota(k,t) \geq \frac{1}{2}$.*

We show that for every $\epsilon > 0$ and all $k \geq 3$ and $t \geq 0$ the set $\mathcal{C}(k,t)$ contains a graph G with $i(G) > \frac{1}{2} - \epsilon$.

Let H be the graph which is obtained from the complete bipartite graph $K_{k,k}$ by subdividing one edge. It is easy to see that H is k-critical. Let H' be the graph obtained from H by applying Meredith extension on the divalent vertex of H and let H_0 be the graph obtained from H' by applying Meredith extension on all vertices of $V_{k-1}(H')$. Hence, $H_0 \in \mathcal{C}(k,0)$. To obtain a graph H_t of $\mathcal{C}(k,t)$ ($t \geq 1$) apply Meredith extension on the terminal vertices of the distinguished paths of H_{t-1} as described in the proof of Lemma 2.7. Starting with $H_t = H_t^0$, construct an infinite sequence $H_t^0, H_t^1 \dots$ of graphs by Meredith extension. By Lemma 2.8, these graphs are in $\mathcal{C}(k,t)$.

If H_t^i is obtained from H by applying Meredith extension n_i times, then $|V(H_t^i)| = 2(k + n_ik - n_i) + 1$ and it has an independent set of $k + n_ik - n_i$ vertices. Hence, $\alpha(H) \geq \frac{1}{2} - \frac{1}{2(2k+2n_i(k-1)+1)}$. Choose n_i such that $2k + 2n_i(k-1) + 1 > \frac{1}{2\epsilon}$ and the claim is proved.

By Theorem 2.10, we have $\iota(k,t) \leq \frac{1}{2} + \frac{1}{4k\varphi(k,t)+2}$, where $\varphi(k,t) = t(k-1)^2 + k - 1$. Since $\varphi(k,t+1) > \varphi(k,t)$ it follows with the Claim 2.11.1 that $\lim_{t\to\infty} \iota(k,t) = \frac{1}{2}$. □

Theorem 2.12. *For every $\epsilon > 0$, there is a set $\mathcal{C}_\epsilon$ of critical graphs such that*

1. *$\iota(G) \leq \frac{1}{2}$ for every $G \in \mathcal{C}$ if and only if $\iota(G) \leq \frac{1}{2}$ for every $G \in \mathcal{C}_\epsilon$.*

2. *If $G \in \mathcal{C}_\epsilon$, then $\iota(G) < \frac{1}{2} + \epsilon$.*

Proof. Let $\epsilon > 0$ be given. We first construct $\mathcal{C}_\epsilon$. Let $\varphi(k,t) = t(k-1)^2 + k - 1$ and for $k = 3$ choose $t_3 \geq 0$ such that $\frac{1}{4k\varphi(k,t_3)+2} = \frac{1}{12\varphi(3,t_3)+2} < \epsilon$. Let $\mathcal{C}_\epsilon = \bigcup_{k=2}^{\infty} \mathcal{C}(k,t_3)$.

We have $\mathcal{C} = \bigcup_{k=2}^{\infty} \mathcal{C}(k)$. For $k \geq 2$ it follows with Theorem 2.9 that $\iota(G) \leq \frac{1}{2}$ for every $G \in \mathcal{C}(k)$ if and only if $\iota(G) \leq \frac{1}{2}$ for every $G \in \mathcal{C}(k,t_3)$. Therefore, $\iota(G) \leq \frac{1}{2}$ for every $G \in \mathcal{C}$ if and only if $\iota(G) \leq \frac{1}{2}$ for every $G \in \mathcal{C}_\epsilon$.

It remains to prove statement 2. Let $G \in \mathcal{C}_\epsilon$. If $G \in \mathcal{C}(2)$, then $\iota(G) < \frac{1}{2}$. Let $k \geq 3$ and $G \in \mathcal{C}(k,t_3)$. We have $\varphi(k+1,t) > \varphi(k,t)$ and thus, $\frac{1}{4k\varphi(k,t_3)+2} \leq \frac{1}{12\varphi(3,t_3)+2} < \epsilon$. It follows with Theorem 2.10 that $\iota(G) < \frac{1}{2} + \frac{1}{4k\varphi(k,t_3)+2} < \frac{1}{2} + \epsilon$. Therefore, if $G \in \mathcal{C}_\epsilon$, then $\iota(G) < \frac{1}{2} + \epsilon$. □

Concluding remark

Let $s \in \{1,\ldots,k-1\}$. The main results (Theorems 2.11 and 2.12) can also be deduced if we ask for the existence of s distinguished paths in Definition 2.2, say to define $\mathcal{C}_s(k,t)$. If we change $\varphi(k,t)$ in Theorem 2.10 to $\varphi_s(k,t) = st(k-1) + s$, then we similarly can deduce that if $G \in \mathcal{C}_s(k,t)$, then $\iota(G) < \frac{1}{2} + \frac{1}{4k\varphi_s(k,t)+2}$. The two natural choices for s are 1 and $k-1$. We took $k-1$ since then the structural properties of 3-critical graphs which are implied by Vizing's Adjacency Lemma are generalized to graphs of $\mathcal{C}(k,0)$.

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