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# Core partitions with $d$-distinct parts

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**Abstract**

In this paper, we study $(s, s+1)$-core partitions with $d$-distinct parts. We obtain results on the number and the largest size of such partitions, so we extend Xiong's paper in which results are obtained about $(s, s+1)$-core partitions with distinct parts. Also we propose a conjecture about $(s, s+r)$-core partitions with $d$-distinct parts for $1 \leq r \leq d$.

## 1 Introduction

A *partition* of $n$ is a finite nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $n = \lambda_1 + \lambda_2 + \ldots + \lambda_l$. A summand in a partition is called *part.* We say that $n$ is the size of $\lambda$ and $l$ is the length of $\lambda$. For example, $\lambda = (6, 3, 3, 2, 1)$ is a partition of $n = 15$. The parts of the partition $\lambda$ are $6, 3, 3, 2$ and 1. The size of $\lambda$ is 15 and the length of $\lambda$ is 5. A partition $\lambda$ is called *a partition with d-distinct parts* if and only if $\lambda_i - \lambda_{i+1} \geq d$i for $1/lei \leq l - 1$.

Partitions can be visualized with a *Young diagram*, which is a finite collection of boxes arranged in left-justified rows, with $\lambda_i$ boxes in the $i$-th row. The pair $(i, j)$ shows the coordinates of the boxes in the Young diagram. The Young diagram of $\lambda = (6, 3, 3, 2, 1)$ is as follows:

| | | | | | |
|---|---|---|---|---|---|
| $(1,1)$ | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(1,5)$ | $(1,6)$ |
| $(2,1)$ | $(2,2)$ | $(2,3)$ | | | |
| $(3,1)$ | $(3,2)$ | $(3,3)$ | | | |
| $(4,1)$ | $(4,2)$ | | | | |
| $(5,1)$ | | | | | |

For each box in the Young diagram in coordinates $(i,j)$, the *hook length* is defined as the sum of the number of boxes exactly to the right, exactly below, and the box itself. So the hook lengths of the partition $\lambda = (6,3,3,2,1)$ can be given as follows:

| | | | | | |
|---|---|---|---|---|---|
| 10 | 8 | 6 | 3 | 2 | 1 |
| 6 | 4 | 2 | | | |
| 5 | 2 | 1 | | | |
| 2 | 1 | | | | |
| 1 | | | | | |

(1.1)

Here $h(i,j)$ will show the entry in coordinate $(i,j)$ of the box, that is, the hook length of the box. If $\lambda = (6,3,3,2,1)$, then $h(1,1) = 10,\ h(2,3) = 2$ and $h(6,1) = 1$, as you can see from (1.1).

A partition $\lambda$ is called an *s-core* partition if $\lambda$ has no boxes of hook length $s$. For example, the partition $\lambda = (6,3,3,2,1)$ is a 7-core but it is not a 5-core, since $\lambda$ has no boxes of hook length 7, but it has a box of hook length 5 (see Diagram (1.1)).

A more general definition: a partition $\lambda$ is called an *$(s_1, s_2, \ldots, s_t)$-core* partition if $\lambda$ has no boxes of hook length $s_1, s_2, \ldots, s_t$. So for example the partition $\lambda = (6,3,3,2,1)$ is a $(7,9)$-core partition.

There are many studies about core partitions, and such partitions are closely related to posets, cranks, Raney numbers, Catalan numbers, Fibonacci numbers, etc.; see [1, 6, 15, 16].

Anderson [3] shows that the number of $(s,t)$-core partitions is finite if and only if $s$ and $t$ are coprime. In this case, this number is

$$\frac{1}{s+t}\binom{s+t}{s}.$$

Olsson and Stanton [9] give the largest size of such partitions. Some results on the number, the largest size and the average size of such partitions are provided in [2, 4, 7, 8, 10, 13, 14, 15]. In particular, the number of $(s,s+1)$-core partitions is the Catalan number

$$C_s = \frac{1}{s+1}\binom{2s}{s}.$$

Amdeberhan [1] conjectures that the number of $(s,s+1)$-core partitions with distinct parts equals the Fibonacci numbers. This conjecture is proved independently by Xiong [12] and Straub [11]. More generally, Straub [11] characterizes the number $N_d(s)$ of $(s,ds-1)$-core partitions with distinct parts by $N_d(1) = 1, N_d(2) = d$ and, for $s \geq 3$,

$$N_d(s) = N_d(s-1) + dN_d(s-2).$$

Xiong [12] also obtain results on the number, the largest size and the avarage size of $(s, s+1)$-core partitions with distinct parts.

In this paper, we consider the problem of counting the number of special partitions which are $s$-core for certain values of $s$. More precisely, we focus on $(s, s+1)$-core partitions with $d$-distinct parts. We obtain results on the number and the largest size of such partitions, and so we extend Xiong's paper in which the results are obtained about $(s, s+1)$-core partitions with distinct parts. Also, we propose the following conjecture about $(s, s+r)$-core partitions with $d$-distinct parts for $1 \leq r \leq d$. That is, we conjecture that the number $N_{d,r}(s)$ of $(s, s+r)$-core partitions with $d$-distinct parts is characterized by $N_{d,r}(s) = s$ for $1 \leq s \leq d$, $N_{d,r}(d+1) = d+r$, and for $s \geq d+2$,

$$N_{d,r}(s) = N_{d,r}(s-1) + N_{d,r}(s-(d+1))$$

for $1 \leq r \leq d$.

## 2 $(s, s+1)$-core partitions with $d$-distinct parts

Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a partition whose corresponding Young diagram has $l$ rows. The set $\beta(\lambda)$ of $\lambda$ is defined to be the set of first column hook length in the Young diagram of $\lambda$, i.e., $\beta(\lambda) = \{h(i,1) : 1 \leq i \leq l\}$. For example, if $\lambda = (6, 3, 3, 2, 1)$, then we get

$$\begin{aligned} \beta(\lambda) &= \{h(1,1), h(2,1), h(3,1), h(4,1), h(5,1)\} \\ &= \{10, 6, 5, 2, 1\} \end{aligned}$$

by using Diagram (1.1).

Now we generalize the definition of the twin-free set in [11].

**Definition 2.1** *Suppose that $d$ is a positive integer such that $d \geq 2$. A set $X \subseteq \mathbb{N}$ is called a $d$-th order twin-free set if there is no $x \in X$ such that*

$$\{x, x+k\} \subseteq X, \quad \text{for } 1 \leq k \leq d.$$

If we take $d = 1$ in Definition 2.1, then we obtain the twin-free set in [11], that is, we obtain the first order twin-free set.

**Example 2.1** Let us take $X = \{10, 5, 2\}$. For $d = 2$, $X$ is a second order twin-free set, since the sets

$$\{2,3\}, \{5,6\}, \{10,11\}, \{2,4\}, \{5,7\}, \{10,12\}$$

are not a subset of $X$. But the set $\{2, 5\}$ is a subset of $X$, so $X$ is not a third order twin-free set.

**Theorem 2.1** *(i) Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a partition. Then*

$$\lambda_i = h(i,1) - l + 1, \quad \textit{for } 1 \le i \le l.$$

*Thus*

$$|\lambda| = \sum_{x \in \beta(\lambda)} x - \binom{l}{2}.$$

*(ii) A partition $\lambda$ is an $s$-core partition if and only if for any $x \in \beta(\lambda)$ with $x > s$, we always have $x - s \in \beta(\lambda)$.*

**Proof.** See [3, 5]. □

**Lemma 2.1** *The partition $\lambda$ is a partition with $d$-distinct parts if and only if $\beta(\lambda)$ is a $d$-th order twin free set.*

**Proof.** Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a partition. Now $\lambda$ is a partition with $d$-distinct parts if and only if $\lambda_i - \lambda_{i+1} \ge d$ for $1 \le i \le l-1$. Then by Theorem 2.1(i),

$$\begin{aligned} h(i,1) - h(i+1,1) &= (\lambda_i + l - 1) - (\lambda_{i+1} + l - 1) \\ &= \lambda_i - \lambda_{i+1} \\ &\ge d. \end{aligned}$$

So we obtain

$$h(i,1) - h(i+1,1) \ge d \quad \text{if and only if} \quad \beta(\lambda) \text{ is a } d\text{-th order twin-free set.}$$

□

**Lemma 2.2** *Suppose $\lambda$ is an $(s, s+1)$-core partition with $d$-distinct parts. Then*

$$\beta(\lambda) \subset \{1, 2, \ldots, s-1\}.$$

**Proof.** Suppose that $\lambda$ is a partition with $d$-distinct parts. Then, $\beta(\lambda)$ is a $d$-th order twin-free set by Lemma 2.1. Since $\lambda$ is an $(s, s+1)$-core partition, we have $s, s+1 \notin \beta(\lambda)$. If $x \ge s+2$ and $x \in \beta(\lambda)$ then, by Theorem 2.1(ii), we know that $x-s, x-(s+1) \in \beta(\lambda)$. But this is a contradiction since $\beta(\lambda)$ is a $d$-th order twin-free set. That is, $x \notin \beta(\lambda)$ and so we get the required result $\beta(\lambda) \subset \{1, 2, \ldots, s-1\}$. □

**Lemma 2.3** *A partition $\lambda$ is an $(s, s+1)$-core partition with $d$-distinct parts if and only if $\beta(\lambda)$ is a $d$-th order twin-free subset of the set $\{1, 2, \ldots, s-1\}$.*

**Proof.** If a partition $\lambda$ is an $(s, s+1)$-core partition with $d$-distinct parts then by Lemma 2.2, $\beta(\lambda)$ must be a subset of $\{1, 2, \ldots, s-1\}$. Also, By Lemma 2.1, $\beta(\lambda)$ must be a $d$-th order twin-free set.

Conversely, suppose that $\beta(\lambda)$ is a $d$-th order twin-free subset of $\{1, 2, \ldots, s-1\}$. By Lemma 2.1, $\lambda$ is a partition with $d$-distinct parts. Also, since $\beta(\lambda)$ is a subset of the set $\{1, 2, \ldots, s-1\}$, all the hook lengths of the corresponding partition are smaller than $s$ and $s+1$. This means that $\lambda$ is an $(s, s+1)$-core partition. □

**Theorem 2.2** *The number $N_d(s)$ of $(s, s+1)$-core partitions with $d$-distinct parts is characterized by $N_d(s) = s$ for $1 \leq s \leq (d+1)$, and for $s \geq d+2$,*

$$N_d(s) = N_d(s-1) + N_d(s-(d+1)).$$

**Proof.** Let $X_k$ denote the set of all $d$-th order twin-free subsets of the set $\{1, 2, \ldots, k-1\}$. A partition $\lambda$ is an $(s, s+1)$-core partition with $d$-distinct parts if and only if $\beta(\lambda)$ is a $d$-th order twin-free subset of the set $\{1, 2, \ldots, s-1\}$ by Lemma 2.3. That is, $N_d(s) = |X_s|$. Suppose that $X \in X_s$. If $s-1 \in X$, then $s-2, s-3, \ldots, s-(d+1) \notin X$, since $X$ is a $d$-th order twin-free set. So

$$|\{X \in X_s : (s-1) \in X\}| = \left|X_{s-(d+1)}\right|,$$

and

$$|\{X \in X_s : (s-1) \notin X\}| = |X_{s-1}|.$$

Thus $|X_s| = |X_{s-1}| + \left|X_{s-(d+1)}\right|$. Notice that

$$\begin{array}{rcl} N_d(1) & = & |X_1| = 1 \\ N_d(2) & = & |X_2| = 2 \\ \vdots & \vdots & \vdots \\ N_d(d) & = & |X_d| = d \\ N_d(d+1) & = & |X_{d+1}| = d+1. \end{array}$$

So we obtain the required result. □

If we take the value $d = 1$ in Theorem 2.2, we find that the number of $(s, s+1)$-core partitions with distinct parts is the Fibonacci number $F_{s+1}$ in [11, 12].

**Example 2.2** For $d = 2$, $N_2(6) = 9$. The seven $(6, 7)$-core partitions with 2-distinct parts are

$$\{\}, \{1\}, \{2\}, \{3\}, \{3, 1\}, \{4\}, \{4, 1\}, \{5\}, \{4, 2\}.$$

We can see in Table 1 the number $N_2(s)$ of $(s, s+1)$-core partitions with 2-distinct parts for $1 \leq s \leq 8$.

| $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $N_2(s)$ | 1 | 2 | 3 | 4 | 6 | 9 | 13 | 19 |

Table 1: The number $N_2(s)$ of $(s, s+1)$-core partitions with 2-distinct parts

The generating function of the sequence $N_2(s)$ is

$$-\frac{x^2 + x + 1}{x^3 + x - 1}.$$

Also, the sequence $N_2(s)$ satisfies the recurrence relation

$$N_2(s) = N_2(s-1) + N_2(s-3).$$

**Theorem 2.3** *If $s \equiv 0, 1$ or $2 \pmod{d+2}$ then the largest size of $(s, s+1)$-core partitions with $d$-distinct parts is*

$$\left[\frac{1}{d+2}\binom{s+1}{2} + \frac{s(d-1)}{2(d+2)}\right],$$

*or otherwise*

$$\left[\frac{1}{d+2}\binom{s+1}{2} + \frac{s(d-1)}{2(d+2)} + 1\right]$$

*where $[x]$ is the largest integer not greater than $x$.*

**Proof.** Let $\lambda$ be an $(s, s+1)$-core partition with $d$-distinct parts. Suppose that $\beta(\lambda) = \{x_1, x_2, \ldots, x_k\}$. We need to maximize $\lambda$ and since $\beta(\lambda)$ is a $d$-th order twin-free set, we need $x_1 = s-1$, $x_2 = s-1-(d-1)$, and generally $x_i = s - d(i-1) - i$, so

$$\begin{aligned} |\lambda| &= \sum_{i=1}^{k} x_i - \binom{k}{2} \\ &\leq \sum_{i=1}^{k} (s - d(i-1) - i) - \binom{k}{2} \\ &= sk + \frac{dk - dk^2}{2} - k^2. \end{aligned}$$

Also, to maximize $\lambda$, we want to take $k$ as large as possible; however we also have to subtract the $\binom{k}{2}$ term. So if $x_k < (k-1) = \binom{k}{2} - \binom{k-1}{2}$, the gain we have made by including $x_k$ is offset by the loss of the second term. So there are sometimes two $(s, s+1)$ cores with $d$-distinct parts and maximal size: this is when we have $x_k = k-1$, and so it makes no difference whether we include this term or not.

When $s = (d+2)n$ for some integer $n$, we obtain

$$|\lambda| \leq sk + \frac{dk - dk^2}{2} - k^2 \leq \frac{(d+2)n^2}{2} + \frac{dn}{2}.$$

When $s = (d+2)n + r$, where $1 \leq r \leq d+1$, for some integer $n$, we obtain

$$|\lambda| \leq sk + \frac{dk - dk^2}{2} - k^2 \leq \frac{(d+2)n^2}{2} + \frac{dn}{2} + rn + (r-1).$$

So we can get the desired result for each case. □

If we take the value $d = 1$ in Theorem 2.3, we find that the largest size of the $(s, s+1)$-core partitions with distinct parts is $\left[\frac{1}{3}\binom{s+1}{2}\right]$ in [12].

**Example 2.3** For $s = 6$ and $d = 2$, since $s \equiv 2 \pmod 4$, the largest size of $(6, 7)$-core partitions with 2-distinct parts is

$$\left[\frac{1}{2+2}\binom{6+1}{2} + \frac{6(2-1)}{2(2+2)}\right] = 6,$$

by Theorem 2.3. Indeed, $(6,7)$-core partitions with 2-distinct parts are

$$\{\}, \{1\}, \{2\}, \{3\}, \{3,1\}, \{4\}, \{4,1\}, \{5\}, \{4,2\}.$$

So the largest size of $(6,7)$-core partitions with 2-distinct parts is $4+2=6$.

For $s=7$ and $d=2$, since $s \equiv 3 \pmod 4$, the largest size of $(7,8)$-core partitions with 2-distinct parts is

$$\left[\frac{1}{2+2}\binom{7+1}{2} + \frac{7(2-1)}{2(2+2)} + 1\right] = 8,$$

by Theorem 2.3. Indeed, $(7,8)$-core partitions with 2-distinct parts are

$$\{\}, \{1\}, \{2\}, \{3\}, \{3,1\}, \{4\}, \{4,1\}, \{5\}, \{4,2\}, \{5,1\}, \{6\}, \{5,2\}, \{5,3\}.$$

So the largest size of $(7,8)$-core partitions with 2-distinct parts is $5+3=8$.

**Theorem 2.4** *If $s \equiv 1 \pmod{(d+2)}$ then there are two $(s,s+1)$-core partitions of largest size with d-distinct parts; otherwise there is only one such partition of largest size.*

**Proof.** Note that if $\lambda$ is an $(s,s+1)$-core partition with $d$-distinct parts which has the largest size, then $\beta(\lambda) = \{s-1, s-(d+2), \ldots, s-((k-1)d+k)\}$ for some integer $k$. When $t=(d+2)n$ for some integer $n$, we see that $\lambda$ has the largest size if and only if $k=n$. When $t=(d+2)n+1$ for some integer $n$, then $\lambda$ has the largest size if and only if $k=n$ or $k=n+1$. For all other cases $t=(d+2)n+r$, where $2 \leq r \leq d+1$, we have that $\lambda$ has the largest size if and only if $k=n+1$. So we obtain the desired result. □

If we take the value $d=1$ in Theorem 2.4, we get the number of the largest size of the $(s,s+1)$-core partitions with distinct parts is $\dfrac{3-(-1)^{s \bmod 3}}{2}$ in [12].

**Example 2.4** For $s=5$ and $d=2$, since $s \equiv 1 \pmod 4$, there are only two $(s,s+1)$-core partitions of largest size with 2-distinct parts by Theorem 2.4. Actually, $(5,6)$-core partitions with 2-distinct parts are

$$\{\}, \{1\}, \{2\}, \{3\}, \{3,1\}, \{4\}.$$

So there are two partitions of the largest size of $(6,7)$-core partitions with 2-distinct parts. These partitions are $\{3,1\}$ and $\{4\}$.

For $s=8$ and $d=3$, since $s \equiv 3 \pmod 5$, there is only one $(s,s+1)$-core partition of the largest size with 3-distinct parts by Theorem 2.4. Indeed, $(8,9)$-core partitions with 3-distinct parts are

$$\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{4,1\}, \{5\}, \{5,1\}, \{6\}, \{5,2\}, \{6,1\}, \{7\}, \{6,2\}, \{6,3\}.$$

So there is only one partition of the largest size of $(8,9)$-core partitions with 3-distinct parts. This partition is $\{6,3\}$.

## 3 $(s, s+r)$-core partitions with $d$-distinct parts

More generally, we propose a conjecture about the number of $(s, s+r)$-core partitions with $d$-distinct parts for $1 \leq r \leq d$. This conjecture is based on experimental evidence and has been verified for $s < 10$ after listing all relevant partitions. We will present some of our experimental results in Tables 2 and 3.

Table 2 shows $(s, s+2)$-core partitions with $d$-distinct partitions for $2 \leq d \leq 7$.

| $d$ \ $(s, s+2)$ | (1, 3) | (2,4) | (3,5) | (4,6) | (5,7) | (6,8) | (7,9) | (8,10) |
|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 2 | 4 | 5 | 7 | 11 | 16 | 23 |
| 3 | 1 | 2 | 3 | 5 | 6 | 8 | 11 | 16 |
| 4 | 1 | 2 | 3 | 4 | 6 | 7 | 9 | 12 |
| 5 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 10 |
| 6 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 |

Table 2: The number of $(s, s+2)$-core partitions with $d$-distinct parts

Table 3 shows $(s, s+3)$-core partitions with $d$-distinct partitions for $3 \leq d \leq 7$. According to our experiments, we present the following conjecture.

| $d$ \ $(s, s+3)$ | (1,4) | (2,5) | (3,6) | (4,7) | (5,8) | (6,9) | (7,10) | (8,11) |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2 | 3 | 6 | 7 | 9 | 12 | 18 |
| 4 | 1 | 2 | 3 | 4 | 7 | 8 | 10 | 13 |
| 5 | 1 | 2 | 3 | 4 | 5 | 8 | 9 | 11 |
| 6 | 1 | 2 | 3 | 4 | 5 | 6 | 9 | 10 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 10 |

Table 3: $(s, s+3)$-core partitions with $d$-distinct parts

**Conjecture 1** *For $1 \leq r \leq d$, the number $N_{d,r}(s)$ of $(s, s+r)$-core partitions with d-distinct parts is characterized by $N_{d,r}(s) = s$ for $1 \leq s \leq d$, $N_{d,r}(d+1) = d+r$, and for $s \geq d+2$,*

$$N_{d,r}(s) = N_{d,r}(s-1) + N_{d,r}(s-(d+1)).$$

**Example 3.1** For $s = 6, d = 3$ and $r = 2$, the eight $(s, s+r)$-core, i.e. the $(6, 8)$-core, partitions with 3-distinct parts are

$$\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 4\}, \{1, 6\}.$$

| $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $N_{3,2}(s)$ | 1 | 2 | 3 | 5 | 6 | 8 | 11 | 16 | 22 |

Table 4: The number $N_{3,2}(s)$ of $(s, s+2)$-core partitions with 3-distinct parts

We can see in Table 4 the number $N_{3,2}(s)$ of $(s, s+2)$-core partitions with 3-distinct parts for $1 \leq s \leq 9$. The generating function of the sequence $N_{3,2}(s)$ is

$$-\frac{2x^3 + x^2 + x + 1}{x^4 + x - 1}.$$

Also, the sequence $N_{3,2}(s)$ satisfies the recurrence relation

$$N_{3,2}(s) = N_{3,2}(s-1) + N_{3,2}(s-4).$$

□

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