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On the connectedness of 3-line graphs

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Abstract

One of the most familiar derived graphs is the line graph. The line graph $L(G)$ of a graph G is that graph whose vertices are the edges of G where two vertices of $L(G)$ are adjacent if the corresponding edges are adjacent in G. Two nontrivial paths P and Q in a graph G are said to be adjacent paths in G if P and Q have exactly one vertex in common and this vertex is an end-vertex of both P and Q. For an integer $\ell \geq 2$, the ℓ-line graph $L_\ell(G)$ of a graph G is the graph whose vertex set is the set of all ℓ-paths (paths of order ℓ) of G where two vertices of $L_\ell(G)$ are adjacent if they are adjacent ℓ-paths in G. Since the 2-line graph is the line graph $L(G)$ for every graph G, this is a generalization of line graphs. We study the 3-line graphs of several well-known classes of graphs. It is shown that $L_3(G) = G$ if and only if $G = C_n$ for some odd integer $n \geq 5$. Several sufficient conditions are presented for the 3-line graph of a connected graph to be connected. While the 3-line graph of a connected bipartite graph is disconnected, it is shown that the 3-line graph of every connected bipartite graph has at most two nontrivial components. Other results and an open question dealing with the connectedness of 3-line graphs are also presented.

1 Introduction

There are many graphs associated with a given graph. We refer to each of these graphs as a "derived graph". For a given graph G, a *derived graph* of G is a graph obtained by performing some operation on G. The study of structural properties of derived graphs is a popular area of research in graph theory. One of the most familiar graph operations on a graph is that of the line graph. The *line graph* $L(G)$ of a graph G is that graph whose vertices are the edges of G where two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. One of the best-known results on the structure of line graphs deals with forbidden subgraphs by Beineke [2] and another deals with isomorphic line graphs by Whitney [7]. A characterization of graphs whose line graph is Hamiltonian is due to Harary and Nash-Williams [5]. Iterated line graphs of almost all connected graphs were shown to be Hamiltonian by Chartrand [3]. Over the years, various generalizations of line graphs have been introduced and studied by many (see [1, 6], for example).

A more general class of derived graphs was inspired by line graphs. Because an edge in a graph G can be considered as a subgraph P_2 or a subgraph K_2 of G, an edge in the definition of line graph can be replaced by another subgraph of G, such as a path, a cycle or a complete graph, for example. Furthermore, we can think of an edge as the edge set of the path P_2 or of the complete graph K_2 and define adjacency of vertices in the resulting graph in terms of a prescribed property involving sets.

Let G be a connected graph of order at least 3. Two nontrivial paths P and Q in G are said to be *adjacent* in G if $V(P) \cap V(Q) = \{x\}$ where x is an end-vertex of both P and Q. For an integer $\ell \geq 2$, the *ℓ-line graph* $L_\ell(G)$ of a graph G is the graph whose vertex set is the set of all ℓ-paths (paths of order ℓ) of G where two vertices of $L_\ell(G)$ are adjacent if they are adjacent ℓ-paths in G. In particular, the standard line graph $L(G)$ of a graph G is the 2-line graph $L_2(G)$ of G. We now study ℓ-line graphs when $\ell = 3$, namely the 3-line graphs. Thus, the vertex set of the 3-line graph $L_3(G)$ of a graph G is the set of 3-paths of G where two vertices of $L_3(G)$ (two 3-paths of G) are adjacent if they have an end-vertex and only an end-vertex in common. Let G be a nontrivial connected graph and v a vertex of G. If $\deg v = 1$, then there is no P_3 in G whose interior vertex is v; while if $\deg v \geq 2$, then there are exactly $\binom{\deg v}{2}$ copies of P_3 whose interior vertex is v. Thus, we have the following result.

Proposition 1.1 *If G is a connected graph of order $n \geq 2$ with degree sequence d_1, d_2, …, d_n, then the order of $L_3(G)$ is $\sum_{i=1}^{n} \binom{d_i}{2}$.*

For example, the graph G of order 6 in Figure 1 is 3-regular and so its 3-line graph $L(G)$ has order 18 by Proposition 1.1. The graph $L(G)$ is also shown in Figure 1, where each 3-path P in G is indicated by the two edges of P.

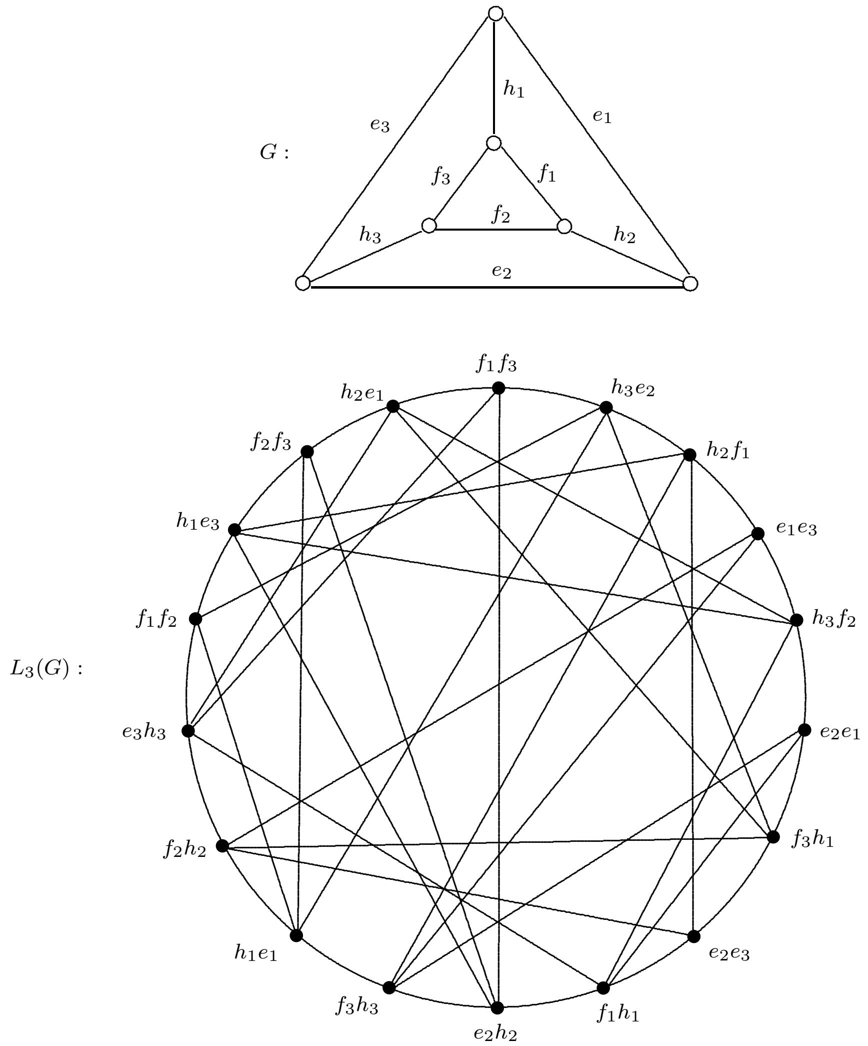

Figure 1: A graph G and $L_3(G)$

2 Preliminary Results on 3-Line Graphs

If G is a connected graph of order 3 or 4, then every two distinct paths of order 3 have two vertices in common and so $L_3(G)$ is empty. On the other hand, there are graphs G of order 5 or more for which $L_3(G)$ is empty. In fact, if G is a connected graph of order at least 5, then $L_3(G)$ is an empty graph if and only if G does not contain P_5 as a subgraph. This observation gives rise to the characterization of those graphs having an empty 3-line graph. For integers a and b with $2 \leq a \leq b$, the *double star* $S_{a,b}$ is that tree of diameter 3 whose two central vertices have degrees a and b.

Proposition 2.1 *Let G be a connected graph of order at least 5. Then $L_3(G)$ is an empty graph if and only if $G \in \{K_{1,t}, K_{1,t}+e, S_{a,b}\}$ where $t \geq 4$ and $a+b \geq 5$.*

It is known that if G is a nontrivial connected graph, then $L(G)$ is also connected. However, this is not the case for 3-line graphs. In fact, $L_3(G)$ is disconnected for all connected bipartite graphs.

Proposition 2.2 *If G is a connected bipartite graph of order at least 5, then $L_3(G)$ is disconnected.*

Proof. Let U and W be the partite sets of G. Since G is a bipartite graph, it follows that, for each 3-path (x, y, z) of G, either $x, z \in U$ or $x, z \in W$. Let $\mathcal{P}_1$ be the set of those 3-paths of G whose two end-vertices are in U and let $\mathcal{P}_2$ be the set of those 3-paths of G whose two end-vertices are in W. If $Q_1 \in \mathcal{P}_1$ and $Q_2 \in \mathcal{P}_2$ are two 3-paths in G, then Q_1 and Q_2 are not adjacent in $L_3(G)$. This implies that there is no path in $L_3(G)$ connecting a 3-path in $\mathcal{P}_1$ and a 3-path in $\mathcal{P}_2$. Therefore, $L_3(G)$ is disconnected. ■

The following result provides the structure of the 3-line graph of complete bipartite graphs. We state this without proof.

Proposition 2.3 *For integers s and t such that $2 \le s \le t$ and $t \ge 3$, the 3-line graph $L_3(K_{s,t})$ of the complete bipartite graph $K_{s,t}$ consists of two graphs H_1 and H_2, where*

- *H_1 is a $[2(t-1)(s-2)]$-regular graph of order $t\binom{s}{2}$ and*
- *H_2 is a $[2(s-1)(t-2)]$-regular graph of order $s\binom{t}{2}$.*

If $s \ge 3$, then $L_3(K_{s,t})$ consists of two nontrivial components, namely H_1 and H_2.

For each integer $n \ge 3$, $L(C_n) = C_n$. In fact, not only is $L(C_n) = C_n$, but it is known that if G is a connected graph of order $n \ge 3$, then $L(G) \cong G$ if and only if $G = C_n$. The corresponding result holds for 3-line graphs of odd cycles as well. To see this, we first determine the 3-line graphs of cycles.

Proposition 2.4 *For an integer $n \ge 5$, the 3-line graph of the cycle C_n of order n is*

$$L_3(C_n) = \begin{cases} 2C_{\frac{n}{2}} & \text{if } n \ge 6 \text{ is even} \\ C_n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n)$ be the cycle of order $n \ge 5$. A 3-path P_3 in C_n is (v_i, v_{i+1}, v_{i+2}) for some integer i with $1 \le i \le n$. The subscript of each vertex is expressed as an integer modulo n. Thus, the vertex set of $L_3(C_n)$ is

$$V(L_3(C_n)) = \{x_i = (v_i, v_{i+1}, v_{i+2}) : \ 1 \le i \le n\},$$

where then $x_{n-1} = (v_{n-1}, v_n, v_1)$ and $x_n = (v_n, v_1, v_2)$. Thus, the order of $L_3(C_n)$ is $n' = n$. For each integer $i \in \{1, 2, \ldots, n\}$, the vertex x_i is adjacent to the two vertices x_{i-2} and x_{i+2} in $L_3(C_n)$. Hence, $L_3(C_n)$ is a 2-regular graph. If n is even (and so

n' is even), then $L_3(C_n)$ is the union of two cycles Q_1 and Q_2 of order $n/2$, where $Q_1 = (x_1, x_3, \ldots, x_{n-1}, x_1)$ and $Q_2 = (x_2, x_4, \ldots, x_n, x_2)$; while if n is odd (and so n' is odd), then $L_3(C_n) = C_n = (x_1, x_3, \ldots, x_n, x_2, x_4, \ldots, x_{n-1}, x_1)$ is also an n-cycle. ■

We now verify the result mentioned above.

Theorem 2.5 *Let G be a connected graph of order at least 3. Then $L_3(G) \cong G$ if and only if $G = C_n$ for some odd integer $n \geq 5$.*

Proof. If $G = C_n$ for some odd integer $n \geq 5$, then $L_3(C_n) = C_n$ by Proposition 2.4. For the converse, suppose that G is a nontrivial connected graph of order n and size m such that $L_3(G) \cong G$. Then $m \geq n - 1$. If $m = n - 1$, then G is a tree. Since $L_3(T) \ncong T$ for every tree T of order at least 3 by Proposition 2.1 and Proposition 2.2, it follows that $m \geq n$. Assume, to the contrary, that $G \ncong C_n$ for some odd integer $n \geq 5$. By Proposition 2.4 then, $G \ncong C_n$ for an even integer $n \geq 6$. Thus, $G \ncong C_n$ for any integer $n \geq 5$. Since $m \geq n$, it follows that $\Delta(G) = \Delta \geq 3$. If $\delta(G) \geq 2$, then the order of $L_3(G)$ exceeds n by Proposition 1.1, a contradiction. Thus, G contains end-vertices. For each integer i with $1 \leq i \leq \Delta$, let n_i be the number of vertices of degree i in G. Since $2m \geq 2n$, it follows that

$$2m = \sum_{i=1}^{\Delta} i n_i \geq 2n = 2(n_1 + n_2 + \cdots + n_\Delta)$$

and so $n_1 \leq n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta$. Hence,

$$\begin{aligned} n &= n_1 + n_2 + \cdots + n_\Delta \\ &\leq [n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta] + (n_2 + n_3 + \cdots + n_\Delta) \\ &= n_2 + 2n_3 + 3n_4 + \cdots + (\Delta - 1)n_\Delta = \sum_{i=2}^{\Delta} (i-1)n_i. \end{aligned}$$

On the other hand, since $L_3(G) \cong G$ and $\binom{i}{2} > i-1$ for each integer i with $3 \leq i \leq \Delta$, it follows that

$$n = \sum_{i=2}^{\Delta} \binom{i}{2} n_i > \sum_{i=2}^{\Delta} (i-1)n_i \geq n,$$

which is impossible. ■

3 Conditions for the Connectedness of 3-Line Graphs

One of the most important structural properties that a graph can possess is that of being connected. The connectedness of line graphs has been studied extensively (see [4, 8], for example). In this section, we discuss the connectedness of the 3-line graph of a graph. First, we provide sufficient conditions for the 3-line graph of a connected graph to be connected. The first condition we consider concerns 2-connected graphs containing odd cycles of sufficiently large length.

Theorem 3.1 *If G is a 2-connected graph containing an odd cycle of length 7 or more, then $L_3(G)$ is connected.*

Proof. Let $C = (u_1, u_2, \ldots, u_p, u_1)$ be a p-cycle of G, where $p \geq 7$ is odd. If $G = C$, then the result follows by Proposition 2.4. Thus, we may assume that $G \neq C$ and so there are 3-paths of G that are not on C. Let C' be the p-cycle in $L_3(G)$ where $C' = L_3(C)$. Let P be a 3-path of G that is not a path on C. We show that the vertex P in $L_3(G)$ is connected to a vertex of C'. We consider four possible locations of P in G. In what follows, the subscripts are expressed as integers modulo p.

Case 1. *All three vertices of P lie on C.* Let $P = (u_i, u_j, u_k)$, where $1 \leq i < j < k \leq p$. Then the vertices u_i, u_j, u_k divide C into three subpaths, namely a $u_i - u_j$ path, a $u_j - u_k$ path and a $u_k - u_i$ path. Since the length of C is 7 or more, at least one of the three subpaths of C has length 3 or more, say the $u_i - u_j$ path has this property. Then the vertex P of $L_3(G)$ is adjacent to the 3-path (u_i, u_{i+1}, u_{i+2}) in C'.

Case 2. *Exactly two vertices of P lie on C.* There are three possibilities here.

Subcase 2.1. *An edge of P is an edge of C, say $P = (u, u_j, u_{j+1})$, where $u \notin V(C)$.* Then the vertex P of $L_3(G)$ is adjacent to the 3-path $(u_{j+1}, u_{j+2}, u_{j+3})$ in C'.

Subcase 2.2. *$P = (u, u_j, u_k)$, where $u \notin V(C)$ and u_ju_k is a chord of C.* Here, either the path $(u_j, u_{j+1}, \ldots, u_k)$ or the path $(u_k, u_{k+1}, \ldots, u_j)$ on C has length 4 or more, say the former. Then the vertex P of $L_3(G)$ is adjacent to the 3-path (u_k, u_{k-1}, u_{k-2}) in C'.

Subcase 2.3. *$P = (u_i, v, u_k)$, where $v \notin V(C)$.* Here as well, either the path $(u_i, u_{i+1}, \ldots, u_k)$ or the path $(u_k, u_{k+1}, \ldots, u_i)$ on C has length 4 or more, say the former. Then the vertex P of $L_3(G)$ is adjacent to the 3-path (u_i, u_{i+1}, u_{i+2}) in C'.

Case 3. *Exactly one vertex of P lies on C.* There are two possibilities here.

Subcase 3.1. *$P = (u_i, v, w)$, where $v, w \notin V(C)$ or $P = (u, v, u_k)$, where $u, v \notin V(C)$.* We may assume that $P = (u_i, v, w)$, where $v, w \notin V(C)$. Here, the vertex P of $L_3(G)$ is adjacent to the 3-path (u_i, u_{i+1}, u_{i+2}) in C'.

Subcase 3.2. *$P = (u, u_j, w)$, where $u, w \notin V(C)$.* Since G is 2-connected, G contains a $u - u_i$ path for some vertex $u_i \in V(C) - \{u_j\}$ that does not contain any other vertex of C. If this path does not contain w, denote this path by P'. If this path contains w, then G contains a $w - u_i$ path P'' for some vertex $u_i \in V(C) - \{u_j\}$ that does not contain u and any other vertices of C. Since the situation for P' and P'' are similar, we may assume that G contains a $u - u_i$ path for some vertex u_i that does not contain any other vertex of C where $i < j$. Let $P' = (u = x_0, x_1, \ldots, x_\ell = u_i)$, where $\ell \geq 1$. Since the length of C is 7 or more, either the path $(u_i, u_{i+1}, \ldots, u_j)$ or the path $(u_j, u_{j+1}, \ldots, u_i)$ on C has length 4 or more, say the former.

Suppose first that $\ell \geq 2$ is even. Then

$$((u = x_0, x_1, x_2), (x_2, x_3, x_4), \ldots, (x_{\ell-2}, x_{\ell-1}, x_\ell = u_i), (u_i, u_{i+1}, u_{i+2}))$$

is a path in $L_3(G)$. If, on the other hand, $\ell \geq 1$ is odd, then

$$((u = x_0, x_1, x_2), (x_2, x_3, x_4), \ldots, (x_{\ell-1}, x_\ell = u_i, u_{i+1}), (u_{i+1}, u_{i+2}, u_{i+3}))$$

is a path in $L_3(G)$. In either situation, $P = (u, u_j, w)$ and $(u = x_0, x_1, x_2)$ are adjacent vertices in $L_3(G)$ and so the vertex P is connected to a vertex of C' in $L_3(G)$.

Case 4. *No vertex of P lies on C.* Then $P = (u, v, w)$, where $u, v, w \notin V(C)$. As we saw in Subcase 3.2, either there is a $u - u_i$ path in G for some vertex $u_i \in V(C)$ that contains neither v nor w or any other vertex of C or there is a $w - u_i$ path in G for some vertex $u_i \in V(C)$ that contains neither u nor w or any other vertex of C, say the former. Denote this path by P'. Then, as we saw in Subcase 3.2, the path P' can be used to show that the vertex P of $L_3(G)$ is connected to a vertex of C'. ■

It is necessary that the length of an odd cycle in the graph of Theorem 3.1 is at least 7. For example, the graph $H = C_5 + e$ is 2-connected and contains a 5-cycle, as shown in Figure 2. The 3-line graph $L_3(H)$ is a disconnected graph of order 9 with three components. This graph is also shown in Figure 2, where a vertex x of H is denoted by ijk if $x = (v_i, v_j, v_k)$ for some 3-element subset of $\{1, 2, 3, 4, 5\}$.

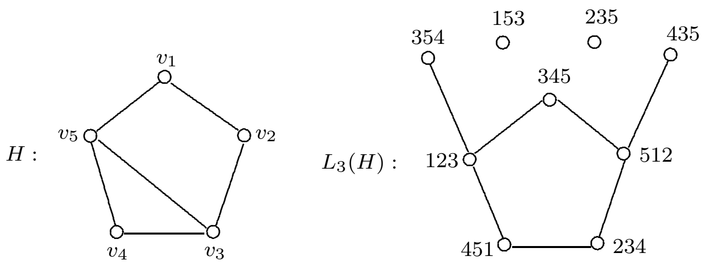

Figure 2: The graph H and its 3-line graph $L_3(H)$

The following is a consequence of Theorem 3.1.

Corollary 3.2 *If G is a Hamiltonian graph of odd order $n \geq 7$ or G is a graph of even order $n \geq 8$ such that $G - v$ is Hamiltonian for some vertex v of G, then $L_3(G)$ is connected.*

Proof. If G is a Hamiltonian graph of odd order $n \geq 7$, then $L_3(G)$ is connected by Theorem 3.1. Thus, we may assume that $n \geq 8$ is even. Let $C = (u_1, u_2, \ldots, u_{n-1}, u_1)$ be an $(n-1)$-cycle in G and let u_n be the vertex of G not on C. Since G is connected, u_n is adjacent to at least one vertex of C. Since $n - 1 \geq 7$ is odd, it again follows by Proposition 2.4 that $L_3(C_{n-1})$ is an odd cycle of order $n - 1$ in $L_3(G)$. Denote the subgraph $L_3(C)$ in $L_3(G)$ by C'. It suffices to show that each vertex of $L_3(G)$ not in C' is adjacent to some vertex of C' in $L_3(G)$. Let $P = (u_i, u_j, u_k)$ be a 3-path of G that is not a path on C. If no vertex of P is u_n, then, as in Case 1, the 3-path P is adjacent to a vertex of C'. Hence, we may assume that P contains u_n. First, suppose that $u_n = u_i$ or $u_n = u_k$, say the former, and so $P = (u_n, u_j, u_k)$. Then at least one of the $u_j - u_k$ paths on C has length 4 or more, say $(u_j, u_{j+1}, \ldots, u_k)$ has this property. Then $P' = (u_k, u_{k-1}, u_{k-2})$ is a vertex of C' that is adjacent to P. Next, suppose that $u_n = u_j$ and so $P = (u_i, u_n, u_k)$. Then at least one of the $u_i - u_k$ paths on C has length 4 or more, say $(u_i, u_{i+1}, \ldots, u_k)$ has this property. Then $P' = (u_i, u_{i+1}, u_{i+2})$ is a vertex of C' that is adjacent to P. Therefore $L_3(G)$ is connected. ■

We now illustrate Corollary 3.2. The Petersen graph P of order 10 has a 9-cycle. In fact, $P - v$ is Hamiltonian for each vertex v of P. By Corollary 3.2 then, $L_3(P)$ is connected. In fact, it can be shown that $L_3(P)$ is Hamiltonian. On the other hand, for each integer $r \geq 3$, the r-regular complete bipartite graph $K_{r,r}$ has no $(2r-1)$-cycle and $L_3(K_{r,r})$ is disconnected by Proposition 2.3 or by Proposition 2.2. Also, it is necessary that the even order of a graph of Corollary 3.2 be at least 8. For example, let F be the graph of order 6 obtained from the graph H of order 5 in Figure 2 by adding a new vertex v_6 and joining v_6 to the vertices v_2 and v_5 of H. Then $F - v_6 = H$ is Hamiltonian. Since the 3-path (v_2, v_3, v_5) is an isolated vertex of $L_3(F)$, it follows that $L_3(F)$ is disconnected.

It can be shown that the 3-line graph $L_3(K_n)$ is connected for $n \geq 5$. The graph $L_3(K_5)$ is shown in Figure 3 where $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$ and a vertex x of $L_3(K_5)$ is denoted by ijk if $x = (v_i, v_j, v_k)$ for some triple (i, j, k) of elements of $\{1, 2, 3, 4, 5\}$. This implies for a given integer $n \geq 5$ that there is a minimum positive integer $f(n)$ such that if G is a connected graph of order $n \geq 5$ with $\delta(G) \geq f(n)$, then $L_3(G)$ is connected. Next, we show that $f(5) = 4$ and $f(n) = \lceil \frac{n+1}{2} \rceil$ if $n \geq 6$.

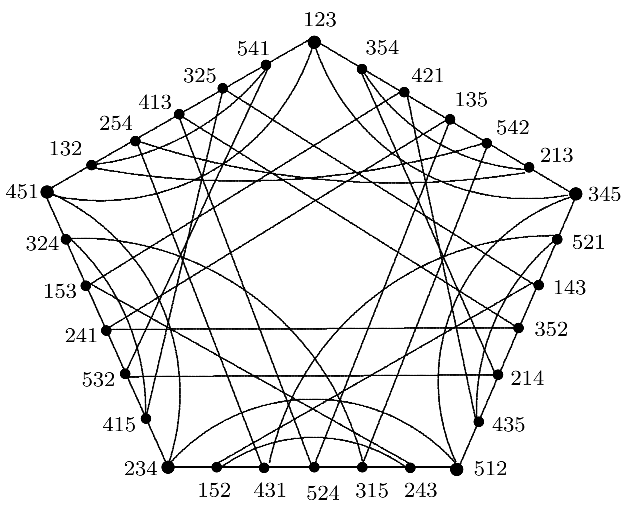

Figure 3: The graph $L_3(K_5)$

Theorem 3.3 *For an integer $n \geq 5$, let $f(n)$ be the minimum integer such that the 3-line graph $L_3(G)$ of every graph G of order n with $\delta(G) \geq f(n)$ is connected. Then*

$$f(n) = \begin{cases} 4 & \text{if } n = 5 \\ \lceil \frac{n+1}{2} \rceil & \text{if } n \geq 6. \end{cases}$$

Proof. We have seen that the 3-line graph $L_3(K_5)$ of K_5 is connected. Hence, $f(5) \leq 4$. Next, we show that $f(5) \geq 4$. The two graphs G_1 and G_2 shown in Figure 4 have order 5 and minimum degree 3. The 3-line graph $L_3(G_1)$ has three components, two of which are trivial components with vertices (v_2, v_1, v_4) and (v_3, v_1, v_5). The 3-line graph $L_3(G_2)$ has four components, three of which are trivial components resulting from the three 3-paths created from the triangle (v_1, v_3, v_4, v_1) in G_2. Thus, $f(5) = 4$.

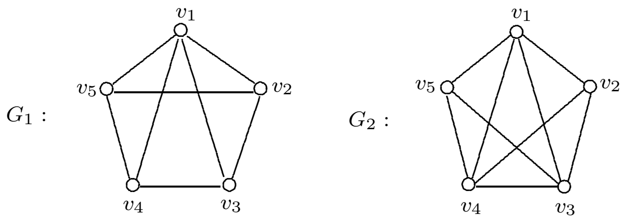

Figure 4: Two graphs of order 5 and minimum degree 3 having a disconnected 3-line graph

Next, suppose that $n \geq 6$. First, we show that $f(n) \leq \lceil \frac{n+1}{2} \rceil$. In order to do this, we show that if G is a graph with $\delta(G) \geq \lceil \frac{n+1}{2} \rceil$, then $L_3(G)$ is connected. Since $\delta(G) \geq \lceil \frac{n+1}{2} \rceil \geq \frac{n+1}{2}$, it follows that G is Hamiltonian-connected and so G is Hamiltonian. If $n \geq 7$ is odd, then G contains an n-cycle and so $L_3(G)$ is connected by Corollary 3.2. Thus, we may assume that $n \geq 6$ is even. Let $v \in V(G)$ and $H = G - v$. Since H is a connected graph of order $n-1$ and $\delta(H) \geq \frac{n+1}{2} - 1 \geq \frac{n-1}{2}$, it follows that H is Hamiltonian and so H contains an $(n-1)$-cycle. Thus, G contains an $(n-1)$-cycle. Again, $L_3(G)$ is connected by Corollary 3.2. Hence, $f(n) \leq \lceil \frac{n+1}{2} \rceil$. On the other hand, by Theorem 2.3, the 3-line graph $L_3(K_{r,r})$ of the r-regular complete bipartite graph $K_{r,r}$ of order $n = 2r \geq 6$ is disconnected for each integer $r \geq 3$. Since $\delta(K_{r,r}) = \lceil \frac{n}{2} \rceil$, it follows that $f(n) \geq \lceil \frac{n+1}{2} \rceil$ for $n \geq 6$. Therefore, $f(n) = \lceil \frac{n+1}{2} \rceil$. ■

4 Conditions for the Disconnectedness of 3-Line Graphs

We have seen that the 3-line graph of many connected graphs has one or two nontrivial components. We now describe a class of graphs, namely paths, whose 3-line graphs always have at most two nontrivial components.

Proposition 4.1 *For an integer $n \geq 5$, the 3-line graph of the path P_n of order n is*

$$L_3(P_n) = \begin{cases} 2P_{\frac{n-2}{2}} & \text{if } n \text{ is even} \\ P_{\lceil \frac{n-2}{2} \rceil} + P_{\lfloor \frac{n-2}{2} \rfloor} & \text{if } n \text{ is odd.} \end{cases}$$

By Proposition 4.1, the 3-line graph $L_3(P_5)$ of the 5-path P_5 has exactly one nontrivial component and the 3-line graph $L_3(P_n)$ of P_n for $n \geq 6$ has exactly two nontrivial components. By Proposition 2.1, the 3-line graphs of stars and double stars of order at least 5 have only trivial components. In the case of more general trees of order 5 or more, the 3-line graph of a tree has at most two nontrivial components. In order to show this, we begin with an observation.

Observation 4.2 *Let $P = (u, v, w)$ be a 3-path in a connected graph G. If (i) u and w are end-vertices of G or (ii) each of u and w is incident with pendant edges, except uv and wv and possibly uw, then P is an isolated vertex of $L_3(G)$.*

Theorem 4.3 *If T is a tree of order $n \geq 5$, then $L_3(T)$ has at most two nontrivial components.*

Proof. If T is a star or double star, then $L_3(T)$ is empty and the result holds. So we proceed by induction on the order $n \geq 5$ of a tree of diameter at least 4. If $n = 5$, then since $\text{diam}(T) \geq 4$, then T is a path of order 5 and $L_3(T)$ has exactly one nontrivial component and so the basis step of the induction holds. Suppose for a fixed integer $n \geq 5$ that the 3-line graph of every tree of order n of diameter at least 4 has at most two nontrivial components. Let T be a tree of order $n + 1$ and $\text{diam}(T) = d \geq 4$. We show that $L_3(T)$ has at most two nontrivial components. We consider two cases.

Case 1. *There exists a diametrical path P in T, say P=($u = u_0$, u_1, ..., $u_d = v$), such that at least one of u and v is adjacent to a vertex of degree at least* 3 *in T.* We may assume that $\deg_T u_{d-1} \geq 3$. Let $T' = T - v$. By the inductive hypothesis, $L_3(T')$ has at most two nontrivial components. Since $\deg_T u_{d-1} \geq 3$, there is at least one end-vertex $w \neq v$ that is adjacent to u_{d-1}. Then (w, u_{d-1}, u_{d-2}) and $(u_{d-2}, u_{d-3}, u_{d-4})$ are adjacent in $L_3(T')$ and belong to a nontrivial component of $L_3(T')$. Consequently, in $L_3(T)$, the vertex (v, u_{d-1}, u_{d-2}) is adjacent to the vertex $(u_{d-2}, u_{d-3}, u_{d-4})$, which belongs to a nontrivial component of $L_3(T)$. The 3-paths that v belongs to are either (v, u_{d-1}, w), (v, u_{d-1}, z) for some end-vertex z of T or (v, u_{d-1}, u_{d-2}). The vertices (v, u_{d-1}, w) and (v, u_{d-1}, z) are isolated vertices in $L_3(T)$ by Observation 4.2, while the vertex (v, u_{d-1}, u_{d-2}) is adjacent to the vertex $(u_{d-2}, u_{d-3}, u_{d-4})$, which belongs to a nontrivial component of $L_3(T)$. Thus, $L_3(T)$ has at most two nontrivial components.

Case 2. *The two end-vertices of every diametrical path in T are adjacent to vertices of degree* 2 *in T.* If T is a path, then the result holds. So we may assume that T is not a path. Let $P = (u = u_0, u_1, u_2, \ldots, u_d = v)$ be a diametrical path in T. Let $T' = T - v$. By the inductive hypothesis, $L_3(T')$ has at most two nontrivial components if $\text{diam}(T') \geq 4$. To show that $L_3(T)$ has at most two nontrivial components, it suffices to show that the vertex (v, u_{d-1}, u_{d-2}) in $L_3(T)$ is adjacent to some vertex in a nontrivial component of $L_3(T')$. If $d \geq 6$, then $(u_{d-2}, u_{d-3}, u_{d-4})$ and $(u_{d-4}, u_{d-5}, u_{d-6})$ are adjacent vertices in $L_3(T')$. Since (v, u_{d-1}, u_{d-2}) and $(u_{d-2}, u_{d-3}, u_{d-4})$ are adjacent, it follows that $L_3(T)$ has at most two nontrivial components. Thus, we may assume that $d = 4$ or $d = 5$. We consider these two subcases.

Subcase 2.1. $d = 4$. If $\text{diam}(T') = d = 4$, then there exist two vertices, say u_5 and u_6, such that (u_2, u_5, u_6) is a path in T'. Then (u_2, u_5, u_6) is adjacent to (u_2, u_1, u_0) in $L_3(T')$. It follows that (u_2, u_5, u_6) belongs to one nontrivial component of $L_3(T')$. Since the vertex (v, u_3, u_2) of $L_3(T)$ is adjacent to (u_2, u_5, u_6), it follows that $L_3(T)$ also has at most two nontrivial components. If $\text{diam}(T') = d - 1 = 3$, then u_2 is adjacent to one or more end-vertices of T. Then T is a tree obtained from P_5 by adding one or more pendant edges to u_2. By Observation 4.2, each such pendant edge lies on a 3-path that is an isolated vertex of $L_3(T)$. Thus, $L_3(T)$ has one nontrivial component, namely the one containing the adjacent vertices (u, u_1, u_2) and (u_2, u_3, v).

Subcase 2.2. $d = 5$. Since T is not a path, at least one of u_2 and u_3 has degree at least 3 in T. If $\text{diam}(T') = d = 5$, then there must exist two vertices, say u_6 and u_7, such that (u_3, u_6, u_7) is a path in T'. Then (u_3, u_6, u_7) is adjacent to (u_3, u_2, u_1) in $L_3(T')$. It follows that (u_3, u_2, u_1) belongs to one nontrivial component of $L_3(T')$. Since the vertex (v, u_4, u_3) of $L_3(T)$ is adjacent to (u_3, u_2, u_1), it follows that $L_3(T)$ also has at most two nontrivial components. If $\text{diam}(T') = d - 1 = 4$, then either u_3 has degree 2 in T or u_3 is adjacent to one or more end-vertices of T. We may assume that u_2 is also adjacent to some end-vertex of T. For otherwise, we consider $T' = T - u$ and by a similar argument as the above proof, the result holds. Therefore, we may assume that T can be obtained from P_5 by adding some pendant edges to u_2 or u_3. In this case, $L_3(T)$ has at most two nontrivial components. ■

Nontrivial trees are connected bipartite graphs, of course. With the aid of Theorem 4.3, we now show that the 3-line graph of every connected bipartite graph of order at least 4 has at most two nontrivial components.

Theorem 4.4 *If G is a connected bipartite graph of size at least 2, then $L_3(G)$ has at most two nontrivial components.*

Proof. The result is obvious if the size of G is 2 or 3. For a connected bipartite graph having size 4 or more, we proceed by induction on the the size m. First, suppose that F is a connected bipartite graph of size 4. Thus, either F is a tree or $F = C_4$ and so $L_3(F)$ has at most one nontrivial component. Hence, the statement is true for all connected bipartite graphs of size 4. Next, suppose for every connected bipartite graph H of size m for some integer $m \geq 4$ that $L_3(H)$ has at most two nontrivial components. Let G be a connected bipartite graph of size $m+1$. We show that $L_3(G)$ has at most two nontrivial components.

If G is a tree, then $L_3(G)$ has at most two nontrivial components by Theorem 4.3. Thus, we may assume that G is not a tree and so G contains an even cycle C. Let e be an edge of G that lies on C. Then $H = G - e$ is a connected bipartite graph of size m. If $G = C$, then the result is true by Proposition 2.4, so we can assume that $G \neq C$. Let $C = (v_1, v_2, v_3, \ldots, v_k, v_1)$ ba a cycle of minimum length in G, where then $k \geq 4$ is an even integer. Since $G \neq C$, at least one vertex of C has degree 3 or more in G, say $\deg(v_k) \geq 3$. Let $e = v_k v_1$ and let $H = G - e$. Since

H is a connected bipartite graph of size m, by the induction hypothesis, $L_3(H)$ has at most two nontrivial components. We show that each of these 3-paths of G that are not in H is either an isolated vertex of $L_3(G)$ or is adjacent to a vertex of a nontrivial component of $L_3(H)$. We now consider two cases, according to whether $k \geq 6$ or $k = 4$.

Case 1. $k \geq 6$. Let $S = \{v_{k-1}v_k, f_1, f_2, \ldots, f_p\}$ be the set of edges incident with the vertex v_k in H. None of theses edges are chords of C. Since the vertex (v_1, v_2, v_3) is adjacent to the vertex (v_3, v_4, v_5) in $L_3(H)$, the vertex (v_1, v_2, v_3) belongs to a nontrivial component of $L_3(H)$. For each edge $f \in S$, the 3-paths fe and (v_{k-1}, v_k, v_1) are adjacent to the vertex (v_1, v_2, v_3) in $L_3(H)$ and so each such vertex belongs to a nontrivial component of $L_3(H)$ containing the vertex (v_1, v_2, v_3). Similarly, let $S' = \{v_2v_1, h_1, h_2, \ldots, h_q\}$ be the set of edges incident with the vertex v_1 in H. None of theses edges are chords of C. Since the vertex (v_k, v_{k-1}, v_{k-2}) is adjacent to the vertex $(v_{k-2}, v_{k-3}, v_{k-4})$ in $L_3(H)$, the vertex (v_k, v_{k-1}, v_{k-2}) belongs to a nontrivial component of $L_3(H)$. For each edge $h \in S'$, the 3-paths he and (v_2, v_1, v_k) are adjacent to the vertex (v_k, v_{k-1}, v_{k-2}) in $L_3(H)$ and so each such vertex belongs to a nontrivial component of $L_3(H)$ containing the vertex (v_k, v_{k-1}, v_{k-2}).

Case 2. $k = 4$. By the same argument as in *Case* 1, if f is an edge of H that is incident with $v_k = v_4$ that is distinct from v_3v_4, then the 3-path fe is adjacent to (v_1, v_2, v_3) in $L_3(H)$. However, the vertex (v_1, v_2, v_3) belongs to a nontrivial component of $L_3(H)$ containing the edge joining (v_1, v_2, v_3) and the 3-path containing the edges v_3v_4 and f.

Now consider the 3-path he. The 3-path he is adjacent to (v_2, v_3, v_4) in $L_3(H)$. However, the vertex (v_2, v_3, v_4) belongs to a nontrivial component of $L_3(H)$ containing the edge joining (v_2, v_3, v_4) and the 3-path consisting of the edges v_1v_2 and h of G.

If the vertex v_2 in H is incident with an edge g not in C, then the 3-path consisting of g and v_2v_3 is adjacent to the 3-path in $L_3(H)$ consisting of v_3v_4 and f. However, the 3-path (v_1, v_4, v_3) in $L_3(H)$ is adjacent to the 3-path consisting of v_2v_3 and g. Hence, we may assume that $\deg_G(v_2) = 2$. If there is either a pendant 3-path P at v_1 or v_3, say the former, then P is adjacent to (v_1, v_2, v_3) in $L_3(H)$. Since (v_1, v_2, v_3) is adjacent to the 3-path containing v_3v_4 and f, it follows that P belongs to a nontrivial component of $L_3(H)$. Thus, the vertex (v_1, v_4, v_3) is adjacent to a nontrivial component of $L_3(H)$. Hence, we may assume that the only edges incident with v_1 and v_3 are either pendant edges or the edges v_1w and v_3w where $w \notin V(C)$. In either case, (v_1, v_4, v_3) is an isolated vertex in $L_3(H)$.

Next, we consider the 3-path (v_4, v_1, v_2). If the vertex v_3 in H is incident with an edge k not in C, then the 3-path consisting of k and v_2v_3 is adjacent to the 3-path in $L_3(H)$ consisting of v_1v_2 and h. However, the 3-path (v_4, v_1, v_2) is adjacent to the 3-path consisting of v_2v_3 and k. Thus, the vertex (v_2, v_1, v_4) is adjacent to a nontrivial component of $L_3(H)$. We may assume that $\deg_G(v_1) = 2$. Let X be the set of two 3-paths consisting of v_2v_3, k and v_4v_3, k. The vertex (v_4, v_1, v_2) is adjacent to each $x \in X$ in $L_3(G)$. If the there is $x \in X$ that is not an isolated vertex of

G, then the 3-path (v_4, v_1, v_2) is adjacent to a vertex of a nontrivial component of $L_3(H)$. Moreover, if there is either a pendant 3-path P at v_4 or v_2, then the 3-path (v_4, v_1, v_2) is adjacent to the 3-path P in $L_3(H)$. However, the 3-path P in $L_3(H)$ is adjacent to the 3-path (v_2, v_3, v_4) in $L_3(H)$. Thus, the path P belongs to a nontrivial component in $L_3(H)$. Hence, we may assume that the only edges incident with v_3 are either pendant edges or edges belonging to pendant 3-paths and the only edges incident with v_4 and v_2 are either pendant edges or the edges v_2w and v_4w where $w \notin V(C)$. In any case, it can be shown that $L_3(G)$ has at most two nontrivial components. ■

While there are many connected graphs whose 3-line graph has exactly two nontrivial components, there are also connected graphs G for which $L_3(G)$ has three nontrivial components. For example, if G is the corona of K_3, then $L_3(G) = 3K_1 + 3K_2$. In fact, more can be said.

Proposition 4.5 *If a graph G is obtained from K_3 by adding $t \geq 1$ pendant edges at each vertex of K_3, then $L_3(G) = 3(\binom{t}{2} + 1)K_1 + 3K_{t,t}$.*

Proof. Let G be the graph obtained from $K_3 = (u_1, u_2, u_3, u_1)$ by adding the t edges $e_1, e_2, \ldots, e_t$ at u_1, the t edges $f_1, f_2, \ldots, f_t$ at u_2 and the t edges $g_1, g_2, \ldots, g_t$ at u_3. Let $f = u_1u_2$, $g = u_2u_3$ and $e = u_1u_3$ be the three edges of K_3. Next, let $H = L_3(G)$ and let X and Y be the two disjoint sets of 3-paths in G defined by

$$\begin{aligned} X &= \{ef, fg, ge\} \cup \{e_ie_j, g_ig_j, f_if_j : 1 \leq i < j \leq t\} \\ Y &= \{ee_i, eg_i, gg_i, gf_i, ff_i, fe_i : 1 \leq i \leq t\}. \end{aligned}$$

Then $V(H) = X \cup Y$. Since $|X| = 3(\binom{t}{2} + 1)$, it follows by Observation 4.2 that $H[X] = 3(\binom{t}{2} + 1)K_1$.

Next, we show that $H[Y] = 3K_{t,t}$. Partition the set Y into the three sets

$$U = U_1 \cup U_2,\ V = V_1 \cup V_2 \text{ and } W = W_1 \cup W_2,$$

where

$$\begin{aligned} &U_1 = \{ee_i : 1 \leq i \leq t\} \text{ and } U_2 = \{gf_i : 1 \leq i \leq t\} \\ &V_1 = \{gg_i : 1 \leq i \leq t\} \text{ and } V_2 = \{fe_i : 1 \leq i \leq t\} \\ &W_1 = \{ff_i : 1 \leq i \leq t\} \text{ and } W_2 = \{eg_i : 1 \leq i \leq t\}. \end{aligned}$$

Then $|U_i| = |V_i| = |W_i| = t$ for $i = 1, 2$. We show that $H[U] = H[V] = H[W] = K_{t,t}$. By the symmetry of the graph G, it suffices to show that $H[U] = K_{t,t}$ with partite sets U_1 and U_2. If P is a 3-path in U_1 and Q is a 3-path in U_2, then $V(P) \cap V(Q) = \{u_3\}$ and u_3 is an end-vertex in both P and Q. Thus P and Q are adjacent in H. If P and Q are two 3-paths in U_1, then P and Q have the edge e in common. Thus P and Q are not adjacent in H. Similarly, if P and Q are two 3-paths in U_2, then P and Q have the edge g in common and so P and Q are not adjacent in H. Therefore,

$H[U] = K_{t,t}$ in H with partite sets U_1 and U_2. Similarly, $H[V] = H[W] = K_{t,t}$, where the partite sets of $H(V)$ are V_1 and V_2 and the partite sets of $H(W)$ are W_1 and W_2.

Next, we claim that $H[U]$, $H[V]$ and $H[W]$ are three components of H, that is, there is no edge between any two of the three sets U, V and W. Again, by the symmetry of the graph G, we may assume that $P \in U$ and $Q \in V$. First, suppose that $P = ee_i$ for some i with $1 \leq i \leq t$. If $Q = gg_j$ where $1 \leq j \leq t$, then an end-vertex of P is the interior vertex of Q; while if $Q = fe_j$ where $1 \leq j \leq t$, then u_1 is the interior vertex of P and Q. Hence, P and Q are not adjacent in H. Next, suppose that $P = gf_i$ for some i with $1 \leq i \leq t$. If $Q = gg_j$ where $1 \leq j \leq t$, then P and Q have the edge g in common; while if $Q = fe_j$ where $1 \leq j \leq t$, then the interior vertex of P is an end-vertex of Q. Again, P and Q are not adjacent in H. Thus, as claimed, $H[U]$, $H[V]$ and $H[W]$ are three components of H and so $H[Y] = 3K_{t,t}$. Therefore, $L_3(G) = 3(\binom{t}{2} + 1)K_1 + 3K_{t,t}$. ■

It is not known whether there is another class of graphs whose 3-line graphs has three nontrivial components. Furthermore, whether there is a connected graph G for which $L_3(G)$ has four or more nontrivial components or more is also not known. We close with this open question.

Problem 4.6 *Does there exist a connected graph G such that $L_3(G)$ has four or more nontrivial components?*

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