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Improved bounds for uniform hypergraphs without property B

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Abstract

A hypergraph is said to be properly 2-colorable if there exists a 2-coloring of its vertices such that no hyperedge is monochromatic. On the other hand, a hypergraph is called non-2-colorable if there exists at least one monochromatic hyperedge in each of the possible 2-colorings of its vertex set. Let $m(n)$ denote the minimum number of hyperedges in a non-2-colorable n-uniform hypergraph. Establishing the lower and upper bounds on $m(n)$ is a well-studied research direction over several decades. In this paper, we present new constructions for non-2-colorable uniform hypergraphs. These constructions improve the upper bounds for $m(8)$, $m(13)$, $m(14)$, $m(16)$ and $m(17)$. We also improve the lower bound for $m(5)$.

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1 Introduction

Hypergraphs are combinatorial structures that are generalizations of graphs. A *hypergraph* consists of a finite set of elements called its vertex set and a hyperedge set formed by some distinct subsets of its vertex set. A hypergraph $H = (V, E)$ with vertex set V is called *n-uniform* if each hyperedge in the hyperedge set E has exactly n vertices in it. A *2-coloring* of such a hypergraph H is an assignment of one of the two colors red and blue to each of the vertices in V. We say a 2-coloring of H to be *proper* if each of its hyperedges has red as well as blue vertices. H is said to be *non-2-colorable* if no proper 2-coloring exists for it; otherwise, it is said to satisfy *Property B*. For an integer $n \geq 1$, let $m(n)$ denote the minimum number of hyperedges present in a non-2-colorable n-uniform hypergraph.

Establishing an upper bound on $m(n)$ is a well-explored combinatorial problem. Erdős [6] gave a non-constructive proof to establish the currently best known upper bound $m(n) = O(n^2 2^n)$. However, there is no known construction for a non-2-colorable n-uniform hypergraph that matches this upper bound. Abbott and Moser [2] constructed a non-2-colorable n-uniform hypergraph with $O((\sqrt{7}+o(1))^n)$ hyperedges. Gebauer [8] improved this result by constructing a non-2-colorable n-uniform hypergraph with $O(2^{(1+o(1))n})$ hyperedges. Even though this is the best construction known for a non-2-colorable n-uniform hypergraph for large n, it is still asymptotically far from the above-mentioned non-constructive upper bound given by Erdős.

Finding upper bounds on $m(n)$ for small values of n is also a well-studied problem and several constructions have been given for establishing these. For example, it can be easily seen that $m(1) \leq 1$, $m(2) \leq 3$ (the corresponding 2-uniform hypergraph is the triangle graph) and $m(3) \leq 7$ (the corresponding 3-uniform hypergraph is known as the Fano plane [10], denoted by H_f in this paper). The previously-mentioned construction of Abbott and Moser shows that $m(4) \leq 27$, $m(6) \leq 147$ and $m(8) \leq 2187$. Moreover, their construction also gives non-trivial upper bounds on $m(n)$ for $n = 9, 10, 12, 14, 15$ and 16. For $n \geq 3$, Abbott and Hanson [1] gave a construction using a non-2-colorable $(n-2)$-uniform hypergraph to show that $m(n) \leq n \cdot m(n-2) + 2^{n-1} + 2^{n-2}((n-1) \mod 2)$. Using the best known upper bounds on $m(n-2)$, this recurrence relation establishes non-trivial upper bounds as well as improves such bounds on $m(n)$ for a few small values of n. For example, it shows that $m(4) \leq 24$, $m(5) \leq 51$ and $m(7) \leq 421$. Seymour [14] further improved the upper bound on $m(4)$ to $m(4) \leq 23$ by constructing a non-2-colorable 4-uniform hypergraph with 23 hyperedges. In this paper, we denote this hypergraph by H_s. For even integers $n \geq 4$, Toft [15] generalized this construction using a non-2-colorable $(n-2)$-uniform hypergraph to improve Abbott and Hanson's result to $m(n) \leq n \cdot m(n-2) + 2^{n-1} + \binom{n}{n/2}/2$. In particular, this led to establishing an upper bound $m(8) \leq 1339$. For a given integer $n \geq 3$ and a non-2-colorable $(n-2)$-uniform hypergraph A, we refer to Abbott–Hanson's construction for odd n and Toft's construction for even n as *Abbott–Hanson–Toft construction* and denote the number of hyperedges in such a hypergraph as $m_A(n)$. We describe this construction in Section 1.1.

$m(n)$	Corresponding construction/recurrence relation
$m(1) = 1$	Single vertex
$m(2) = 3$	Triangle graph
$m(3) = 7$	Fano plane [10]
$m(4) = 23$	[12], [14]
$m(5) \leq 51$	$m(5) \leq 2^4 + 5m(3)$
$m(6) \leq 147$	$m(6) \leq m(2)m(3)^2$
$m(7) \leq 421$	$m(7) \leq 2^6 + 7m(5)$
$m(8) \leq 1269$	[11]
$m(9) \leq 2401$	$m(9) \leq m(3)^4$
$m(10) \leq 7803$	$m(10) \leq m(2)m(5)^2$
$m(11) \leq 25449$	$m(11) \leq 15 \cdot 2^8 + 9m(9)$
$m(12) \leq 55223$	$m(12) \leq m(3)^4 m(4)$
$m(13) \leq 297347$	$m(13) \leq 17 \cdot 2^{10} + 11m(11)$
$m(14) \leq 531723$	$m(14) \leq m(2)m(7)^2$
$m(15) \leq 857157$	$m(15) \leq m(3)^5 m(5)$
$m(16) \leq 4831083$	$m(16) \leq m(2)m(8)^2$
$m(17) \leq 13201419$	$m(17) \leq 21 \cdot 2^{14} + 15m(15)$

Table 1: The best known upper bounds on $m(n)$ for small values of n [11]

It can be easily observed that $m(n) \leq m_A(n)$ for any non-2-colorable $(n-2)$-uniform hypergraph A. In fact, we have already seen that the above-mentioned upper bounds $m(4) \leq 23$, $m(5) \leq 51$, $m(7) \leq 421$ and $m(8) \leq 1339$ are obtained by Abbott–Hanson–Toft constructions using the best known constructions for non-2-colorable 2, 3, 5 and 6-uniform hypergraphs, respectively. A construction given by Mathews et al. [11] improved the upper bound on $m(8)$ to $m(8) \leq 1269$. In addition, they modified the Abbott–Hanson–Toft construction to improve the upper bounds on $m(n)$ for $n = 11, 13$ and 17. The currently best known upper bounds on $m(n)$ for $n \leq 17$ are given in Table 1.

In the other direction, Erdős [6] showed the lower bound on $m(n)$ to be $m(n) = \Omega(2^n)$, which was later improved by Beck [3] to $m(n) = \Omega(n^{1/3-o(1)}2^n)$. The currently best known lower bound $m(n) = \Omega(\sqrt{\frac{n}{\ln n}}2^n)$ was given by Radhakrishnan and Srinivasan [13]. A simpler proof for the same result has been given by Cherkashin and Kozik [4]. Note that there is a significant asymptotic gap between the currently best known lower and upper bounds on $m(n)$. Even for small values of n, we are only aware of a few lower bounds for $m(n)$ that match the corresponding upper bounds. It can be easily seen that $m(1) \geq 1$, $m(2) \geq 3$ and $m(3) \geq 7$ and therefore $m(1) = 1$, $m(2) = 3$ and $m(3) = 7$. Östergård [12] showed that $m(4) \geq 23$ and established $m(4) = 23$ as a result. The exact values of $m(n)$ are not yet known for $n \geq 5$, even though it can be easily observed that $m(n+1) \geq m(n)$ for any $n \geq 1$.

1.1 Abbott-Hanson-Toft Construction

As mentioned above, Abbott-Hanson's construction [1] for odd n along with Toft's construction [15] for even n is referred to as Abbott-Hanson-Toft construction. For a given $n \geq 3$, this construction is built using a non-2-colorable $(n-2)$-uniform hypergraph, which we call the *core hypergraph* and denote by $H'_c = (V'_c, E'_c)$. Let its hyperedge set be $E'_c = \{e_1, e_2, \ldots, e_{m_c}\}$.

Let A and B be two disjoint sets of vertices where $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, each disjoint with V'_c. For a given $K \subset \{1, 2, \ldots, n\}$, we define $A_K = \bigcup_{i \in K} \{a_i\}$, $B_K = \bigcup_{i \in K} \{b_i\}$, $\overline{A}_K = A \setminus A_K$ and $\overline{B}_K = B \setminus B_K$.

The construction of the non-2-colorable n-uniform hypergraph $H = (V, E)$ is as follows. The vertex set is $V = V'_c \cup A \cup B$ and the hyperedge set E consists of the following hyperedges:

(i) $e_i \cup \{a_j\} \cup \{b_j\}$ for every pair i, j satisfying $1 \leq i \leq m_c$ and $1 \leq j \leq n$;

(ii) $A_K \cup \overline{B}_K$ for each K such that $|K|$ is odd and $1 \leq |K| \leq \lfloor n/2 \rfloor$;

(iii) $\overline{A}_K \cup B_K$ for each K such that $|K|$ is even and $2 \leq |K| \leq \lfloor n/2 \rfloor$;

(iv) A.

It is easy to observe that the number $m_{H'_c}(n)$ of hyperedges in H is $2^{n-1} + nm_c$ for odd n and $2^{n-1} + nm_c + \binom{n}{n/2}/2$ for even n.

1.2 Our Contributions

In this paper, we give constructions that improve the best known upper bounds on $m(8)$, $m(13)$, $m(14)$, $m(16)$ and $m(17)$. We also establish a non-trivial lower bound on $m(5)$.

In Section 2, we provide a construction that improves the upper bounds for $m(8)$, $m(14)$ and $m(17)$.

Theorem 1.1. *Consider an integer k satisfying $0 < k < n$. Let $w = \lfloor n/k \rfloor$, $x = n \mod k$, $y = \lfloor k/x \rfloor$ and $z = k \mod x$. (Note that y and z are defined if and only if $x > 0$.)*

(a) If $x > 0$ and $z > 0$, $m(n) \leq w \cdot m(n-k)m(k) + y \cdot m(k)^w m(x) + \binom{x+z-1}{z} m(n-k)m(x)^y + \binom{x+z-1}{x} m(k)^w$.

(b) If $x > 0$ and $z = 0$, $m(n) \leq w \cdot m(n-k)m(k) + y \cdot m(k)^w m(x) + m(n-k)m(x)^y$.

(c) If $x = 0$, $m(n) \leq w \cdot m(n-k)m(k) + m(k)^w$.

In Section 3, we give a construction to prove the following result that improves the upper bounds for $m(13)$ and $m(16)$ by substituting H_f and H_s, respectively, for F.

Theorem 1.2. *Consider an integer $k \geq 2$ and a non-2-colorable $(k-1)$-uniform hypergraph F. Then,* $m(3k+1) \leq (m(k-1)+2^{k-1})m(k+1)^2 + 2m_F(k+1)m(k)^2 + 4m(k+1)m(k)^2$.

In Section 4, we improve the currently best known lower bound $m(5) \geq 28$.

Theorem 1.3. $m(5) \geq 29$.

2 Multi-Core Construction

Consider an integer k satisfying $0 < k < n$. We define $w = \lfloor n/k \rfloor, x = n \mod k, y = \lfloor k/x \rfloor$ and $z = k \mod x$. As mentioned above, y and z are defined if and only if $x > 0$. A multi-core construction makes use of a non-2-colorable $(n-k)$-uniform hypergraph $H_c = (V_c, E_c)$, a total of w identical non-2-colorable k-uniform hypergraphs $H_1 = (V_1, E_1), \ldots, H_w = (V_w, E_w)$ and a total of y identical non-2-colorable x-uniform hypergraphs $H'_1 = (V'_1, E'_1), \ldots, H'_y = (V'_y, E'_y)$. The vertex sets of the hypergraphs H_c, $H_1, \ldots, H_w$, $H'_1, \ldots, H'_y$ are pairwise disjoint. Let us denote $E_c = \{e_1, e_2, \ldots, e_{m_c}\}$, $E_1 = \{e^1_1, e^1_2, \ldots, e^1_{m_k}\}$, …, $E_w = \{e^w_1, e^w_2, \ldots, e^w_{m_k}\}$, $E'_1 = \{e'^1_1, e'^1_2, \ldots, e'^1_{m_x}\}$, …, $E'_y = \{e'^y_1, e'^y_2, \ldots, e'^y_{m_x}\}$. Consider a vertex set $A = \{a_1, a_2, \ldots, a_{x+z-1}\}$, disjoint with each of V_c, $V_1, \ldots, V_w$, $V'_1, \ldots, V'_y$. We define $\mathcal{A}_p$ as the collection of all p-element subsets of the vertex set A. Let $\mathcal{E} = \{j_1 \cup j_2 \cup \cdots \cup j_w : (j_1, j_2, \ldots, j_w) \in E_1 \times E_2 \times \cdots \times E_w\}$ and $\mathcal{E}' = \{j'_1 \cup j'_2 \cup \cdots \cup j'_y : (j'_1, j'_2, \ldots, j'_y) \in E'_1 \times E'_2 \times \cdots \times E'_y\}$.

We define the construction of a non-2-colorable n-uniform hypergraph $H = (V, E)$ as follows. The vertex set is $V = V_c \cup A \cup V_1 \cup \cdots \cup V_w \cup V'_1 \cup \cdots \cup V'_y$. The construction of the hyperedges belonging to E depends on the values of x and z as follows.

Case 1. For $x > 0$ and $z > 0$, E contains the following hyperedges:

(i) $e_i \cup e^l_j$ for every triple i, j, l satisfying $1 \leq i \leq m_c$, $1 \leq j \leq m_k$ and $1 \leq l \leq w$;

(ii) $e'^j_i \cup e$ for every triple i, j, e satisfying $1 \leq i \leq m_x$, $1 \leq j \leq y$ and $e \in \mathcal{E}$;

(iii) $e_i \cup e' \cup S$ for every triple i, e, S satisfying $1 \leq i \leq m_c$, $e' \in \mathcal{E}'$ and $S \in \mathcal{A}_z$;

(iv) $e \cup S$ for every pair e, S satisfying $e \in \mathcal{E}$ and $S \in \mathcal{A}_x$.

Case 2. For $x > 0$ and $z = 0$, E contains the following hyperedges:

(i) $e_i \cup e^l_j$ for every triple i, j, l satisfying $1 \leq i \leq m_c$, $1 \leq j \leq m_k$ and $1 \leq l \leq w$;

(ii) $e'^j_i \cup e$ for every triple i, j, e satisfying $1 \leq i \leq m_x$, $1 \leq j \leq y$ and $e \in \mathcal{E}$;

(iii) $e_i \cup e'$ for every pair i, e' satisfying $1 \leq i \leq m_c$ and $e' \in \mathcal{E}'$.

Case 3. For $x = 0$, E contains the following hyperedges:

(i) $e_i \cup e_j^l$ for every triple i, j, l satisfying $1 \leq i \leq m_c$, $1 \leq j \leq m_k$ and $1 \leq l \leq w$;

(ii) e for each $e \in \mathcal{E}$.

The number of hyperedges in H is given by

$$|E| = \begin{cases} wm_cm_k + ym_x(m_k)^w + \binom{x+z-1}{z}m_c(m_x)^y + \binom{x+z-1}{x}(m_k)^w & \text{if } x > 0, z > 0; \\ wm_cm_k + ym_x(m_k)^w + m_c(m_x)^y & \text{if } x > 0, z = 0; \\ wm_cm_k + (m_k)^w & \text{if } x = 0. \end{cases}$$

Proof of Theorem 1.1. For the sake of contradiction, let us assume that χ is a proper 2-coloring of H. Without loss of generality, let the hypergraph H_c contain a red hyperedge in the coloring χ. The hyperedges formed in Step (i) in each of the cases ensure that each hypergraph H_j contains a blue hyperedge for each $j \in \{1, \ldots, w\}$.

Case 1. If $x > 0$ and $z > 0$, the hyperedges formed in Step (ii) ensure that each hypergraph H'_l contains a red hyperedge for each $l \in \{1, 2, \ldots, y\}$. It can be noted from the hyperedges generated in Step (iii) that there are at most $z - 1$ red vertices in the set A. This implies that A has at least x blue vertices. The hyperedges formed in Step (iv) ensure that there are at most $x-1$ blue vertices in A. Thus, we have a contradiction.

Case 2. If $x > 0$ and $z = 0$, the hyperedges formed in Step (ii) ensure that each hypergraph H'_l contains a red hyperedge for each $l \in \{1, 2, \ldots, y\}$. It follows that the hyperedges generated in Step (iii) include a red hyperedge. Thus, we have a contradiction.

Case 3. If $x = 0$, it immediately follows that we have a blue hyperedge among the hyperedges generated in Step (ii) of the construction. This leads to a contradiction.

Thus, we have the following result on $m(n)$.

If $x > 0$ and $z > 0$,

$$m(n) \leq w \cdot m(n-k)m(k) + y \cdot m(k)^w m(x) + \binom{x+z-1}{z}m(n-k)m(x)^y + \binom{x+z-1}{x}m(k)^w.$$

If $x > 0$ and $z = 0$,

$$m(n) \leq w \cdot m(n-k)m(k) + y \cdot m(k)^w m(x) + m(n-k)m(x)^y.$$

If $x = 0$,

$$m(n) \leq w \cdot m(n-k)m(k) + m(k)^w.$$

□

These recurrence relations give improvements on $m(n)$ for $n = 8, 14$ and 17 as follows.

- For $n = 8$ and $k = 5$, we have $m(8) \leq m(3)m(5) + m(5)m(3) + \binom{4}{2}m(3)m(3) + \binom{4}{3}m(5) \leq 1212$ by using $m(3) = 7$ and $m(5) \leq 51$ from Table 1.
- For $n = 14$ and $k = 5$, the recurrence relation gives $m(14) \leq 2m(9)m(5) + m(5)^2m(4) + \binom{4}{1}m(9)m(4) + \binom{4}{4}m(5)^2 \leq 528218$ by using $m(4) = 23$, $m(5) \leq 51$ and $m(9) \leq 2401$ from Table 1.
- For $n = 17$ and $k = 7$, we obtain $m(17) \leq 2m(10)m(7) + 2m(7)^2m(3) + \binom{3}{1}m(10)m(3)^2 + \binom{3}{3}m(7)^2 \leq 10375782$ by using $m(3) = 7$, $m(7) \leq 421$ and $m(10) \leq 7803$ from Table 1.

3 Block Construction

For an integer $k > 0$, we describe the construction of a collection $\mathcal{H}$ of non-2-colorable n-uniform hypergraphs. Any hypergraph $H = (V, E)$ belonging to this collection is constructed using a non-2-colorable $(n - 2k)$-uniform hypergraph denoted by $H_c = (V_c, E_c)$ and two disjoint collections of hypergraphs $\mathcal{A}$ and $\mathcal{B}$. Let $E_c = \{e_1, e_2, \ldots, e_{m_c}\}$. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_t\}$ and $\mathcal{B} = \{H'_1, H'_2, \ldots, H'_t\}$ be the collection of hypergraphs such that each of $H_i = (V_i, E_i)$ and $H'_i = (V'_i, E'_i)$ is an identical copy of a non-2-colorable k_i-uniform hypergraph satisfying $k_i \geq k$ and $\sum_{i=1}^{t} k_i \geq n$. Note that the sets $V_c, V_1, V_2, \ldots, V_t, V'_1, V'_2, \ldots V'_t$ are pairwise disjoint.

Let $P = \{i_1, i_2, \ldots, i_p\} \subseteq \{1, 2, \ldots, t\}$ such that $1 \leq i_1 < i_2 < \ldots < i_p \leq t$. Using the Cartesian products $\mathcal{C}_P = E_{i_1} \times E_{i_2} \times \cdots \times E_{i_p}$ and $\mathcal{C}'_P = E'_{i_1} \times E'_{i_2} \times \cdots \times E'_{i_p}$, let us define the collection of hyperedges $\mathcal{A}_P$ and $\mathcal{B}_P$ as $\mathcal{A}_P = \{j_1 \cup j_2 \cup \cdots \cup j_p : (j_1, j_2, \ldots, j_p) \in \mathcal{C}_P\}$ and $\mathcal{B}_P = \{j'_1 \cup j'_2 \cup \cdots \cup j'_p : (j'_1, j'_2, \ldots, j'_p) \in \mathcal{C}'_P\}$, respectively. Also, let $\overline{P} = \{1, 2, \ldots, t\} \setminus P$.

The hypergraph H has the vertex set $V = V_c \cup V_1 \cup \cdots \cup V_t \cup V'_1 \cup \cdots \cup V'_t$ and the hyperedge set E is generated from the following hyperedges, each containing at least n vertices.

(i) For each j satisfying $1 \leq j \leq t$, $e_i \cup e_{H_j} \cup e_{H'_j}$ for every triple $i, e_{H_j}, e_{H'_j}$ satisfying $1 \leq i \leq m_c$, $e_{H_j} \in E_j$ and $e_{H'_j} \in E'_j$

(ii) For each $P \subset \{1, 2, \ldots, t\}$ such that $|P|$ is odd and $1 \leq |P| \leq \lfloor t/2 \rfloor$, $e_H \cup e_{H'}$ for every pair $e_H, e_{H'}$ satisfying $e_H \in \mathcal{A}_P$ and $e_{H'} \in \mathcal{B}_{\overline{P}}$

(iii) For each $P \subset \{1, 2, \ldots, t\}$ such that $|P|$ is even and $0 \leq |P| \leq \lfloor t/2 \rfloor$, $e_H \cup e_{H'}$ for every pair $e_H, e_{H'}$ satisfying $e_H \in \mathcal{A}_{\overline{P}}$ and $e_{H'} \in \mathcal{B}_P$

We select an arbitrary set of n vertices from each of the hyperedges generated above to form the hyperedge set E. In case a hyperedge is included more than once in E by this process, we keep only one of those to ensure that E is not a multi-set. Let us count the number of hyperedges added to the hyperedge set E. Step (i) adds at most $|E_c| \sum_{i=1}^{t} |E_i||E'_i| = \sum_{i=1}^{t} |E_i|^2 |E_c|$ hyperedges, whereas Steps (ii) and (iii) together add at most $\prod_{i=1}^{t} |E_i| \big(1 + \binom{t}{1} + \ldots + \binom{t}{\lfloor t/2 \rfloor}\big)$ hyperedges. Note that $|E| \le \sum_{i=1}^{t}|E_i|^2|E_c| + 2^{t-1}\prod_{i=1}^{t}|E_i|$ when t is odd, and $|E| \le \sum_{i=1}^{t}|E_i|^2|E_c| + \left(2^{t-1} + \binom{t}{t/2}/2\right)\prod_{i=1}^{t}|E_i|$ when t is even. In the following lemma, we prove that H is non-2-colorable by showing that any proper 2-coloring of H can be used to obtain a proper 2-coloring of any t-uniform hypergraph constructed by Abbott–Hanson–Toft construction.

Lemma 3.1. *H is non-2-colorable.*

Proof. Consider any t-uniform hypergraph $H_{AHT} = (V_{AHT}, E_{AHT})$ constructed by Abbott–Hanson–Toft construction using a non-2-colorable $(t-2)$-uniform core hypergraph and two disjoint vertex sets $\{p_1, \ldots, p_t\}$ and $\{q_1, \ldots, q_t\}$. Assuming for the sake of contradiction that a proper 2-coloring exists for H, we give a proper 2-coloring for H_{AHT} as follows.

- Color all vertices of the non-2-colorable $(t-2)$-uniform core hypergraph of H_{AHT} with the color of the monochromatic hyperedge of H_c used in the construction of H.
- Color each vertex p_i with the color of the monochromatic hyperedge of H_i used in the construction of H.
- Similarly, color each vertex q_i with the color of the monochromatic hyperedge of H_i' used in the construction of H.

Since H_{AHT} is non-2-colorable, we have a contradiction. As a result, we have the following recurrence relation:

$$m(n) \le \begin{cases} m(n-2k)\sum_{i=1}^{t} m(k_i)^2 + 2^{t-1}\prod_{i=1}^{t} m(k_i) & \text{if } t \text{ is odd;} \\ m(n-2k)\sum_{i=1}^{t} m(k_i)^2 + \left(2^{t-1} + \binom{t}{t/2}/2\right)\prod_{i=1}^{t} m(k_i) & \text{if } t \text{ is even.} \end{cases} \qquad \square$$

Consider the special case when $n = 3k+1$. Setting the values of t and k_i's as $t = 3$, $k_1 = k+1$ and $k_2 = k_3 = k$ in this special case, we obtain the following recurrence relation:

$$m(3k+1) \le m(k+1)^3 + 6m(k)^2m(k+1). \tag{1}$$

We give an improvement of this result below.

Modified Block Construction

Let us first repeat the detailed description for the special case mentioned above, i.e., the construction of a non-2-colorable $(3k+1)$-uniform hypergraph $H = (V, E)$ belonging to $\mathcal{H}$. We construct H using a non-2-colorable $(k+1)$-uniform hypergraph $H_c = (V_c, E_c)$ along with non-2-colorable $(k+1)$-uniform hypergraphs $H_1 = (V_1, E_1)$, $H_1' = (V_1', E_1')$ and non-2-colorable k-uniform hypergraphs $H_2 = (V_2, E_2)$, $H_2' = (V_2', E_2')$, $H_3 = (V_3, E_3)$, $H_3' = (V_3', E_3')$. Note that each H_i' is an identical copy of H_i for $1 \le i \le 3$.

For the modified construction described below, we set H_1 as the Abbott–Hanson–Toft construction that uses a non-2-colorable $(k-1)$-uniform core hypergraph $H_{1c} = (V_{1c}, E_{1c})$ and disjoint vertex sets $A = \{a_1, a_2, \dots, a_{k+1}\}$, $B = \{b_1, b_2, \dots, b_{k+1}\}$. Note that H'_1 is not necessarily identical to H_1 in this modified block construction, whereas each H'_i is an identical copy of H_i for $2 \le i \le 3$.

Using the notations introduced above, the vertex set of the hypergraph H is $V = V_c \cup V_{1c} \cup A \cup B \cup V'_1 \cup V_2 \cup V'_2 \cup V_3 \cup V'_3$. The hyperedge set E is generated from the following hyperedges:

(a) $e_{H_c} \cup e_{H_1} \cup e_{H'_1}$ for every triple $e_{H_c}, e_{H_1}, e_{H'_1}$ satisfying $e_{H_c} \in E_c$, $e_{H_1} \in E_1$ and $e_{H'_1} \in E'_1$;

(b) $e_{H_c} \cup e_{H_2} \cup e_{H'_2}$ for every triple $e_{H_c}, e_{H_2}, e_{H'_2}$ satisfying $e_{H_c} \in E_c$, $e_{H_2} \in E_2$ and $e_{H'_2} \in E'_2$;

(c) $e_{H_c} \cup e_{H_3} \cup e_{H'_3}$ for every triple $e_{H_c}, e_{H_3}, e_{H'_3}$ satisfying $e_{H_c} \in E_c$, $e_{H_3} \in E_3$ and $e_{H'_3} \in E'_3$;

(d) $e_{H_1} \cup e_{H'}$ for every pair $e_{H_1}, e_{H'}$ satisfying $e_{H_1} \in E_1$ and $e_{H'} \in \{j'_2 \cup j'_3 : (j'_2, j'_3) \in E'_2 \times E'_3\}$;

(e) $e_{H_2} \cup e_{H'}$ for every pair $e_{H_2}, e_{H'}$ satisfying $e_{H_2} \in E_2$ and $e_{H'} \in \{j'_1 \cup j'_3 : (j'_1, j'_3) \in E'_1 \times E'_3\}$;

(f) $e_{H_3} \cup e_{H'}$ for every pair $e_{H_3}, e_{H'}$ satisfying $e_{H_3} \in E_3$ and $e_{H'} \in \{j'_1 \cup j'_2 : (j'_1, j'_2) \in E'_1 \times E'_2\}$;

(g) All elements of the set $\{j_1 \cup j_2 \cup j_3 : (j_1, j_2, j_3) \in E_1 \times E_2 \times E_3\}$.

Note that each of the hyperedges formed in Steps (b) to (g) has $3k+1$ vertices. However, the hyperedges formed in Step (a) have $3k+3$ vertices in each of them. We can remove any two vertices from each of these hyperedges to obtain the following recurrence relation; recall that $m_{H_{1c}}(k+1)$ denotes the number of hyperedges in the non-2-colorable $(k+1)$-uniform hypergraph constructed by Abbott–Hanson–Toft construction that uses H_{1c} as its core hypergraph:

$$m(3k+1) \le m_{H_{1c}}(k+1)m(k+1)^2 + 2m_{H_{1c}}(k+1)m(k)^2 + 4m(k+1)m(k)^2. \quad (2)$$

Whenever $m(k+1) < m_{H_{1c}}(k+1)$, it is evident that the upper bound on $m(3k+1)$ that this recurrence relation gives is worse than the one given by (1). However, we observe that we can improve (2) by carefully selecting the two vertices to be removed from each hyperedge formed in Step (a). Recall that each of these hyperedges is a union of three hyperedges $e_{H_c} \in E_c$, $e_{H_1} \in E_1$ and $e_{H'_1} \in E'_1$. In the following paragraph, we describe a process to create a set of $k-1$ vertices from each hyperedge in the $(k+1)$-uniform hypergraph $H_1 = (V_1, E_1)$. For each hyperedge $e_{H_c} \cup e_{H_1} \cup e_{H'_1}$ formed in Step (a), we use this process to remove two vertices from e_{H_1}.

Given a hyperedge $h \in E_1$, we create a set h' containing $k-1$ vertices as follows.

Case 1. If h is created by Step (i) of Abbott–Hanson–Toft construction, i.e., if h is of the form $e \cup \{a_i\} \cup \{b_i\}$ for some $e \in E_{1c}$, $a_i \in A$ and $b_i \in B$, we define $h' = e$. In other words, we remove a_i and b_i from h to create h'.

Case 2. If h is created in Step (ii) of Abbott–Hanson–Toft construction, i.e., if h is of the form $A_K \cup \overline{B}_K$ for some $K \subset \{1, \ldots, k+1\}$ such that $|K|$ is odd and $1 \leq |K| \leq \lfloor (k+1)/2 \rfloor$, we define $h' = A_K \cup \overline{B}_K \setminus \{a_k, a_{k+1}, b_k, b_{k+1}\}$.

Case 3. If h is created in Step (iii) of Abbott–Hanson–Toft construction, i.e., if h is of the form $\overline{A}_K \cup B_K$ for some $K \subset \{1, \ldots, k+1\}$ such that $|K|$ is even and $2 \leq |K| \leq \lfloor (k+1)/2 \rfloor$, we define $h' = \overline{A}_K \cup B_K \setminus \{a_k, a_{k+1}, b_k, b_{k+1}\}$.

Case 4. If h is formed in Step (iv) of Abbott–Hanson–Toft construction, i.e., if $h = A$, we define $h' = A \setminus \{a_k, a_{k+1}\}$.

This completes the construction of the $(3k+1)$-uniform hypergraph H.

Proof of Theorem 1.2. We improve the recurrence relation given in (2) as a result of selecting $k-1$ vertices from each $h \in E_1$, as described above. Since this process generates multiple copies of some $(k-1)$-element vertex sets, the number of distinct hyperedges formed in Step (a) in the construction of H is reduced. Let us determine the cardinality of the set $\{h' : h' \text{ is generated from some } h \in E_1\}$.

It is easy to observe that the number of distinct h''s formed in Case 1 is $|E_{1c}|$. On the other hand, the total number of distinct h''s formed in Cases 2, 3 and 4 is at most 2^{k-1}. It follows from the fact that there are 2^{k-1} subsets of $A \setminus \{a_k, a_{k+1}\}$ and each h' formed in one of the Cases 2, 3 and 4 is a union of the sets $\bigcup_{i \in P}\{a_i\}$ and $\bigcup_{i \in \{1,\ldots,k-1\} \setminus P}\{b_i\}$ for some $P \subseteq \{1, \ldots, k-1\}$.

Since we have shown in Lemma 3.1 that H is non-2-colorable, we have the following improvement over (2):

$$m(3k+1) \leq (m(k-1) + 2^{k-1})m(k+1)^2 + 2m_{H_{1c}}(k+1)m(k)^2 + 4m(k+1)m(k)^2.$$

The theorem follows by setting F to be the hypergraph H_{1c}. □

This result improves the upper bounds on $m(n)$ for $n = 13$ and 16 as follows.

- For $n = 13$, we have $k = 4$. Note that $m_F(5) = m_{H_{1c}}(5) = 51$, when the Fano plane [10] H_f having 7 hyperedges is used as the core hypergraph H_{1c}. Therefore, we obtain $m(13) \leq (m(3) + 2^3)m(5)^2 + 2m_{H_f}(5)m(4)^2 + 4m(5)m(4)^2 \leq 200889$ by using $m(3) = 7$, $m(4) = 23$ and $m(5) \leq 51$ from Table 1.

- For $n = 16$, we have $k = 5$. Note that $m_F(6) = m_{H_{1c}}(6) = 180$, when the non-2-colorable 4-uniform hypergraph H_s with 23 hyperedges is used as the core hypergraph H_{1c}. Therefore, we obtain $m(16) \leq (m(4) + 2^4)m(6)^2 + 2m_{H_s}(6)m(5)^2 + 4m(6)m(5)^2 \leq 3308499$ by using $m(4) = 23$, $m(5) \leq 51$ and $m(6) \leq 147$ from Table 1.

4 Improved Lower Bound for $m(5)$

We begin this section with Lemma 4.1 and Lemma 4.2 that are used to prove Lemma 4.3, which gives the best known lower bound on $m(5)$. Let $m_l(n)$ be the minimum number of hyperedges in a non-2-colorable n-uniform hypergraph with l vertices.

Lemma 4.1. *[7]* $m_{2n-1}(n) = m_{2n}(n) = \binom{2n-1}{n}$.

Lemma 4.2. *[5] (Schönheim bound) Consider positive integers $l \geq n \geq t \geq 1$ and $\lambda \geq 1$. Any n-uniform hypergraph with l vertices such that every t-subset of its vertices is contained in at least λ hyperedges has at least* $\left\lceil \frac{l}{n} \left\lceil \frac{l-1}{n-1} \cdots \left\lceil \frac{\lambda(l-t+1)}{n-t+1} \right\rceil \cdots \right\rceil \right\rceil$ *hyperedges.*

Lemma 4.3. *[9] If $n \geq 4$, then*

$$m(n) \geq \min_{x>2n, x\in\mathbb{N}} \left\{ \max \left\{ \left\lceil \frac{\binom{x}{\lfloor x/2 \rfloor}}{\binom{x-n}{\lfloor x/2\rfloor - n} + \binom{x-n}{\lceil x/2 \rceil - n}} \right\rceil, \left\lceil \frac{x}{n} \left\lceil \frac{x-1}{n-1} \right\rceil \right\rceil \right\} \right\}.$$

Lemma 4.3 implies that $m(5) \geq 28$, which is obtained when $x = 23$. We improve this to $m(5) \geq 29$ using the following lemma.

Lemma 4.4. *[13] Consider a positive integer γ and a real number $p \in [0, 1]$. Any n-uniform hypergraph $H = (V, E)$ satisfying $|\{\{e_1, e_2\} : e_1, e_2 \in E,\ |e_1 \cap e_2| = 1\}| \leq \gamma$ is properly 2-colorable if* $2^{-n+1}(1-p)^n|E| + 4\gamma\left(2^{-2n+1}p\int_0^1 (1-(xp)^2)^{n-1}\mathrm{d}x\right) < 1$.

Proof of Theorem 1.3. Let us consider a 5-uniform hypergraph $H = (V, E)$ with at most 28 hyperedges. We show that it is properly 2-colorable.

Case 1. If $5 \leq |V| \leq 8$, any *balanced* 2-coloring (a coloring in which the number of blue vertices and the number of red vertices differ by at most 1) of its vertex set is a proper 2-coloring of H.

Case 2. If $|V| = 9$ or $|V| = 10$, it follows from Lemma 4.1 that $m_9(5) = m_{10}(5) = 126$. Since $|E| \leq 28$, H has a proper 2-coloring.

Case 3. If $11 \leq |V| \leq 22$, consider a balanced 2-coloring of H. We observe that a red hyperedge blocks $\binom{|V|-5}{\lfloor |V|/2 \rfloor - 5}$ and a blue hyperedge blocks $\binom{|V|-5}{\lceil |V|/2 \rceil - 5}$ such colorings. In order to ensure that none of these balanced 2-colorings is a proper 2-coloring of H, we need at least $\left\lceil \frac{\binom{|V|}{\lfloor |V|/2 \rfloor}}{\binom{|V|-5}{\lfloor |V|/2\rfloor - 5} + \binom{|V|-5}{\lceil |V|/2 \rceil - 5}} \right\rceil$ hyperedges. Since $11 \leq |V| \leq 22$, it implies that we need at least 29 hyperedges to ensure that no balanced 2-coloring of H is a proper 2-coloring.

Case 4. If $|V| = 23$ and there exists a pair of vertices $\{v_i, v_j\}$ not contained together in any hyperedge of H, we construct a new hypergraph $H' = (V', E')$ by merging vertices v_i and v_j into a new vertex v. We observe that H' is 5-uniform with 22 vertices and $|E|$ hyperedges. It follows from Case 3 that H' is properly 2-colorable. This coloring of H' can be extended to a proper 2-coloring of H by assigning the color of v to v_i and v_j. If $|E| \leq 27$, note that Lemma 4.2 ensures that there exists a pair of vertices not contained together in any hyperedge of H. Therefore, we would complete the proof by assuming that $|E| = 28$ and every pair of vertices is contained in at least one hyperedge of H. For such a hypergraph, we show that the cardinality of the set $\{\{e_1, e_2\} : e_1, e_2 \in E,\ |e_1 \cap e_2| = 1\}$ is at most 335. Setting $p = 0.3, \gamma = 335, n = 5$ and $|E| = 28$ in Lemma 4.4, we observe that H is properly 2-colorable since $2^{-n+1}(1-p)^n|E| + 4\gamma \cdot 2^{-2n+1} p \int_0^1 (1-(xp)^2)^{n-1} \mathrm{d}x < 1$.

In order to show that the cardinality of the set $\{\{e_1, e_2\} : e_1, e_2 \in E,\ |e_1 \cap e_2| = 1\}$ is at most 335, we consider the degree sequence of H. Note that the *degree* of a vertex is defined as the number of hyperedges it is contained in and the *degree sequence* of a hypergraph is the ordering of the degrees of its vertices in a non-increasing order. Consider an arbitrary vertex u of H. Observe that there are 22 distinct vertex pairs involving u and any hyperedge containing u has 4 such pairs in it. Therefore, the degree of u is at least 6 and there exists another vertex u' such that $\{u, u'\}$ is contained in at least two different hyperedges of H. Since the sum of the degrees of the vertices of H is 140, the only possible degree sequences of H are $\langle 8, 6, \ldots, 6\rangle$ and $\langle 7, 7, 6, \ldots, 6\rangle$. For the first sequence, the cardinality of the set $\{\{e_1, e_2\} : e_1, e_2 \in E,\ |e_1 \cap e_2| = 1\}$ is upper bounded by $(\binom{6}{2} - 1)\cdot 22 + (\binom{8}{2} - 1) = 335$. For the second sequence, it is upper bounded by $(\binom{6}{2} - 1) \cdot 21 + (\binom{7}{2} - 1) \cdot 2 = 334$.

Case 5. If $|V| \geq 24$, assume the induction hypothesis that any 5-uniform hypergraph with $|V| - 1$ vertices and $|E|$ hyperedges is properly 2-colorable. The base case $|V| = 23$ is proved in Case 4. If there exists a pair of vertices $\{v_i, v_j\}$ not contained together in any hyperedge of H, consider a new hypergraph $H' = (V', E')$ constructed by merging v_i and v_j into a new vertex v. Since H' is 5-uniform with $|V'| = |V| - 1$ and $|E'| = |E|$, we know from the induction hypothesis that H' is properly 2-colorable. This coloring of H' can be extended to a proper 2-coloring of H by assigning the color of v to v_i and v_j. Since it follows from Lemma 4.2 that the minimum number of hyperedges required to ensure that each pair of vertices is contained in at least one hyperedge is $\left\lceil \frac{|V|}{5} \left\lceil \frac{|V|-1}{4} \right\rceil \right\rceil \geq 29$, we are guaranteed to have a pair of vertices $\{v_i, v_j\}$ not contained together in any hyperedge of H. □

5 Conclusion

In this paper, we have established the lower bound $m(5) \geq 29$ which is still far from the best known upper bound $m(5) \leq 51$. We have also established improved upper bounds for $m(8)$, $m(13)$, $m(14)$, $m(16)$ and $m(17)$. In Table 2, we have highlighted these improved bounds on $m(n)$ for $n \leq 17$. It would be interesting to determine the exact values of $m(n)$ for $n \geq 5$.

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$m(n)$	Corresponding construction/recurrence relation
$m(1) = 1$	Single vertex
$m(2) = 3$	Triangle graph
$m(3) = 7$	Fano plane [10]
$m(4) = 23$	[12], [14]
$m(5) \leq 51$	$m(5) \leq 2^4 + 5m(3)$
$m(6) \leq 147$	$m(6) \leq m(2)m(3)^2$
$m(7) \leq 421$	$m(7) \leq 2^6 + 7m(5)$
$m(8) \leq \mathbf{1212}$	$m(8) \leq 2m(3)m(5) + \binom{4}{2}m(3)m(3) + \binom{4}{3}m(5)$
$m(9) \leq 2401$	$m(9) \leq m(3)^4$
$m(10) \leq 7803$	$m(10) \leq m(2)m(5)^2$
$m(11) \leq 25449$	$m(11) \leq 15 \cdot 2^8 + 9m(9)$
$m(12) \leq 55223$	$m(12) \leq m(3)^4 m(4)$
$m(13) \leq \mathbf{200889}$	$m(13) \leq (m(3) + 2^3)m(5)^2 + 2m_{H_f}(5)m(4)^2 + 4m(5)m(4)^2$
$m(14) \leq \mathbf{528218}$	$m(14) \leq 2m(9)m(5) + m(5)^2 m(4) + \binom{4}{1}m(9)m(4) + \binom{4}{4}m(5)^2$
$m(15) \leq 857157$	$m(15) \leq m(3)^5 m(5)$
$m(16) \leq \mathbf{3308499}$	$m(16) \leq (m(4) + 2^4)m(6)^2 + 2m_{H_s}(6)m(5)^2 + 4m(6)m(5)^2$
$m(17) \leq \mathbf{10375782}$	$m(17) \leq 2m(10)m(7) + 2m(7)^2 m(3) + \binom{3}{1}m(10)m(3)^2 + \binom{3}{3}m(7)^2$

Table 2: Improved upper bounds on $m(n)$ for small values of n

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