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Reducing the maximum degree of a graph by deleting edges

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Abstract

We investigate the smallest number $\lambda_{\mathrm{e}}(G)$ of edges that can be removed from a non-empty graph G so that the resulting graph has a smaller maximum degree. We prove that if m is the number of edges, k is the maximum degree, and t is the number of vertices of degree k, then $\lambda_{\mathrm{e}}(G) \leq \frac{m+(k-1)t}{2k-1}$. We also show that $\lambda_{\mathrm{e}}(G) \leq \frac{m}{k}$ if G is a tree. For each of these two bounds, we determine the graphs which attain the bound. We provide other sharp bounds for $\lambda_{\mathrm{e}}(G)$, relations with other graph parameters, and structural observations.

1 Introduction

Unless stated otherwise, we shall use small letters such as x to denote non-negative integers or functions or elements of a set, and capital letters such as X to denote sets or graphs. The set $\{1, 2, \dots\}$ of positive integers is denoted by $\mathbb{N}$. For any $n \in \mathbb{N}$, the set $\{1, \dots, n\}$ is denoted by $[n]$. For a set X, the set $\{\{x, y\}: x, y \in X, x \neq y\}$ (of all 2-element subsets of X) is denoted by $\binom{X}{2}$. All arbitrary sets are assumed to be finite.

A *graph* G is a pair (X, Y), where X is a set, called the *vertex set of* G, and Y is a subset of $\binom{X}{2}$ and is called the *edge set of* G. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. An element of $V(G)$ is called a *vertex of* G, and an element of $E(G)$ is called an *edge of* G. We may represent an edge $\{v, w\}$ by vw. If vw is an edge of G, then v and w are said to be *adjacent in* G, and we say that w is a *neighbour of* v *in* G (and vice-versa). An edge vw is said to be *incident to* x if $x = v$ or $x = w$.

For $v \in V(G)$, $N_G(v)$ denotes the set of neighbours of v in G, $N_G[v]$ denotes $N_G(v) \cup \{v\}$ and is called the *closed neighbourhood of* v *in* G, $E_G(v)$ denotes the set of edges of G that are incident to v, and $d_G(v)$ denotes $|N_G(v)|$ $(= |E_G(v)|)$ and is

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called the *degree of* v *in* G. For $X \subseteq V(G)$, we denote $\bigcup_{v \in X} N_G(v)$, $\bigcup_{v \in X} N_G[v]$, and $\bigcup_{v \in X} E_G(v)$ by $N_G(X)$, $N_G[X]$, and $E_G(X)$, respectively. The *minimum degree of* G is $\min\{d_G(v) \colon v \in V(G)\}$ and is denoted by $\delta(G)$. The *maximum degree of* G is $\max\{d_G(v) \colon v \in V(G)\}$ and is denoted by $\Delta(G)$. Let $M(G)$ denote the set of vertices of G of degree $\Delta(G)$. Let G_{e} denote the subgraph of G given by $(\bigcup_{v \in M(G)} E_G(v), E_G(M(G)))$ $(= (N_G[M(G)], E_G(M(G))))$.

If H and G are graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is called a *subgraph of* G, and we say that G *contains* H. For $X \subseteq V(G)$, $(X, E(G) \cap \binom{X}{2})$ is called the *subgraph of* G *induced by* X and is denoted by $G[X]$. For $S \subseteq V(G)$, $G - S$ denotes the subgraph of G induced by $V(G) \backslash S$. We may abbreviate $G - \{v\}$ to $G - v$. For $L \subseteq E(G)$, $G - L$ denotes the subgraph of G obtained by removing from G the edges in L, that is, $G - L = (V(G), E(G) \backslash L)$. We may abbreviate $G - \{e\}$ to $G - e$.

In [3], we investigated the smallest number of vertices that can be removed from a graph so that the new graph obtained has a smaller maximum degree. In the present paper, we investigate the smallest number of edges that can be removed from a graph for the same purpose. The first problem is of *domination* type (see [3]), whereas the second problem is of *edge-covering* type (see below).

We call a subset L of $E(G)$ a Δ*-reducing edge set of* G if $\Delta(G - L) < \Delta(G)$ or $\Delta(G) = 0$. We denote the size of a smallest Δ-reducing edge set of G by $\lambda_{\mathrm{e}}(G)$.

We provide several bounds and equations for $\lambda_{\mathrm{e}}(G)$. Before stating our results, we need to add some definitions and notation, and make a few observations.

For $L \subseteq E(G)$ and $X \subseteq V(G)$, we say that L *is an edge cover of* X *in* G if for each $v \in X$ with $d_G(v) > 0$, at least one edge in L is incident to v. Note that L is a Δ-reducing edge set of G if and only if L is an edge cover of $M(G)$ in G. Thus,

$$\lambda_{\mathrm{e}}(G) = \min\{|L| \colon L \text{ is an edge cover of } M(G) \text{ in } G\}.$$

Consequently, we immediately obtain

$$\lambda_{\mathrm{e}}(G) = \lambda_{\mathrm{e}}(G_{\mathrm{e}}). \tag{1}$$

If $G, G_1, \dots, G_r$ are graphs such that $V(G) = \bigcup_{i=1}^r V(G_i)$ and $E(G) = \bigcup_{i=1}^r E(G_i)$, then we say that G is the *union of* $G_1, \dots, G_r$.

If $X_1, \dots, X_s$ are sets such that no r of $X_1, \dots, X_s$ have a common element, then $X_1, \dots, X_s$ are said to be r*-wise disjoint*. Graphs $G_1, \dots, G_s$ are said to be r*-wise vertex-disjoint* if $V(G_1), \dots, V(G_s)$ are r-wise disjoint. Graphs $G_1, \dots, G_s$ are said to be r*-wise edge-disjoint* if $E(G_1), \dots, E(G_s)$ are r-wise disjoint. We may use the term *pairwise* instead of 2*-wise*.

If $v_1, v_2, \dots, v_n$ are the distinct vertices of a graph G with $E(G) = \{v_i v_{i+1} \colon i \in [n-1]\}$, then G is called a $v_1 v_n$*-path* or simply a *path*. The path $([n], \{\{1,2\}, \dots, \{n-1, n\}\})$ is denoted by P_n. For a path P, the *length of* P, denoted by $l(P)$, is $|V(P)| - 1$.

For a graph G and $u, v \in V(G)$, the *distance of* v *from* u, denoted by $d_G(u, v)$, is given by $d_G(u, v) = 0$ if $u = v$, $d_G(u, v) = \min\{l(P) \colon P \text{ is a } uv\text{-path}, G \text{ contains } P\}$ if G contains a uv-path, and $d_G(u, v) = \infty$ if G contains no uv-path.

A graph H is *connected* if for every $u, v \in V(H)$ with $u \neq v$, H contains a uv-path. A *component* of a graph G is a maximal connected subgraph of G (that is, one that is not a subgraph of any other connected subgraph of G). It is easy to see that if $G_1, \dots, G_r$ are the distinct components of G, then $G_1, \dots, G_r$ are pairwise vertex-disjoint and hence pairwise edge-disjoint, and G is the union of $G_1, \dots, G_r$.

Let H be a graph. A graph G is a *copy of* H if there exists a bijection $f : V(G) \to V(H)$ such that $E(H) = \{f(u)f(v) \colon uv \in E(G)\}$.

If $n \geq 3$ and $v_1, v_2, \dots, v_n$ are the distinct vertices of a graph G with $E(G) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$, then G is called a *cycle*. The cycle $([n], \{\{1,2\}, \dots, \{n-1,n\}, \{n,1\}\})$ is denoted by C_n. A *triangle* is a copy of C_3.

A *tree* is a connected graph that contains no cycles. A *forest* is a graph whose components are trees. For $k \geq 1$, the tree $(\{0\} \cup [k], \{\{0,i\} \colon i \in [k]\})$ is denoted by $K_{1,k}$. A copy of $K_{1,k}$ will be called a *k-star* or simply a *star*.

A graph G is *complete* if every two vertices of G are adjacent (that is, $E(G) = \binom{V(G)}{2}$). A graph G is *empty* if no two vertices of G are adjacent (that is, $E(G) = \emptyset$). A graph G is a *singleton* if $|V(G)| = 1$, in which case G is complete and empty.

If $k \in \{0\} \cup \mathbb{N}$ and each vertex of a graph G has degree k, then G is called *k-regular* or simply *regular*.

We are now ready to state our main results, given in the next section. In Section 3, we investigate $\lambda_{\mathrm{e}}(G)$ from a structural point of view; we obtain equations for $\lambda_{\mathrm{e}}(G)$ in terms of certain parameters of certain subgraphs of G, and observe how $\lambda_{\mathrm{e}}(G)$ changes with the deletion of edges. Some of the structural results are then used in the proofs of the main upper bounds presented in the next section; these proofs are given in Section 4.

2 Main results

In this section, we present our main results, most of which are bounds for $\lambda_{\mathrm{e}}(G)$ in terms of basic paramaters of G. We start with a lower bound.

Proposition 2.1 *If G is a graph, $n = |V(G)|$, $m = |E(G)|$, $k = \Delta(G) \geq 1$, and $t = |M(G)|$, then*

$$\lambda_{\mathrm{e}}(G) \geq \max\left\{\left\lceil \frac{2m-(k-1)n}{2} \right\rceil, \left\lceil \frac{t}{2} \right\rceil\right\}.$$

Moreover, equality holds if G is complete.

Proof. Let L be a Δ-reducing edge set of G of size $\lambda_{\mathrm{e}}(G)$. Since $\Delta(G-L) \leq k-1$, the handshaking lemma (applied to $G-L$) gives us $|E(G-L)| \leq \frac{(k-1)n}{2}$. Since $m = |E(G-L)| + |L| \leq \frac{(k-1)n}{2} + \lambda_{\mathrm{e}}(G)$, $\lambda_{\mathrm{e}}(G) \geq \left\lceil \frac{2m-(k-1)n}{2} \right\rceil$.

Since L is a Δ-reducing edge set of G, each vertex in $M(G)$ is contained in some edge in L. Thus, $M(G) \subseteq \bigcup_{e \in L} e$. Therefore, $t \leq \sum_{e \in L} |e| = 2|L|$, and hence $\lambda_{\mathrm{e}}(G) \geq \left\lceil \frac{t}{2} \right\rceil$.

Suppose that G is a complete graph. Then $t = n$, $k = n - 1$, and $m = \frac{n(n-1)}{2}$. Let $v_1, \dots, v_n$ be the vertices of G. Let $X = \{v_{2i-1}v_{2i} \colon i \in \mathbb{N},\, i \leq \frac{n}{2}\}$. If n is even, then X is a Δ-reducing edge set of G of size $\frac{n}{2} = \lceil \frac{t}{2} \rceil = \left\lceil \frac{2m-(k-1)n}{2} \right\rceil$. If n is odd, then $X \cup \{v_n v_1\}$ is a Δ-reducing edge set of G of size $\frac{n+1}{2} = \lceil \frac{t}{2} \rceil = \left\lceil \frac{2m-(k-1)n}{2} \right\rceil$. □

In the rest of this section, we present upper bounds for $\lambda_{\mathrm{e}}(G)$, the proofs of which are given in Section 4. For this purpose, we shall first introduce a class of graphs that attain each of these upper bounds.

For $k \geq 1$, we will call a graph G a *special k-star union* if $\Delta(G) = k$ and each non-singleton component of G is the union of k-stars that are pairwise edge-disjoint and k-wise vertex-disjoint. In Section 4, we prove the following.

Lemma 2.2 *If G is a special k-star union, $m = |E(G)|$, and $t = |M(G)|$, then $m = kt$ and $\lambda_{\mathrm{e}}(G) = t$.*

Theorem 2.3 *If G is a graph, $m = |E(G)|$, $k = \Delta(G) \geq 1$, and $t = |M(G)|$, then*

$$\lambda_{\mathrm{e}}(G) \leq \frac{m + (k-1)t}{2k-1}.$$

Moreover, equality holds if and only if G is a special k-star union or each non-singleton component of G is a 2-star or a triangle.

Remark 2.4 By (1), we may take $m = |E(G_{\mathrm{e}})|$ in each of the results above, and $n = |V(G_{\mathrm{e}})|$ in Proposition 2.1. Note that $\Delta(G) = \Delta(G_{\mathrm{e}})$ and $M(G) = M(G_{\mathrm{e}})$. Thus, we actually have the following immediate consequence.

Corollary 2.5 *If G is a graph, $n = |V(G_{\mathrm{e}})|$, $m = |E(G_{\mathrm{e}})|$, $k = \Delta(G) \geq 1$, and $t = |M(G)|$, then*

$$\max\left\{ \left\lceil \frac{2m-(k-1)n}{2} \right\rceil, \left\lceil \frac{t}{2} \right\rceil \right\} \leq \lambda_{\mathrm{e}}(G) \leq \frac{m + (k-1)t}{2k-1}.$$

Moreover, the bounds are sharp.

Consider the numbers m, k, and t in Corollary 2.5. By the definition of G_{e}, $m \leq kt$. Let $H = G_{\mathrm{e}}$. By the handshaking lemma, $2m = \sum_{v \in V(H)} d_H(v) \geq \sum_{v \in M(G)} d_H(v) = kt$ (and equality holds if and only if G_{e} is regular). Thus,

$$\frac{kt}{2} \leq m \leq kt. \tag{2}$$

Using a probabilistic argument similar to that used by Alon in [1], we prove the following bound.

Theorem 2.6 *If G is a graph, $m = |E(G_e)|$, $k = \Delta(G) \geq 2$, and $t = |M(G)|$, then*

$$\lambda_e(G) \leq m\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{\frac{1}{k-1}}\right).$$

Moreover, equality holds if G_e is a special k-star union.

As we also show in Section 4, a slight adjustment of the proof of Theorem 2.6 yields the following weaker but simpler (and still sharp) result.

Theorem 2.7 *If G is a graph, $m = |E(G_e)|$, $k = \Delta(G) \geq 1$, and $t = |M(G)|$, then*

$$\lambda_e(G) \leq \frac{m}{k}\left(1 + \ln\left(\frac{kt}{m}\right)\right).$$

Moreover, equality holds if G_e is a special k-star union.

A set of pairwise disjoint edges of G is called a *matching of G*. The *matching number of G* is the size of a largest matching of G and is denoted by $\alpha'(G)$. In the next section, we prove the following result.

Theorem 2.8 *For every non-empty graph G,*

$$\lambda_e(G) = |M(G)| - \alpha'(G[M(G)]).$$

If G is a regular non-empty graph, then $M(G) = V(G)$, and hence, by Theorem 2.8, $\lambda_e(G) = |V(G)| - \alpha'(G)$. Thus, for a regular graph G, a lower bound for $\alpha'(G)$ yields an upper bound for $\lambda_e(G)$, and vice-versa. For $k \geq 3$, Henning and Yeo [4] established a lower bound for $\alpha'(G)$ for all k-regular graphs G, and showed that the bound is attained for infinitely many k-regular graphs. Biedl, Demaine, Duncan, Fleischer, and Kobourov [2] had proved the bound for $k = 3$ and several other interesting lower bounds for $\alpha'(G)$. Another important lower bound for k-regular graphs with $k \geq 4$ is given by O and West [6]. The 2-regular graphs are the cycles. It is easy to see that $\{n, 1\} \cup \{\{2i, 2i+1\} \colon 1 \leq i \leq \lceil n/2 \rceil - 1\}$ is a smallest Δ-reducing edge set of C_n, so

$$\lambda_e(C_n) = \left\lceil \frac{n}{2} \right\rceil. \tag{3}$$

For $k \geq 1$, we will call a tree T an *edge-disjoint k-star union* if T is the union of pairwise edge-disjoint k-stars. In Section 4, we prove the following sharp bound for trees.

Theorem 2.9 *If T is a tree, $n = |V(T)|$, $m = |E(T)|$, and $k = \Delta(T) \geq 1$, then*

$$\lambda_e(T) \leq \frac{n-1}{k} = \frac{m}{k}.$$

Moreover, equality holds if and only if T is an edge-disjoint k-star union.

The trees of maximum degree at most 2 are the paths. It is easy to see that $\{\{2i, 2i+1\}: 1 \le i \le \lceil (n-2)/2 \rceil\}$ is a smallest Δ-reducing edge set of P_n, so

$$\lambda_{\mathrm{e}}(P_n) = \left\lceil \frac{n-2}{2} \right\rceil. \tag{4}$$

Theorem 2.9 yields the following generalization.

Theorem 2.10 *If F is a forest, $m = |E(F)|$, and $k = \Delta(F) \ge 1$, then*

$$\lambda_{\mathrm{e}}(F) \le \frac{m}{k}.$$

Moreover, equality holds if and only if each non-singleton component of F is an edge-disjoint k-star union.

Proof. Let $\mathcal{C}$ be the set of components of F. Let $\mathcal{D} = \{C \in \mathcal{C}: \Delta(C) = k\}$. Since $\Delta(F) = k$, $\mathcal{D} \ne \emptyset$. For each $D \in \mathcal{D}$, D is a tree, so $\lambda_{\mathrm{e}}(D) \le \frac{|E(D)|}{k}$ by Theorem 2.9. By Proposition 3.7 (given in the next section), $\lambda_{\mathrm{e}}(F) = \sum_{D \in \mathcal{D}} \lambda_{\mathrm{e}}(D) \le \sum_{D \in \mathcal{D}} \frac{|E(D)|}{k} \le \frac{m}{k}$. If each non-singleton component of F is an edge-disjoint k-star union, then, by Theorem 2.9, $\lambda_{\mathrm{e}}(F) = \sum_{D \in \mathcal{D}} \frac{|E(D)|}{k} = \frac{m}{k}$. Now suppose $\lambda_{\mathrm{e}}(F) = \frac{m}{k}$. Then, by the above, $m = \sum_{D \in \mathcal{D}} |E(D)|$ and $\lambda_{\mathrm{e}}(D) = \frac{|E(D)|}{k}$ for each $D \in \mathcal{D}$. Thus, each non-singleton component of F is a member of $\mathcal{D}$, and, by Theorem 2.9, it is an edge-disjoint k-star union. $\Box$

By the observations in Remark 2.4, we may take $m = |E(G_{\mathrm{e}})|$ in Theorem 2.10. Thus, for the case where G is a forest, Theorem 2.10 improves each of the upper bounds in Corollary 2.5, Theorem 2.6, and Theorem 2.7. Indeed, since $m \le kt$ (by (2)), we have $\frac{m+(k-1)t}{2k-1} \ge \frac{m+(k-1)(m/k)}{2k-1} = \frac{m}{k}$, $m\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{\frac{1}{k-1}}\right) \ge m\left(1 - \frac{k-1}{k}\right) = \frac{m}{k}$, and $\frac{m}{k}\left(1 + \ln\left(\frac{kt}{m}\right)\right) \ge \frac{m}{k}$.

3 Structural results

In this section, we take a close look at how $\lambda_{\mathrm{e}}(G)$ is determined by the structure of G and at how it is affected by removing edges from G. Some of the following observations are used in the proofs given in the next section.

Let $M_1(G)$ denote $\{v \in M(G): vw \in E(G)$ for some $w \in M(G)\backslash\{v\}\}$. Let $M_2(G)$ denote $M(G)\backslash M_1(G)$. Thus, $M_2(G) = \{v \in M(G): d_G(v, w) \ge 2$ for each $w \in M(G)\backslash\{v\}\}$.

Recall the definition of an edge cover, given in Section 1. An edge cover of $V(G)$ in G is called an *edge cover of G*. The *edge-covering number of G* is the size of a smallest edge cover of G and is denoted by $\beta'(G)$. Clearly, $\lambda_{\mathrm{e}}(G) = \beta'(G)$ if G is regular. In general, we have the following.

Theorem 3.1 *For every non-empty graph G,*

$$\lambda_e(G) = |M_2(G)| + \beta'(G[M_1(G)]).$$

Proof. We start with a few observations. Let $k = \Delta(G)$. Since G is non-empty, $k \geq 1$. For each $v \in M(G)$, G has exactly k edges incident to v. By definition of $M_2(G)$,

$$\text{for any } v \in M_2(G) \text{ and any } e \in E_G(v),\ e \notin E_G(w) \text{ for each } w \in M(G)\backslash\{v\}. \quad (5)$$

For any $v \in M_1(G)$, $vw \in E(G)$ for some $w \in M(G)\backslash\{v\}$, and therefore $w \in M_1(G)$ and $vw \in G[M_1(G)]$. In other words,

$$\text{for any } v \in M_1(G),\ G[M_1(G)] \text{ has at least one edge incident to } v. \quad (6)$$

Thus, $G[M_1(G)]$ has an edge cover.

Let K be an edge cover of $G[M_1(G)]$ of size $\beta'(G[M_1(G)])$. For each $v \in M_2(G)$, let $e_v \in E_G(v)$. Let $K' = \{e_v : v \in M_2(G)\} \cup K$. Then K' is a Δ-reducing edge set of G. By (5), $|K'| = |M_2(G)| + |K|$. Thus, $\lambda_e(G) \leq |M_2(G)| + \beta'(G[M_1(G)])$.

Now let L be a Δ-reducing edge set of G of size $\lambda_e(G)$. For each $v \in M(G)$, there exists some $e_v \in E_G(v)$ such that $e_v \in L$. Let $L_1 = \{e_v : v \in M_1(G)\}$ and $L_2 = \{e_v : v \in M_2(G)\}$. Then $L_1 \cup L_2$ is a Δ-reducing edge set of G. Thus, since $L_1 \cup L_2 \subseteq L$ and $|L| = \lambda_e(G)$, $L = L_1 \cup L_2$. By (5), $|L_1 \cup L_2| = |L_1| + |M_2(G)|$. Let $X = \{v \in M_1(G) : e_v \notin E(G[M_1(G)])\}$. By (6), for each $v \in M_1(G)$, there exists some $e'_v \in E_G(v)$ such that $e'_v \in E(G[M_1(G)])$. Let $L'_1 = (L_1\backslash\{e_v : v \in X\}) \cup \{e'_v : v \in X\}$. For each $v \in X$, $e_v \cap M_1(G) = \{v\}$. Thus, L'_1 is an edge cover of $G[M_1(G)]$, and $|L'_1| \leq |L_1|$. We have $\lambda_e(G) = |L| = |M_2(G)| + |L_1| \geq |M_2(G)| + |L'_1| \geq |M_2(G)| + \beta'(G[M_1(G)])$. Since $\lambda_e(G) \leq |M_2(G)| + \beta'(G[M_1(G)])$, the result follows. □

We now prove Theorem 2.8. Using a well-known result of Gallai [5], we then show that Theorems 2.8 and 3.1 are equivalent, meaning that they imply each other.

Proof of Theorem 2.8. Let $H = G[M(G)]$. Let K be a matching of H of size $\alpha'(H)$. Let $X = \bigcup_{e \in K} e$. Then $X \subseteq M(G)$ and $|X| = 2|K|$. For each $v \in M(G)\backslash X$, let $e_v \in E_G(v)$. Let $K' = \{e_v : v \in M(G)\backslash X\}$. Then $K \cup K'$ is a Δ-reducing edge set of G. Thus, $\lambda_e(G) \leq |K| + |K'| \leq |K| + |M(G)\backslash X| = |K| + |M(G)| - |X| = |M(G)| - |K| = |M(G)| - \alpha'(H)$.

Now let L be a Δ-reducing edge set of G of size $\lambda_e(G)$. Then, for each $v \in M(G)$, there exists some $e'_v \in E_G(v)$ such that $e'_v \in L$. Let J be a largest subset of L that is a matching of H. Let $Y = \bigcup_{e \in J} e$. Then $Y \subseteq M(G)$ and $|Y| = 2|J|$. Let $Y' = M(G)\backslash Y$. Let $J' = \{e'_v : v \in Y'\}$. If we assume that $e'_u = e'_v$ for some $u, v \in Y'$ with $u \neq v$, then we obtain that $e'_u = e'_v = uv$ and that $J \cup \{uv\}$ is a matching of H of size $|J| + 1$, which contradicts the choice of J. Thus, $|J'| = |Y'|$. Now $J \cup J' \subseteq L$ and $J \cap J' = \emptyset$. We have $\lambda_e(G) = |L| \geq |J \cup J'| = |J| + |J'| = |J| + |Y'| = |J| + |M(G)| - |Y| = |M(G)| - |J| \geq |M(G)| - \alpha'(H)$. Since $\lambda_e(G) \leq |M(G)| - \alpha'(H)$, the result follows. □

Proposition 3.2 *Theorems 2.8 and 3.1 are equivalent.*

Proof. By (6), $\delta(G[M_1(G)]) \geq 1$. A result of Gallai [5] tells us that $\alpha'(H) + \beta'(H) = |V(H)|$ for every graph H with $\delta(H) \geq 1$. Thus,

$$\alpha'(G[M_1(G)]) + \beta'(G[M_1(G)]) = |V(G[M_1(G)])| = |M_1(G)|.$$

If $v, w \in M(G)$ such that $vw \in E(G)$, then $vw \in M_1(G)$. Thus, $E(G[M(G)]) = E(G[M_1(G)])$, and hence $\alpha'(G[M_1(G)]) = \alpha'(G[M(G)])$. Therefore, since $|M(G)| = |M_1(G)| + |M_2(G)|$, Theorem 2.8 implies Theorem 3.1, and vice-versa. □

From Theorem 3.1 we immediately obtain the next two results.

Proposition 3.3 *If G is a non-empty graph, then $\lambda_e(G) \leq |M(G)|$, and equality holds if and only if $M_2(G) = M(G)$.*

Proof. For each $v \in M(G)$, let $e_v \in E_G(v)$. Since $\{e_v : v \in M(G)\}$ is a Δ-reducing edge set of G, $\lambda_e(G) \leq |\{e_v : v \in M(G)\}| \leq |M(G)|$. By Theorem 3.1, $\lambda_e(G) = |M(G)|$ if $M_2(G) = M(G)$. Suppose $M_2(G) \neq M(G)$. Then $M_1(G) \neq \emptyset$. Let $x \in M_1(G)$. By (6), $xy \in E(G[M_1(G)])$ for some $y \in M_1(G)\backslash\{x\}$. Also by (6), for each $v \in M_1(G)\backslash\{x, y\}$, there exists some $e'_v \in E_G(v)$ such that $e'_v \in E(G[M_1(G)])$. Let $L = \{xy\} \cup \{e'_v : v \in M_1(G)\backslash\{x, y\}\}$. Since L is an edge cover of $G[M_1(G)]$, $\beta'(G[M_1(G)]) \leq |L| \leq |M_1(G)| - 1$. Thus, by Theorem 3.1, $\lambda_e(G) \leq |M_2(G)| + |M_1(G)| - 1 < |M(G)|$. □

Proposition 3.4 *If G is a graph with $M_2(G) \neq M(G)$, then $\Delta(G - M_2(G)) = \Delta(G)$ and $\lambda_e(G) = |M_2(G)| + \lambda_e(G - M_2(G))$.*

Proof. Let $H = G - M_2(G)$. Since $M_2(G) \neq M(G)$, $M_1(G) \neq \emptyset$. By (5), $E_G(M_1(G)) \subseteq E(H)$. Together with $M(G) = M_1(G) \cup M_2(G)$, this gives us $M(H) = M_1(G)$. Let K be an edge cover of $G[M_1(G)]$ of size $\beta'(G[M_1(G)])$ (K exists by (6)). Then K is a Δ-reducing edge set of H, and hence $\lambda_e(H) \leq \beta'(G[M_1(G)])$. By Theorem 3.1, $\lambda_e(G) \geq |M_2(G)| + \lambda_e(H)$. Now let L_1 be a Δ-reducing edge set of H of size $\lambda_e(H)$, and let L_2 be as in the proof of Theorem 3.1. Then $L_1 \cup L_2$ is a Δ-reducing edge set of G. Thus, $\lambda_e(G) \leq |L_1| + |L_2| = \lambda_e(H) + |M_2(G)|$. The result follows. □

In the rest of the section, we take a look at how $\lambda_e(H)$ relates to $\lambda_e(G)$ for a subgraph H of G, or rather, how $\lambda_e(G)$ is affected by removing edges from G.

Lemma 3.5 *If G is a graph, H is a subgraph of G with $\Delta(H) = \Delta(G)$, and L is a Δ-reducing edge set of G, then $L \cap E(H)$ is a Δ-reducing edge set of H.*

Proof. Let $J = L \cap E(H)$. It is sufficient to show that for each $v \in M(H)$, $e \in E_H(v)$ for some $e \in J$. Let $v \in M(H)$. Since $\Delta(H) = \Delta(G)$, $v \in M(G)$ and

$E_H(v) = E_G(v)$. Since $v \in M(G)$, $e \in E_G(v)$ for some $e \in L$. Since $E_G(v) = E_H(v)$, $e \in E(H)$. Therefore, $e \in J$. □

We point out that $|L| = \lambda_e(G)$ does not guarantee that $|L \cap E(H)| = \lambda_e(H)$. Indeed, let $k \geq 2$, let G_1 and G_2 be copies of $K_{1,k}$ with $V(G_1) \cap V(G_2) = \emptyset$, and let G be the union of G_1 and G_2. Let $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$. Let $e \in E(G_2) \backslash \{e_2\}$. Let $H = (V(G), E(G) \backslash \{e\})$. Let $L = \{e_1, e_2\}$. Then L is a Δ-reducing edge set of G of size $\lambda_e(G)$, $L \cap E(H) = \{e_1, e_2\} = L$, but $\{e_1\}$ is a Δ-reducing edge set of H of size $\lambda_e(H)$. Thus, $L \cap E(H)$ is not a smallest Δ-reducing edge set of H.

Corollary 3.6 *If H is a subgraph of G such that $\Delta(H) = \Delta(G)$, then $\lambda_e(H) \leq \lambda_e(G)$.*

Proof. Let L be a Δ-reducing edge set of G of size $\lambda_e(G)$. Let $J = L \cap E(H)$. By Lemma 3.5, J is a Δ-reducing edge set of H. Therefore, $\lambda_e(H) \leq |J| \leq |L| = \lambda_e(G)$. □

Proposition 3.7 *If G is a graph and $G_1, \dots, G_r$ are the distinct components of G whose maximum degree is $\Delta(G)$, then $\lambda_e(G) = \sum_{i=1}^{r} \lambda_e(G_i)$.*

Proof. Let L be a Δ-reducing edge set of G of size $\lambda_e(G)$. For each $i \in [r]$, let $L_i = L \cap E(G_i)$. Then $L_1, \dots, L_r$ partition L, so $|L| = \sum_{i=1}^{r} |L_i|$. By Lemma 3.5, for each $i \in [r]$, L_i is a Δ-reducing edge set of G_i, so $\lambda_e(G_i) \leq |L_i|$. Suppose $\lambda_e(G_j) < |L_j|$ for some $j \in [r]$. Let L'_j be a Δ-reducing edge set of G_j of size $\lambda_e(G_j)$. Then $L'_j \cup \bigcup_{i \in [r] \backslash \{j\}} L_i$ is a Δ-reducing edge set of G that is smaller than L, a contradiction. Thus, $\lambda_e(G_i) = |L_i|$ for each $i \in [r]$. We have $\lambda_e(G) = |L| = \sum_{i=1}^{r} |L_i| = \sum_{i=1}^{r} \lambda_e(G_i)$. □

Proposition 3.8 *If G is a graph, $u, v \in V(G) \backslash M(G)$, and $uv \in E(G)$, then $\lambda_e(G - uv) = \lambda_e(G)$.*

Proof. Let $e = uv$. Since $u, v \notin M(G)$, $\Delta(G - e) = \Delta(G)$. By Corollary 3.6, $\lambda_e(G - e) \leq \lambda_e(G)$. Let L be a Δ-reducing edge set of $G - e$ of size $\lambda_e(G - e)$. Since $u, v \notin M(G)$, $M(G - e) = M(G)$. Thus, L is a Δ-reducing edge set of G, and hence $\lambda_e(G) \leq \lambda_e(G - e)$. Since $\lambda_e(G - e) \leq \lambda_e(G)$, the result follows. □

Proposition 3.9 *If G is a graph and $e \in E(G)$, then $\lambda_e(G) \leq 1 + \lambda_e(G - e)$.*

Proof. If $\Delta(G - e) < \Delta(G)$, then $\lambda_e(G) = 1$. Suppose $\Delta(G - e) = \Delta(G)$. Then $M(G - e) \subseteq M(G) \cup e$. Let L be a Δ-reducing edge set of $G - e$ of size $\lambda_e(G - e)$. Then $L \cup \{e\}$ is a Δ-reducing edge set of G. Thus, $\lambda_e(G) \leq |L \cup \{e\}| = 1 + \lambda_e(G - e)$. □

Corollary 3.10 *If $e_1, \dots, e_t$ are edges of a graph G, then $\lambda_e(G) \leq t + \lambda_e(G - \{e_1, \dots, e_t\})$.*

Proof. The result follows by repeated application of Proposition 3.9. □

4 Proofs of the main upper bounds

We now prove Lemma 2.2 and Theorems 2.3, 2.6, 2.7, and 2.9.

Proof of Lemma 2.2. Since G is a special k-star union, $\Delta(G) = k$ and $E(G) = E(G_1) \cup \cdots \cup E(G_r)$ for some k-stars $G_1, \ldots, G_r$ that are pairwise edge-disjoint and k-wise vertex-disjoint. Thus, $m = kr$, and for $i \in [r]$, there exist $u_i, v_{i,1}, \ldots, v_{i,k} \in V(G)$ such that $G_i = (\{u_i, v_{i,1}, \ldots, v_{i,k}\}, \{u_i v_{i,1}, \ldots, u_i v_{i,k}\})$. For $i \in [r]$, $|E_{G_i}(u_i)| = k = \Delta(G)$, so we have $E_G(u_i) = E_{G_i}(u_i) = E(G_i)$. Thus, since $E(G_1), \ldots, E(G_r)$ are pairwise disjoint, $u_1, \ldots, u_r$ are distinct. Consider any $w \in V(G) \backslash \{u_1, \ldots, u_r\}$. For each $i \in [r]$ such that $w \in V(G_i)$, $E_G(w) \cap E(G_i) = \{u_i w\}$. Thus, $d_G(w) = |\{i \in [r] \colon w \in V(G_i)\}|$, and hence, since $G_1, \ldots, G_r$ are k-wise vertex-disjoint, $d_G(w) < k$. Thus, $M(G) = \{u_1, \ldots, u_r\}$, and hence $t = r$. Since $m = kr$, $m = kt$.

Now let L be a Δ-reducing edge set of G of size $\lambda_{\mathrm{e}}(G)$. For $i \in [r]$, there exists some $e_i \in E_G(u_i)$ such that $e_i \in L$. Let $L' = \{e_1, \ldots, e_r\}$. For $i, j \in [r]$ with $i \neq j$, $E_G(u_i) \cap E_G(u_j) = E(G_i) \cap E(G_j) = \emptyset$, so $e_i \neq e_j$. Thus, $|L'| = r$. Now L' is a Δ-reducing edge set of G and $L' \subseteq L$, so $\lambda_{\mathrm{e}}(G) \leq |L'| \leq |L|$. Since $\lambda_{\mathrm{e}}(G) = |L|$, we obtain $L' = L$, so $\lambda_{\mathrm{e}}(G) = r$. Since $t = r$, the result is proved. □

Proof of Theorem 2.3. If G is a special k-star union, then, by Lemma 2.2, we have $m = kt$ and $\lambda_{\mathrm{e}}(G) = t = \frac{m+(k-1)t}{2k-1}$. If G has exactly $c_1 + c_2 + c_3$ components, c_1 components of G are singletons, c_2 components of G are 2-stars, and c_3 components of G are triangles, then $m = 2c_2 + 3c_3$, $k = 2$, $t = c_2 + 3c_3$, and, by Proposition 3.7, $\lambda_{\mathrm{e}}(G) = c_2 \lambda_{\mathrm{e}}(P_2) + c_3 \lambda_{\mathrm{e}}(C_3) = c_2 + 2c_3 = \frac{m+(k-1)t}{2k-1}$.

We now prove the bound in the theorem and show that it is attained only in the cases above. If $m = 1$, then $k = 1$, and the result follows immediately. We now proceed by induction on m. Thus, suppose $m \geq 2$. If $k = 1$, then the edges of G are pairwise disjoint, G is a special 1-star union, and $\lambda_{\mathrm{e}}(G) = m = \frac{m+(k-1)t}{2k-1}$. Suppose $k \geq 2$.

Suppose $M_2(G) = M(G)$. Let $v_1, \ldots, v_t$ be the vertices in $M_2(G)$. By (5), $E_G(v_1)$, $\ldots$, $E_G(v_t)$ are pairwise disjoint, so $|E_G(M_2(G))| = \sum_{i=1}^{t} |E_G(v_i)| = \sum_{i=1}^{t} k = kt$. Thus, $m \geq kt$, and equality holds only if $E(G) = \bigcup_{i=1}^{t} E_G(v_i)$. By Proposition 3.3, $\lambda_{\mathrm{e}}(G) = t = \frac{kt+(k-1)t}{2k-1} \leq \frac{m+(k-1)t}{2k-1}$. Suppose $\lambda_{\mathrm{e}}(G) = \frac{m+(k-1)t}{2k-1}$. Then $m = kt$, and hence $E(G) = \bigcup_{i=1}^{t} E_G(v_i)$. For $i \in [t]$, let G_i be the k-star $(N_G[v_i], E_G(v_i))$. Then $G_1, \ldots, G_t$ are pairwise edge-disjoint. For $i \in [t]$, we have $d_{G_i}(v_i) = \Delta(G)$, so $v_i \notin V(G_j)$ for $j \in [t] \backslash \{i\}$. Consider any $w \in \bigcup_{i=1}^{t} V(G_i) \backslash \{v_1, \ldots, v_t\}$. Then $w \notin M(G)$, and hence $d_G(w) < k$. For $i \in [t]$ such that $w \in V(G_i)$, $E_G(w) \cap E(G_i) = \{v_i w\}$. Thus, $|\{i \in [t] \colon w \in V(G_i)\}| = d_G(w) < k$. We have therefore shown that $G_1, \ldots, G_t$ are k-wise vertex-disjoint. Since $E(G) = \bigcup_{i=1}^{t} E_G(v_i) = \bigcup_{i=1}^{t} E(G_i)$, G is a special k-star union.

Now suppose $M_2(G) \neq M(G)$. Then $xy \in E(G)$ for some $x, y \in M(G)$. Let $H = G - xy$. We have $m \geq |E_G(x) \cup E_G(y)| = |E_G(x)| + |E_G(y)| - |E_G(x) \cap E_G(y)| = 2k - |\{xy\}| = 2k - 1$. If $\Delta(H) < k$, then $M(G) = \{x, y\}$ and $\lambda_{\mathrm{e}}(G) = 1 < \frac{m+(k-1)t}{2k-1}$.

Suppose $\Delta(H) = k$. Then $M(H) = M(G)\backslash\{x, y\}$. By the induction hypothesis, $\lambda_{\mathrm{e}}(H) \leq \frac{(m-1)+(k-1)(t-2)}{2k-1}$. By Proposition 3.9,

$$\lambda_{\mathrm{e}}(G) \leq 1 + \lambda_{\mathrm{e}}(H) \leq 1 + \frac{(m-1)+(k-1)(t-2)}{2k-1} = \frac{m+(k-1)t}{2k-1}.$$

Suppose $\lambda_{\mathrm{e}}(G) = \frac{m+(k-1)t}{2k-1}$. Then $\lambda_{\mathrm{e}}(G) = 1+\lambda_{\mathrm{e}}(H)$ and $\lambda_{\mathrm{e}}(H) = \frac{(m-1)+(k-1)(t-2)}{2k-1}$. By the induction hypothesis, H is a special k-star union or each non-singleton component of H is a 2-star or a triangle.

Suppose that H is a special k-star union. We have $|M(H)| = t - 2$. Let $u_1, \dots, u_{t-2}$ be the distinct vertices in $M(H)$. By the proof of Lemma 2.2, $E_H(u_1)$, $\dots, E_H(u_{t-2})$ partition $E(H)$, and $\lambda_{\mathrm{e}}(H) = |M(H)|$. Since $d_H(x) = |E_G(x)\backslash\{xy\}| = k-1 > 0$, $u_px \in E(H)$ for some $p \in [t-2]$. Similarly, $u_qy \in E(H)$ for some $q \in [t-2]$. For each $i \in [t-2]\backslash\{p, q\}$, let $e_i \in E_H(u_i)$. Since $M(G) = \{u_1, \dots, u_{t-2}\} \cup \{x, y\}$, $\{e_i \colon i \in [t-2]\backslash\{p, q\}\} \cup \{u_px, u_qy\}$ is a Δ-reducing edge set of G. Together with $t-2 = |M(H)| = \lambda_{\mathrm{e}}(H)$, this gives us $\lambda_{\mathrm{e}}(G) \leq \lambda_{\mathrm{e}}(H)$, which contradicts $\lambda_{\mathrm{e}}(G) = 1 + \lambda_{\mathrm{e}}(H)$.

Therefore, each non-singleton component of H is a 2-star or a triangle. Thus, $k = 2$. For $v \in \{x, y\}$, let H_v be the component of H such that $v \in V(H_v)$. Since $2 = k = d_G(x) = |E_{H_x}(x) \cup \{xy\}| = d_{H_x}(x)+1$, we have $d_{H_x}(x) = 1$, so H_x is a 2-star and x is a leaf of H_x. Suppose $H_x \neq H_y$. Then there are 6 distinct vertices $a_1, \dots, a_6$ of H such that $H_x = (\{a_1, a_2, a_3\}, \{a_1a_2, a_2a_3\})$, $H_y = (\{a_4, a_5, a_6\}, \{a_4a_5, a_5a_6\})$, $a_3 = x$, and $a_4 = y$. Let L be a smallest Δ-reducing edge set of H. Since H_x and H_y are components of H, we have $M(H) \cap (V(H_x) \cup V(H_y)) = \{a_2, a_5\}$ and $L \cap E(H_x) \neq \emptyset \neq L \cap E(H_y)$. Let $e_x \in L \cap E(H_x)$ and $e_y \in L \cap E(H_y)$. Let $L' = (L\backslash\{e_x, e_y\}) \cup \{a_2a_3, a_4a_5\}$. Then L' is a Δ-reducing edge set of G. Thus, we have $\lambda_{\mathrm{e}}(G) \leq |L'| = |L| = \lambda_{\mathrm{e}}(H)$, which contradicts $\lambda_{\mathrm{e}}(G) = 1 + \lambda_{\mathrm{e}}(H)$. Therefore, $H_x = H_y$. Let $G_x = (V(H_x), E(H_x) \cup \{xy\})$. Then G_x is a component of G. Since x and y are the two leaves of the 2-star H_x, G_x is a triangle. Consequently, each non-singleton component of G is a 2-star or a triangle. □

Proof of Theorem 2.6. We may assume that $E_G(M(G)) = [m]$. By (2), $m \leq kt$. Let $p = 1 - \left(\frac{m}{kt}\right)^{\frac{1}{k-1}}$. We set up m independent random experiments, and in each experiment an edge is chosen with probability p. More formally, for $i \in [m]$, let (Ω_i, P_i) be given by $\Omega_i = \{0, 1\}$, $P_i(\{1\}) = p$, and $P_i(\{0\}) = 1 - p$. Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ and let $P \colon 2^\Omega \to [0, 1]$ (where $[0, 1]$ denotes $\{x \in \mathbb{R} \colon 0 \leq x \leq 1\}$) such that $P(\{\omega\}) = \prod_{i=1}^m P_i(\{\omega_i\})$ for each $\omega = (\omega_1, \dots, \omega_m) \in \Omega$, and $P(A) = \sum_{\omega \in A} P(\{\omega\})$ for each $A \subseteq \Omega$. Then (Ω, P) is a probability space.

For each $\omega = (\omega_1, \dots, \omega_m) \in \Omega$, let $S_\omega = \{i \in [m] \colon \omega_i = 1\}$ and $T_\omega = \{v \in M(G) \colon$ no edge incident to v is a member of $S_\omega\}$.

Let $X \colon \Omega \to \mathbb{R}$ be the random variable given by $X(\omega) = |S_\omega|$. For $i \in [m]$, let $X_i \colon \Omega \to \mathbb{R}$ such that, for $\omega = (\omega_1, \dots, \omega_m) \in \Omega$,

$$X_i(\omega) = \begin{cases} 1 & \text{if } i \in S_\omega; \\ 0 & \text{otherwise.} \end{cases}$$

Then $X = \sum_{i=1}^{m} X_i$. For $i \in [m]$, $P(X_i = 1) = P_i(\{1\}) = p$.

Let $Y : \Omega \to \mathbb{R}$ be the random variable given by $Y(\omega) = |T_\omega|$. For $v \in M(G)$, let $Y_v : \Omega \to \mathbb{R}$ such that, for $\omega = (\omega_1, \dots, \omega_m) \in \Omega$,

$$Y_v(\omega) = \begin{cases} 1 & \text{if } v \in T_\omega; \\ 0 & \text{otherwise.} \end{cases}$$

Then $Y = \sum_{v \in M(G)} Y_v$. For $v \in M(G)$, $P(Y_v = 1) = (1-p)^k$.

For any random variable Z, let $\mathrm{E}[Z]$ denote the expected value of Z. By linearity of expectation,

$$\begin{aligned} \mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y] &= \sum_{i=1}^{m} \mathrm{E}[X_i] + \sum_{v \in M(G)} \mathrm{E}[Y_v] \\ &= \sum_{i=1}^{m} P(X_i = 1) + \sum_{v \in M(G)} P(Y_v = 1) = mp + t(1-p)^k. \end{aligned}$$

Thus, by the probabilistic pigeonhole principle, there exists some $\omega^* \in \Omega$ such that $X(\omega^*) + Y(\omega^*) \le mp + t(1-p)^k$. For $v \in T_{\omega^*}$, let $e_v \in E_G(v)$. Let $L_{\omega^*} = S_{\omega^*} \cup \{e_v : v \in T_{\omega^*}\}$. Then L_{ω^*} is a Δ-reducing edge set of G. Thus, $\lambda_{\mathrm{e}}(G) \le |L_{\omega^*}| \le |S_{\omega^*}| + |T_{\omega^*}| = X(\omega^*) + Y(\omega^*) \le mp + t(1-p)^k = m\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{\frac{1}{k-1}}\right)$. If G_{e} is a special k-star union, then, by Lemma 2.2, we have $m = kt$ and $\lambda_{\mathrm{e}}(G) = t$, and hence $\lambda_{\mathrm{e}}(G) = m\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{\frac{1}{k-1}}\right)$. □

Remark 4.1 Note that the minimum value of the function $f : [0,1] \to \mathbb{R}$ given by $f(p) = mp + t(1-p)^k$ occurs at $p = 1 - \left(\frac{m}{kt}\right)^{\frac{1}{k-1}}$, hence the choice of p in the proof above.

Proof of Theorem 2.7. Let $p^* = 1 - \left(\frac{m}{kt}\right)^{\frac{1}{k-1}}$ and $q = \frac{1}{k}\ln\left(\frac{kt}{m}\right)$. By (2), $kt/2 \le m \le kt$. Thus, $0 \le q \le \frac{1}{k}\ln 2 < 1$. Let f be as in Remark 4.1. Thus, $f(p^*) \le f(q)$. By the proof of Theorem 2.6, $\lambda_{\mathrm{e}}(G) \le f(p^*) \le f(q) = mq + t(1-q)^k$. Since $1 - q \le e^{-q}$, we obtain $\lambda_{\mathrm{e}}(G) \le mq + te^{-qk} = \frac{m}{k}\ln\left(\frac{kt}{m}\right) + te^{-\ln\left(\frac{kt}{m}\right)} = \frac{m}{k}\left(1 + \ln\left(\frac{kt}{m}\right)\right)$. If G_{e} is a special k-star union, then, by Lemma 2.2, we have $m = kt$ and $\lambda_{\mathrm{e}}(G) = t$, and hence $\lambda_{\mathrm{e}}(G) = \frac{m}{k}\left(1 + \ln\left(\frac{kt}{m}\right)\right)$. □

We now prove Theorem 2.9, making use of the following well-known facts.

Lemma 4.2 *Let x be a vertex of a tree T. Let $m = \max\{d_T(x,y) : y \in V(T)\}$, and let $D_i = \{y \in V(T) : d_T(x,y) = i\}$ for each $i \in \{0\} \cup [m]$. For each $i \in [m]$ and each $v \in D_i$, $N_T(v) \cap \bigcup_{j=0}^{i} D_j = \{u\}$ for some $u \in D_{i-1}$.*

Indeed, let $v \in D_i$. By definition of D_i, v can only be adjacent to vertices of distance $i-1$, i or $i+1$ from x. If v is adjacent to a vertex w of distance i from x, then, by considering an xv-path and an xw-path, we obtain that T contains a cycle, a contradiction. We obtain the same contradiction if we assume that v is adjacent to two vertices of distance $i-1$ from x.

If a vertex v of a graph G has only one neighbour in G, then v is called a *leaf of* G.

Corollary 4.3 *If T is a tree, $x, z \in V(T)$, and $d_T(x,z) = \max\{d_T(x,y)\colon y \in V(T)\}$, then z is a leaf of T.*

Proof. Let $D_0, D_1, \ldots, D_m$ be as in Lemma 4.2. Then $z \in D_m$. By Lemma 4.2, $N_T(z) = \{u\}$ for some $u \in D_{m-1}$. □

Proof of Theorem 2.9. The result is trivial for $n \leq 2$. We now proceed by induction on n. Thus, consider $n \geq 3$. Since T is connected, $k \geq 2$.

Suppose that T has a leaf z whose neighbour is not in $M(T)$. Let w be the neighbour of z in T. Let $T' = T - z$. By (1), $\lambda_{\mathrm{e}}(T) = \lambda_{\mathrm{e}}(T')$ as $T_{\mathrm{e}} = T'_{\mathrm{e}}$. By the induction hypothesis, $\lambda_{\mathrm{e}}(T') \leq \frac{n-2}{k} < \frac{n-1}{k}$. Thus, $\lambda_{\mathrm{e}}(T) < \frac{n-1}{k}$. Suppose T is an edge-disjoint k-star union. Then T contains a k-star S such that $z \in V(S)$. Since $N_S(z) \subseteq N_T(z) = \{w\}$, z is a leaf of S and $S = (\{w, z'_1, \ldots, z'_k\}, \{wz'_1, \ldots, wz'_k\})$, where $z'_1 = z$ and $z'_2, \ldots, z'_k$ are distinct elements of $V(T)\backslash\{w, z\}$. Thus, we have $d_T(w) = k$, contradicting $w \notin M(T)$. Therefore, T is not an edge-disjoint k-star union.

Now suppose that each leaf of T has its neighbour in $M(T)$. Let x, m, and $D_0, \ldots, D_m$ be as in Lemma 4.2. Let $z \in V(T)$ such that $d_T(x,z) = m$. By Corollary 4.3, z is a leaf of T. Let w be the neighbour of z in T. By Lemma 4.2, $w \in D_{m-1}$.

Suppose $w = x$. Then $m = 1$ and $T = (\{x, z_1, \ldots, z_k\}, \{xz_1, \ldots, xz_k\})$ for some distinct vertices $z_1, \ldots, z_k$ in D_m. Thus, T is a k-star. Since xz_1 is a Δ-reducing edge set of T, $\lambda_{\mathrm{e}}(T) = 1 = \frac{n-1}{k}$.

Now suppose $w \neq x$. Together with Lemma 4.2, this implies that $N_T(w) = \{v, z_1, \ldots, z_{k-1}\}$ for some $v \in D_{m-2}$ and some distinct vertices $z_1, \ldots, z_{k-1}$ in D_m. By Corollary 4.3, $z_1, \ldots, z_{k-1}$ are leaves of T. Let $e = wv$. Let

$$T_1 = T - \{w, z_1, \ldots, z_{k-1}\} \quad \text{and} \quad T_2 = (\{w, z_1, \ldots, z_{k-1}\}, \{wz_1, \ldots, wz_{k-1}\}).$$

Clearly, T_1 and T_2 are the components of $T - e$, and they are trees. Let $T'_2 = (\{v\} \cup V(T_2), \{e\} \cup E(T_2))$. If $T = T'_2$, then $\Delta(T-e) < k$, and hence $\lambda_{\mathrm{e}}(T) = 1 = \frac{n-1}{k}$. We have $\Delta(T_2) < k$.

Suppose $\Delta(T_1) < k$. Then $\Delta(T-e) < k$, and hence $\lambda_{\mathrm{e}}(T) = 1 \leq \frac{n-1}{k}$. Suppose $\lambda_{\mathrm{e}}(T) = \frac{n-1}{k}$. Then $n = k+1 = |V(T_2)| + 1$. Since $n = |V(T_1)| + |V(T_2)|$, we obtain $|V(T_1)| = 1$, so $V(T_1) = \{v\}$. Thus, T is the k-star T'_2.

Finally, suppose $\Delta(T_1) = k$. By Proposition 3.7, $\lambda_{\mathrm{e}}(T-e) = \lambda_{\mathrm{e}}(T_1)$. By the induction hypothesis, $\lambda_{\mathrm{e}}(T_1) \leq \frac{n-k-1}{k}$, and equality holds if and only if T_1 is an edge-disjoint k-star union. By Proposition 3.9, $\lambda_{\mathrm{e}}(T) \leq 1 + \lambda_{\mathrm{e}}(T-e) \leq 1 + \frac{n-k-1}{k} = \frac{n-1}{k}$.

Suppose $\lambda_{\rm e}(T) = \frac{n-1}{k}$. Then $\lambda_{\rm e}(T_1) = \frac{n-k-1}{k}$, and hence T_1 is an edge-disjoint k-star union. Since T is the union of T_1 and T_2', T is an edge-disjoint k-star union.

We now prove the converse. Thus, suppose that T is an edge-disjoint k-star union. Then there exist pairwise edge-disjoint k-stars $G_1, \dots, G_r$ such that $z_1 \in V(G_r)$ and T is the union of $G_1, \dots, G_r$. Since $N_{G_r}(z_1) \subseteq N_T(z_1) = \{w\}$, $G_r = (\{w, z_1, y_1, \dots, y_{k-1}\}, \{wz_1, wy_1, \dots, w_{y_{k-1}}\})$ for some $y_1, \dots, y_{k-1} \in V(T)$. Since $d_{G_r}(w) = k = d_T(w)$, $N_{G_r}(w) = N_T(w)$. Thus, $\{z_1, y_1, \dots, y_{k-1}\} = \{z_1, \dots, z_{k-1}, v\}$, and hence $G_r = T_2'$. Consequently, T_1 is the union of $G_1, \dots, G_{r-1}$, and hence $\lambda_{\rm e}(T_1) = \frac{n-k-1}{k}$. Let L be a Δ-reducing edge set of T of size $\lambda_{\rm e}(T)$. Let $L_1 = L \cap E(T_1)$ and $L_2 = L \cap E(T_2')$. Since $E(T_1)$ and $E(T_2')$ partition $E(T)$, L_1 and L_2 partition L. Since $w \in M(T)$ and $E_T(w) = E(T_2')$, $L_2 \neq \emptyset$. Suppose that L_1 is not a Δ-reducing edge set of T_1. Then, since $\Delta(T_1) = k$, there exists some $u \in V(T_1)$ such that $d_{T_1}(u) = k$ and $E_T(u) \cap L \subseteq L_2$. Since $V(T_1) \cap V(T_2') = \{v\}$ and $L_2 \subseteq V(T_2')$, $u = v$. Now $k \geq |E_T(v)| = |E_{T_1}(v) \cup \{e\}| > |E_{T_1}(v)| = d_{T_1}(v)$, which contradicts $d_{T_1}(v) = d_{T_1}(u) = k$. Thus, L_1 is a Δ-reducing edge set of T_1. We have $\frac{n-1}{k} \geq \lambda_{\rm e}(T) = |L| = |L_1| + |L_2| \geq \lambda_{\rm e}(T_1) + 1 = \frac{n-k-1}{k} + 1 = \frac{n-1}{k}$, so $\lambda_{\rm e}(T) = \frac{n-1}{k}$.

A basic result in the literature is that $|E(G)| = |V(G)| - 1$ if G is a tree. This completes the proof. □

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