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Upper bounds on the k-tuple domination number and k-tuple total domination number of a graph

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Abstract

Given a positive integer k, a subset S of vertices of a graph G is called a *k-tuple dominating set* in G if for every vertex $v \in V(G)$, $|N[v] \cap S| \geq k$. The minimum cardinality of a k-tuple dominating set in G is the *k-tuple domination number* $\gamma_{\times k}(G)$ of G. A subset S of vertices of a graph G is called a *k-tuple total dominating set* in G if for every vertex $v \in V(G)$, $|N(v) \cap S| \geq k$. The minimum cardinality of a k-tuple total dominating set in G is the *k-tuple total domination number* $\gamma_{\times k,t}(G)$ of G. We present probabilistic upper bounds for the k-tuple domination number of a graph as well as for the k-tuple total domination number of a graph, and improve previous bounds given in [J. Harant and M.A. Henning, *Discuss. Math. Graph Theory* 25 (2005), 29–34], [E.J. Cockayne and A.G. Thomason, *J. Combin. Math. Combin. Comput.* 64 (2008), 251–254], and [M.A. Henning and A.P. Kazemi, *Discrete Appl. Math.* 158 (2010), 1006–1011] for graphs with sufficiently large minimum degree under certain assumptions.

1 Introduction

For graph theory notation and terminology not given here we refer to [10], and for the probabilistic methods notation and terminology we refer to [1]. We consider finite, undirected and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices of G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to G), is the cardinality of its open

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neighborhood. We denote by $\delta(G)$ and $\Delta(G)$, the minimum and maximum degrees among all vertices of G, respectively. For a subset S of $V(G)$, the subgraph of G induced by S is denoted by $G[S]$. A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A set $S \subseteq V$ is a *total dominating set* if each vertex in V is adjacent to at least one vertex of S, while the minimum cardinality of a total dominating set is the *total domination number* $\gamma_t(G)$ of G.

For a positive integer k, a set $S \subseteq V(G)$ is called a *k-tuple dominating set* in G if for every vertex $v \in V(G)$, $|N[v] \cap S| \geq k$. The minimum cardinality of a k-tuple dominating set in G is the *k-tuple domination number* $\gamma_{\times k}(G)$ of G. For the case $k = 2$, the k-tuple domination is also called *double domination*. The concept of k-tuple domination number was introduced by Harary and Haynes [9], and further studied for example in [4, 6, 7, 8, 14, 15, 17]. Henning and Kazemi [11] introduced the concept of k-tuple total domination in graphs. For a positive integer k, a subset S of V is a *k-tuple total dominating set* of G if for every vertex $v \in V$, $|N(v) \cap S| \geq k$. The *k-tuple total domination number* $\gamma_{\times k,t}(G)$ is the minimum cardinality of a k-tuple total dominating set of G. The concept of k-tuple total domination number was further studied for example in [2, 3, 5, 12, 13, 16]. We note that if a graph G has a k-tuple dominating set, then clearly, $\delta \geq k - 1$, and if a graph G has a k-tuple total dominating set then $\delta \geq k$.

Harant and Henning obtained the following probabilistic upper bound on the double domination number of a graph.

Theorem 1.1 (Harant and Henning, [8]) *If G is a graph of order n with minimum degree $\delta \geq 1$ and average degree d, then*

$$\gamma_{\times 2}(G) \leq \left(\frac{\ln(1+d) + \ln \delta + 1}{\delta} \right) n.$$

Cockayne and Thomason [4] improved Theorem 1.1.

Theorem 1.2 (Cockayne and Thomason [4]) *If G is a graph of order n with minimum degree $\delta \geq 1$, then*

$$\gamma_{\times 2}(G) \leq \left(\frac{\ln(1+\delta) + \ln \delta + 1}{\delta} \right) n.$$

They also presented the following probabilistic upper bound on the k-tuple domination number of a graph.

Theorem 1.3 (Cockayne and Thomason [4]) *Let G be a graph of order n with minimum degree $\delta \geq 1$. If k is fixed and δ is sufficiently large, then*

$$\gamma_{\times k}(G) \leq n \left(\frac{\ln \delta + (k - 1 + o(1)) \ln \ln \delta}{\delta} \right).$$

Henning and Kazemi proved the following.

Theorem 1.4 (Henning and Kazemi [11]) *If G is a graph of order n with minimum degree $\delta \geq 2$, then*

$$\gamma_{\times 2,t}(G) \leq \left(\frac{\ln(2+\delta)+\ln\delta+1}{\delta}\right)n.$$

Theorem 1.5 (Henning and Kazemi [11]) *Let G be a graph of order n with minimum degree δ. If k is fixed and δ is sufficiently large, then*

$$\gamma_{\times k,t}(G) \leq n\left(\frac{\ln\delta+(k-1+o(1))\ln\ln\delta}{\delta}\right).$$

In the proof of Theorems 1.2, 1.3, 1.4 and 1.5 it is assumed that δ is sufficiently large and k is fixed. In this paper, we first obtain new probabilistic upper bounds for the k-tuple domination number of a graph with sufficiently large δ, explicitly, when $\delta \geq 3k-4$, and we improve both Theorems 1.2 and 1.3 under some certain assumptions. We next obtain new probabilistic upper bounds for the k-tuple total domination number of a graph with sufficiently large δ, explicitly, when $\delta \geq 3k-2$, and we improve both Theorems 1.4 and 1.5 in such a case and under some certain assumptions. The main probabilistic methods are similar to those presented in the proof of Theorems 1.2, 1.3, 1.4 and 1.5.

For two subset A and B of vertices of G, and an integer k, we say that A *k-tuple dominates* B if for any vertex $v \in B$, $|N[v] \cap A| \geq k$. Similarly, we say that A *k-tuple total dominates* B if for any vertex $v \in B$, $|N(v)] \cap A| \geq k$. For a random variable X, we denote by $\mathbb{E}(X)$ the expectation of X.

2 Bounds for the k-tuple domination number

We first prove the following important lemma.

Lemma 2.1 *Let $k \geq 1$ be a positive integer and G be a graph on n vertices with minimum degree $\delta \geq 3k-4$ and maximum degree Δ. Let $A \subseteq V(G)$ be a set obtained by choosing each vertex $v \in V(G)$ independently with probability $p \in (0,1)$, $A' = \{v \in A : |N_G(v) - A| \leq k-2\}$, and $A'' = \{v \in A' : |N_G(v) - A'| \leq 2k-3\}$. Then there is a subset $S \subseteq A'$ such that S k-tuple dominates A'' and $|S| \leq t|A'|$, where*

$$t = p + \sum_{i=0}^{k-1}(k-i)\binom{\delta+1}{i}p^i(1-p)^{\delta-2k+4-i}.$$

Proof. Let $\delta_1 = \min\{\deg_{G[A']}(v) : v \in A''\}$. For any vertex $v \in A''$ we have $\deg_{G[A']}(v) = \deg_G(v) - |N_G(v) - A'| \geq \deg_G(v) - (2k-3) \geq \delta - (2k-3)$. Thus $\delta_1 \geq \delta - (2k-3) \geq k-1$. For each vertex $v \in A''$, pick a set N_v comprising v and δ_1 of its neighbors in A', so $|N_v| = \delta_1 + 1$.

Create a subset $A_1 \subseteq A'$ by choosing each vertex $v \in A'$ independently with probability p. Let $V_i = \{v \in A'' : |N_v \cap A_1| = i\}$, for $0 \leq i \leq k-1$. Form the set X_i by placing within it $k-i$ members of $N_v - A_1$ for each $v \in V_i$. Note that $|X_i| \leq (k-i)|V_i|$. Let $B_1 = \bigcup_{i=0}^{k-1} X_i$. Then the set $D = A_1 \cup B_1$, k-tuple-dominates any vertex of A''. We now compute the expectation of $|D|$. Clearly, $\mathbb{E}(|A_1|) = |A'|p$, since $|A_1|$ can be denoted as the sum of $|A'|$ random variables. For each vertex $v \in A''$, $Pr(v \in V_i) = \binom{\delta_1+1}{i} p^i(1-p)^{\delta_1+1-i}$. Thus by the linearity property of the expectation,

$$\begin{aligned}
\mathbb{E}(|D|) &= \mathbb{E}(|A_1|) + \mathbb{E}(|B_1|) \\
&\leq \mathbb{E}(|A_1|) + \sum_{i=0}^{k-1} \mathbb{E}(|X_i|) \\
&\leq \mathbb{E}(|A_1|) + \sum_{i=0}^{k-1} (k-i)\mathbb{E}(|V_i|) \\
&\leq |A'|p + |A'| \sum_{i=0}^{k-1} (k-i) \binom{\delta_1+1}{i} p^i(1-p)^{\delta_1+1-i} \\
&= |A'| \left[p + \sum_{i=0}^{k-1} (k-i) \binom{\delta_1+1}{i} p^i(1-p)^{\delta_1+1-i} \right] \\
&\leq |A'| \left[p + \sum_{i=0}^{k-1} (k-i) \binom{\delta+1}{i} p^i(1-p)^{\delta-2k+4-i} \right] = t|A'|.
\end{aligned}$$

Hence, by the pigeonhole property of the expectation there is a subset $S \subseteq A'$ such that S k-tuple dominates A'' and $|S| \leq t|A'|$. ■

Theorem 2.2 *Let $k \geq 1$ be a positive integer and $p \in (0,1)$ be a real number. For any graph G on n vertices with minimum degree $\delta \geq 3k-4$ and maximum degree Δ,*

$$\begin{aligned}
\gamma_{\times k}(G) &\leq n\left(p + \sum_{i=0}^{k-1} (k-i) \binom{\delta+1}{i} p^i(1-p)^{\delta+1-i} \right) \\
&- n\left[1 - p - \sum_{i=0}^{k-1} (k-i) \binom{\delta+1}{i} p^i(1-p)^{\delta-2k+4-i} \right] \binom{\delta}{k-2} p^{3+\Delta-k}.
\end{aligned}$$

Proof. Let $k \geq 1$ be a positive integer, and let G be a graph on n vertices with minimum degree $\delta \geq 3k-4$ and maximum degree Δ. Create a subset $A \subseteq V(G)$ by choosing each vertex $v \in V(G)$ independently with probability p. Let $A' = \{v \in A : |N(v) - A| \leq k-2\}$, and $A'' = \{v \in A' : |N(v) - A'| \leq 2k-3\}$. For any vertex $v \in A' - A''$, $|N(v) \cap (A - A')| = |N(v) - A'| - |N(v) - A| \geq 2k-2-(k-2) = k$. Thus any vertex of $A' - A''$ is k-tuple-dominated by some vertex of $A - A'$. Let $V_i = \{v \in V : |N[v] \cap A| = i\}$ for $0 \leq i \leq k-1$. Clearly $V_i \cap A' = \emptyset$, since $|N(v) \cap A| \geq \deg(v) - |N(v) - A| \geq \delta - (k-2) \geq 3k-4-(k-2) = 2(k-1) > k$ for any vertex $v \in A'$. Thus, $V_i \subseteq V(G) - A'$. For each vertex $v \in V_i$, pick a set N_v comprising v and δ of its neighbors in $V(G) - A'$, so $|N_v| = \delta + 1$. Form the set X_i by placing within it $k-i$ members of $N_v - A$ for each $v \in V_i$. Note that $|X_i| \leq (k-i)|V_i|$. Let $B = \bigcup_{i=0}^{k-1} X_i$. For each vertex $v \in V(G)$, $Pr(v \in V_i) = \binom{\delta+1}{i} p^i (1-p)^{\delta+1-i}$.

By Lemma 2.1, there is a set $S \subseteq A'$ such that S k-tuple-dominates any vertex of A'', and $|S| \leq t|A'|$, where

$$t = p + \sum_{i=0}^{k-1} (k-i) \binom{\delta+1}{i} p^i (1-p)^{\delta-2k+4-i}.$$

Evidently, $D = (A - A') \cup B \cup S$ is a k-tuple dominating set in G. We compute the expectation of $|D|$ as follows. Note that

$$\begin{aligned} |D| &= |(A - A') \cup B \cup S| \\ &= |A - A'| + |B| + |S| \\ &= |A| - |A'| + |B| + |S| \\ &\leq |A| + |B| - |A'| + t|A'| \\ &= |A| + |B| - (1-t)|A'|. \end{aligned}$$

By the linearity property of the expectation, $\gamma_{\times k}(G) \leq \mathbb{E}(|D|) \leq \mathbb{E}(|A|) + \mathbb{E}(|B|) - (1-t)\mathbb{E}(|A'|)$. It is routine to see that $\mathbb{E}(|A|) = np$ and $\mathbb{E}(|B|) \leq n \sum_{i=0}^{k-1} (k-i) \binom{\delta+1}{i} p^i (1-p)^{\delta+1-i}$. For a vertex v, if $v \in A'$ then $v \in A$ and at least $\deg(v) - (k-2)$ of its neighbors belong to A. Thus,

$$\begin{aligned} Pr(v \in A') &= \binom{\deg(v)}{\deg(v)-(k-2)} p^{1+\deg(v)-(k-2)} \\ &= \binom{\deg(v)}{k-2} p^{1+\deg(v)-(k-2)} \geq \binom{\delta}{k-2} p^{3+\Delta-k}. \end{aligned}$$

Thus $\mathbb{E}(|A'|) \geq n \binom{\delta}{k-2} p^{3+\Delta-k}$. Now a simple calculation yields the result. ■

Using the fact that $1 - x \leq e^{-x}$, for $0 \leq x \leq 1$ from Theorem 2.2, we obtain the following.

Corollary 2.3 *Let $k \geq 1$ be a positive integer and $p \in (0,1)$ be a real number. For any graph G on n vertices with minimum degree $\delta \geq 3k-4$ and maximum degree Δ,*

$$\gamma_{\times k}(G) \leq$$
$$n\left(\frac{\ln\delta+(k-1+o(1))\ln\ln\delta}{\delta}\right)-n\left\{\binom{\delta}{k-2}\left(\frac{\delta-\ln\delta-(k-1+o(1))\ln\ln\delta}{\delta}\right)\left(\frac{\ln\delta+(k-1+o(1))\ln\ln\delta}{\delta}\right)^{3+\Delta-k}\right\}.$$

Proof. Let $\varepsilon > 0$ and $p = (\ln\delta + (k-1+\varepsilon)\ln\ln\delta)/(\delta - k + 2)$. By Theorem 2.2,

$$\begin{aligned}
\gamma_{\times k}(G) &\leq n\left(p+\sum_{i=0}^{k-1}(k-i)\binom{\delta+1}{i}p^i(1-p)^{\delta+1-i}\right)\\
&\quad -n\left[1-p-\sum_{i=0}^{k-1}(k-i)\binom{\delta+1}{i}p^i(1-p)^{\delta-2k+4-i}\right]\binom{\delta}{k-2}p^{3+\Delta-k}\\
&\leq n\left(p+\sum_{i=0}^{k-1}k(\delta+1)^ip^i(1-p)^{\delta+1-i}\right)\\
&\quad -n\left[1-p-\sum_{i=0}^{k-1}k(\delta+1)^ip^i(1-p)^{\delta-2k+4-i}\right]\binom{\delta}{k-2}p^{3+\Delta-k}\\
&\leq n\left(p+k^2((\delta+1)p)^{k-1}e^{-p(\delta-k+2)}\right) \qquad (1-x\leq e^{-x})\\
&\quad -n\left(1-p-k^2((\delta+1)p)^{k-1}e^{-p(\delta-3k+5)}\right)\binom{\delta}{k-2}p^{3+\Delta-k}.
\end{aligned}$$

But if δ is large, then

$$\begin{aligned}
((\delta+1)p)^{k-1}e^{-p(\delta-k+2)} &= (1+o(1))(\ln\delta)^{k-1}(\ln\delta)^{-(k-1+\varepsilon)}(\delta)^{-1}\\
&= (1+o(1))\frac{1}{\delta(\ln\delta)^{\varepsilon}} < \frac{\varepsilon}{\delta},
\end{aligned}$$

and also

$$((\delta+1)p)^{k-1}e^{-p(\delta-3k+5)} = (1+o(1))(\ln\delta)^{k-1}(\ln\delta)^{-(k-1+\varepsilon)}(\delta)^{-1} < \frac{\varepsilon}{\delta}.$$

Thus $p + k^2((\delta+1)p)^{k-1}e^{-p(\delta-k+2)} \leq p + \frac{k^2\varepsilon}{\delta}$, and

$$p + k^2((\delta+1)p)^{k-1}e^{-p(\delta-3k+5)} \leq p + \frac{k^2\varepsilon}{\delta}.$$

Since $\varepsilon > 0$ is arbitrary, we find that $p + k^2((\delta+1)p)^{k-1}e^{-p(\delta-k+2)} \leq p$, and $p + k^2((\delta+1)p)^{k-1}e^{-p(\delta-3k+5)} \leq p$. Now the result follows. ■

Similarly, letting $p = \dfrac{\ln(1+\delta)+\ln\delta}{\delta}$, we obtain the following.

Corollary 2.4 *For any graph G on n vertices with minimum degree $\delta \geq 2$ and maximum degree Δ, $\gamma_{\times 2}(G) \leq$*

$$\left(\frac{\ln(1+\delta)+\ln\delta+1}{\delta}\right)n - n\left(\frac{\delta-\ln(1+\delta)-\ln\delta-1}{\delta}\right)\left(\frac{\ln(0+\delta)+\ln\delta}{\delta}\right)^{1+\Delta}.$$

We note that Corollary 2.3 improves Theorem 1.3 if δ is sufficiently large and $\delta-\ln\delta-(k-1+o(1))\ln\ln\delta > 0$ (for example if k is fixed or $k = o(\delta)$), and Corollary 2.4 improves Theorem 1.2 if δ is sufficiently large and $\delta-\ln(1+\delta)-\ln\delta-1 > 0$ (for example if k is fixed or $k = o(\delta)$).

3 Bounds for the k-tuple total domination number

We begin with the following important lemma.

Lemma 3.1 *Let $k \geq 1$ be a positive integer and G be a graph on n vertices with minimum degree $\delta \geq 3k-2$ and maximum degree Δ. Let $A \subseteq V(G)$ be a set obtained by choosing each vertex $v \in V(G)$ independently with probability $p \in (0,1)$, $A' = \{v \in V(G) : |N(v) - A| \leq k-1\}$, and $A'' = \{v \in A' : |N_G(v) - A'| \leq 2k-2\}$. Then there is a subset $S \subseteq A'$ such that S k-tuple total dominates A'' and $|S| \leq t|A'|$, where*

$$t = p + \sum_{i=0}^{k-1}(k-i)\binom{\delta}{i}p^i(1-p)^{\delta-(2k-2)-i}.$$

Proof. Let $\delta_1 = \min\{\deg_{G[A']}(v) : v \in A''\}$. For any vertex $v \in A''$ we have $\deg_{G[A']}(v) = \deg_G(v) - |N_G(v) - A'| \geq \deg_G(v) - (2k-3) \geq \delta - (2k-2)$. Thus $\delta_1 \geq \delta - (2k-2) \geq k$. For each vertex $v \in A''$, pick a set N_v consisting of δ_1 of its neighbors in A', so $|N_v| = \delta_1$.

Create a subset $A_1 \subseteq A'$ by choosing each vertex $v \in A'$ independently with probability p. Let $V_i = \{v \in A'' : |N_v \cap A_1| = i\}$, for $0 \leq i \leq k-1$. Form the set X_i by placing within it $k-i$ members of $N_v - A_1$ for each $v \in V_i$. Note that $|X_i| \leq (k-i)|V_i|$. Let $B_1 = \bigcup_{i=0}^{k-1} X_i$. Then the set $D = A_1 \cup B_1$, k-tuple-dominates any vertex of A''. We now compute the expectation of $|D|$. Clearly, $\mathbb{E}(|A_1|) = |A'|p$. For each vertex $v \in A''$, $Pr(v \in V_i) = \binom{\delta_1}{i}p^i(1-p)^{\delta_1-i}$. Thus by the linearity property of the expectation,

$$\begin{aligned}\mathbb{E}(D) &= \mathbb{E}(|A_1|) + \mathbb{E}(|B_1|)\\ &\leq \mathbb{E}(|A_1|) + \sum_{i=0}^{k-1}\mathbb{E}(|X_i|)\end{aligned}$$

$$\begin{aligned}
&\le \mathbb{E}(|A_1|) + \sum_{i=0}^{k-1}(k-i)\mathbb{E}(|V_i|)\\
&\le |A'|p + |A'|\sum_{i=0}^{k-1}(k-i)\binom{\delta_1}{i}p^i(1-p)^{\delta_1-i}\\
&= |A'|\left[p + \sum_{i=0}^{k-1}(k-i)\binom{\delta_1}{i}p^i(1-p)^{\delta_1-i}\right]\\
&\le |A'|\left[p + \sum_{i=0}^{k-1}(k-i)\binom{\delta}{i}p^i(1-p)^{\delta-(2k-2)-i}\right] = t|A'|.
\end{aligned}$$

Hence, there is a subset $S \subseteq A'$ such that S k-tuple dominates A'' and $|S| \le t|A'|$. ■

Theorem 3.2 *Let $k \ge 1$ be a positive integer and $p \in (0,1)$ be a real number. For any graph G on n vertices with minimum degree $\delta \ge 3k-2$ and maximum degree Δ,*

$$\begin{aligned}
\gamma_{\times k,t}(G) \;\le\; & n\left(p + \sum_{i=0}^{k-1}(k-i)\binom{\delta}{i}p^i(1-p)^{\delta-i}\right)\\
& - n\left[1-p-\sum_{i=0}^{k-1}(k-i)\binom{\delta}{i}p^i(1-p)^{\delta-(2k-2)-i}\right]\binom{\delta}{k-1}p^{1+\Delta-(k-1)}.
\end{aligned}$$

Proof. Let $k \ge 1$ be a positive integer and let G be a graph on n vertices with minimum degree $\delta \ge 3k-2$ and maximum degree Δ. Create a subset $A \subseteq V(G)$ by choosing each vertex $v \in V(G)$ independently with probability p. Let $A' = \{v \in A : |N(v) - A| \le k-1\}$, and $A'' = \{v \in A' : |N(v) - A'| \le 2k-2\}$. For any vertex $v \in A' - A''$, $|N(v) \cap (A - A')| = |N(v) - A'| - |N(v) - A| \ge 2k-1-(k-1) = k$. Thus any vertex of $A' - A''$ is k-tuple total-dominated by some vertex of $A - A'$. Let $V_i = \{v \in V : |N[v] \cap A| = i\}$ for $0 \le i \le k-1$. Clearly $V_i \cap A' = \emptyset$, since $|N(v) \cap A| \ge \deg(v) - |N(v) - A| \ge \delta - (k-1) > k$ for any vertex $v \in A'$. Thus, $V_i \subseteq V(G) - A'$. For each vertex $v \in V_i$, pick a set N_v consisting of δ of its neighbors in $V(G) - A'$, so $|N_v| = \delta$. Form the set X_i by placing within it $k-i$ members of $N_v - A$ for each $v \in V_i$. Note that $|X_i| \le (k-i)|V_i|$. Let $B = \bigcup_{i=0}^{k-1} X_i$. For each vertex $v \in V(G)$, $Pr(v \in V_i) = \binom{\delta}{i}p^i(1-p)^{\delta-i}$.

By Lemma 3.1, there is a set $S \subseteq A'$ such that S k-tuple-dominates any vertex of A'', and $|S| \le t|A'|$, where

$$t = p + \sum_{i=0}^{k-1}(k-i)\binom{\delta}{i}p^i(1-p)^{\delta-(2k-2)-i}.$$

Evidently, $D = (A - A') \cup B \cup S$ is a k-tuple total dominating set in G. We compute the expectation of $|D|$ as follows. Note that

$$\begin{aligned} |D| &= |(A - A') \cup B \cup S| \\ &= |A - A'| + |B| + |S| \\ &= |A| - |A'| + |B| + |S| \\ &\leq |A| + |B| - |A'| + t|A'| \\ &= |A| + |B| - (1-t)|A'|. \end{aligned}$$

By the linearity property of the expectation, $\gamma_{\times k}(G) \leq \mathbb{E}(|D|) \leq \mathbb{E}(|A|) + \mathbb{E}(|B|) - (1-t)\mathbb{E}(|A'|)$. It is routine to see that $\mathbb{E}(|A|) = np$ and

$$\mathbb{E}(|B|) \leq n \sum_{i=0}^{k-1} (k-i) \binom{\delta}{i} p^i (1-p)^{\delta-i}.$$

For a vertex v,

$$\begin{aligned} Pr(v \in A') &= \binom{\deg(v)}{\deg(v) - (k-1)} p^{1+\deg(v)-(k-1)} \\ &= \binom{\deg(v)}{k-1} p^{1+\deg(v)-(k-1)} \geq \binom{\delta}{k-1} p^{1+\Delta-(k-1)}. \end{aligned}$$

Thus $\mathbb{E}(|A'|) \geq n \binom{\delta}{k-1} p^{1+\Delta-(k-1)}$. Now a simple calculation yields the result. ∎

Using the fact that $1 - x \leq e^{-x}$, for $0 \leq x \leq 1$ from Theorem 3.2, we obtain the following by letting $p = (\ln\delta + (k-1+\varepsilon)\ln\ln\delta)/(\delta - k + 2)$ for $\varepsilon > 0$.

Corollary 3.3 *Let $k \geq 1$ be a positive integer. For any graph G on n vertices with minimum degree $\delta \geq 3k-2$ and maximum degree Δ,*

$$\begin{aligned} \gamma_{\times k,t}(G) \leq n &\left(\frac{\ln\delta + (k-1+o(1))\ln\ln\delta}{\delta} \right) - n \binom{\delta}{k-1} \\ &\left(\frac{\delta - \ln\delta - (k-1+o(1))\ln\ln\delta}{\delta} \right) i \left(\frac{\ln\delta + (k-1+o(1))\ln\ln\delta}{\delta} \right)^{1+\Delta-(k-1)}. \end{aligned}$$

Proof. Let $\varepsilon > 0$ and $p = (\ln\delta + (k-1+\varepsilon)\ln\ln\delta)/(\delta - k + 2)$. By Theorem 3.2,

$$\begin{aligned} \gamma_{\times k}(G) &\leq n\left(p + \sum_{i=0}^{k-1} (k-i) \binom{\delta}{i} p^i (1-p)^{\delta-i} \right) \\ &\quad - n\left[1 - p - \sum_{i=0}^{k-1} (k-i) \binom{\delta}{i} p^i (1-p)^{\delta-(2k-2)-i} \right] \binom{\delta}{k-1} p^{1+\Delta-(k-1)} \\ &\leq n\left(p + k^2 (\delta p)^{k-1} e^{-p(\delta-k+1)} \right) \qquad \left(1 - x \leq e^{-x}, \binom{\delta}{i} \leq \delta^i\right) \end{aligned}$$

$$-n\left(1-p-k^2(\delta p)^{k-1}e^{-p(\delta-3k+5)}\right)\binom{\delta}{k-1}p^{1+\Delta-(k-1)}.$$

But if δ is large, then $(\delta p)^{k-1}e^{-p(\delta-k+1)} = (1+o(1))(\ln\delta)^{k-1}(\ln\delta)^{-(k-1+\varepsilon)}(\delta)^{-1} < \frac{\varepsilon}{\delta}$, and also $(\delta p)^{k-1}e^{-p(\delta-3k+5)} = (1+o(1))(\ln\delta)^{k-1}(\ln\delta)^{-(k-1+\varepsilon)}(\delta)^{-1} < \frac{\varepsilon}{\delta}$. Thus $p+k^2(\delta p)^{k-1}e^{-p(\delta-k+1)} \le p+\frac{k^2\varepsilon}{\delta}$, and $p+k^2((\delta p)^{k-1}e^{-p(\delta-3k+5)} \le p+\frac{k^2\varepsilon}{\delta}$. Since $\varepsilon > 0$ is arbitrary, we have $p+k^2(\delta p)^{k-1}e^{-p(\delta-k+1)} \le p$, and $p+k^2((\delta p)^{k-1}e^{-p(\delta-3k+5)} \le p$. Now the result follows. ■

Similarly, letting $p = \dfrac{\ln(2+\delta)+\ln\delta}{\delta}$, we obtain the following.

Corollary 3.4 *For any graph G on n vertices with minimum degree $\delta \ge 4$ and maximum degree Δ,*

$$\gamma_{\times 2,t}(G) \le \left(\frac{\ln(2+\delta)+\ln\delta+1}{\delta}\right)n - n(\delta-\ln(2+\delta)-\ln\delta+1)\left(\frac{\ln(1+\delta)+\ln\delta}{\delta}\right)^{\Delta}.$$

We note that Corollary 3.3 improves Theorem 1.5 if δ is sufficiently large and $\delta - \ln\delta - (k-1+o(1))\ln\ln\delta > 0$ (for example, if k is fixed or $k = o(\delta)$), and Corollary 3.4 improves Theorem 1.4.

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