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On the double Roman domination number in trees

S. Nazari-Moghaddam

Department of Mathematics, Dehloran Branch
University of Applied Science and Technology, Dehloran
Iran
sakine.nazari.m@gmail.com

M. Chellali

LAMDA-RO Laboratory, Department of Mathematics
University of Blida, B.P. 270, Blida
Algeria
m_chellali@yahoo.com

Abstract

For a graph G, let $\gamma_{dR}(G)$ and $\gamma_R(G)$ denote the double Roman domination number and the Roman domination number, respectively. In this paper, we show that for every tree T of order $n \geq 3$, with $\ell(T)$ leaves and $s(T)$ support vertices,

$$\begin{aligned}\gamma_R(T) + \lceil \frac{\ell(T) - s(T)}{\Delta(T)} \rceil &\leq \gamma_{dR}(T) \\ &\leq \min\{\lfloor \frac{11n - \ell(T) + 4s(T)}{10} \rfloor, 2\gamma_R(T) - \lceil \frac{s(T)}{\Delta(T)} \rceil\}.\end{aligned}$$

The upper and lower bounds improve previous bounds given by Beeler, Haynes and Hedetniemi [*Discrete Appl. Math.* 211 (2016), 23–29].

1 Introduction

Throughout this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V, E). The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg(v) = |N(v)|$. The *maximum degree* of a graph G is denoted by $\Delta = \Delta(G)$. A *leaf* of G is a vertex with degree one, a *support vertex* is a vertex adjacent to a leaf, and a *strong support vertex* is a support vertex adjacent to at least two leaves. The set of all leaves adjacent to a vertex v is denoted by $L(v)$,

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while the set of leaves of a graph G is denoted by $L(G)$. A *path* of order n is denoted by P_n. The corona $cor(H)$ of a graph H is the graph obtained from H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added. A *double star* $DS_{p,q}$, with $q \geq p \geq 1$, is a graph consisting of the union of two stars $K_{1,q}$ and $K_{1,p}$ together with an edge joining their centers. The *subdivision graph* $S_b(G)$ of a graph G is that graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. A *healthy spider* is the subdivision graph of a star $K_{1,k}$ for $k \geq 2$. The *distance* $d_G(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest uv-path in G. The *diameter* $\text{diam}(G)$ of a graph G is the greatest distance between two vertices of G. A *diametral path* of a graph G is a shortest path whose length is equal to $\text{diam}(G)$. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v, $D(v)$ denotes the set of descendants of v and $D[v] = D(v) \cup \{v\}$. Also, the *depth* of v, $\text{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *grandchild* of v is the descendant of v at distance 2 from v.

A *Roman dominating function* on G, abbreviated RDF, is a function $f : V \to \{0,1,2\}$ such that every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF f is the value $f(V) = \sum_{u \in V} f(u)$, and the *Roman domination number* $\gamma_R(G)$ of G is the minimum weight of an RDF on G. Roman domination was introduced by Cockayne et al. in [6] and was inspired by the work of ReVelle and Rosing [8] and Stewart [9]. Several new varieties of Roman domination have been introduced since 2004, among them, we quote the double Roman domination introduced by Beeler, Haynes and Hedetniemi in [5] and studied for example in [1, 2, 3, 4, 7, 10].

A *double Roman dominating function* (DRDF) on a graph G is a function $f : V \to \{0,1,2,3\}$ having the property that if $f(v) = 0$, then vertex v has at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then vertex v has at least one neighbor w with $f(w) \geq 2$. The *weight* of a DRDF f is the value $f(V) = \sum_{u \in V} f(u)$. The *double Roman domination number* $\gamma_{dR}(G)$ of a graph G is the minimum weight of a DRDF on G. A DRDF of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$-function. For a DRDF f, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0,1,2,3$. Since these four sets determine f, we can equivalently write $f = (V_0, V_1, V_2, V_3)$ (or $f = (V_0^f, V_1^f, V_2^f, V_3^f)$ to refer f). We note that $\omega(f) = |V_1| + 2|V_2| + 3|V_3|$.

In this paper, we show that for every tree T of order $n \geq 3$, with $\ell(T)$ leaves and $s(T)$ support vertices, $\gamma_R(T) + \lceil \frac{\ell(T)-s(T)}{\Delta(T)} \rceil \leq \gamma_{dR}(T) \leq \min\{\lfloor \frac{11n-\ell(T)+4s(T)}{10} \rfloor, 2\gamma_R(T) - \lceil \frac{s(T)}{\Delta(T)} \rceil\}$. All these bounds improve previous bounds given in [5].

We make use of the following results.

Proposition 1.1 ([5]). *In a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the value 1.*

Observation 1.2. *If v is a strong support of a graph G, then there exists a $\gamma_{dR}(G)$-function f with $f(v) = 3$.*

Proof. Let $f = (V_0, \emptyset, V_2, V_3)$ be a $\gamma_{dR}(G)$-function such that $f(v)$ is as large as possible. If $f(v) = 2$, then $f(x) = 1$ for all $x \in L(v)$ which contradicts the choice of f. If $f(v) = 0$, then $f(x) = 2$ for all $x \in L(v)$ and thus the function g defined on $V(G)$ by $g(v) = 3$, $f(x) = 0$ for all $x \in L(v)$ and $g(x) = f(x)$ elsewhere, is a DRDF of G of weight less than $\gamma_{dR}(G)$ which is a contradiction. Hence, $f(v) = 3$, as desired. □

2 Upper bounds

Our main results in this section are two new upper bounds on the double Roman domination number of a tree. It was shown in [5] that every tree T of order $n \geq 3$ satisfied $\gamma_{dR}(T) \leq \frac{5n}{4}$. The first bound we present improves this upper bound for trees T with $n > \frac{8s(T)-2\ell(T)}{3}$.

Let L_t consist of the disjoint union of t copies of P_4 plus a path through a support vertices of these copies, as illustrated in Figure 1.

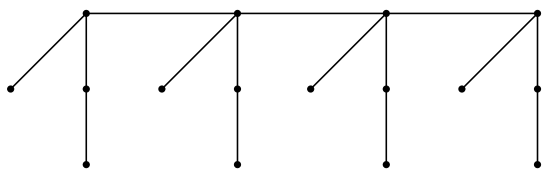

Figure 1: The tree L_4.

Let H_k consist of the disjoint union of k copies of P_5 plus a path through a support vertices of these copies, as illustrated in Figure 2.

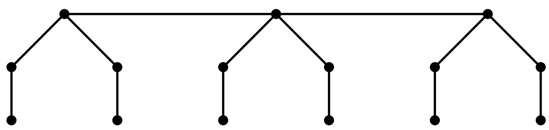

Figure 2: The tree H_3.

Theorem 2.1. *If T is a tree of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, then*

$$\gamma_{dR}(T) \leq \lfloor \frac{11n - \ell(T) + 4s(T)}{10} \rfloor.$$

This bound is sharp for trees L_t with $t \geq 1$ and H_k with $k \in \{1, \ldots, 9\}$.

Proof. The proof is by induction on n. The statement holds for all trees of order $n \in \{3, 4\}$. Suppose $n \geq 5$ and let the result hold for all trees T of order less than n. Let T be a tree of order n. If $\text{diam}(T) = 2$, then T is a star and we have $\gamma_{dR}(T) = 3 < \lfloor \frac{11n-\ell(T)+4s(T)}{10} \rfloor$. If $\text{diam}(T) = 3$, then T is a double star with at least three leaves (because of $n \geq 5$), and thus assigning a 3 to each support vertex and a 0 to the leaves is a DRDF of T of weight 6. Clearly, $\gamma_{dR}(T) \leq 6 \leq$

$\lfloor\frac{11n-\ell(T)+4s(T)}{10}\rfloor$. Hence, we may assume that $\mathrm{diam}(T) \geq 4$. Suppose there are two adjacent vertices x and y, each of degree at least three. Let T' and T'' be the subtrees of T containing x and y, respectively, obtained from the deletion of the edge xy. Clearly, each of T' and T'' has order at least three. By the induction hypothesis, $\gamma_{dR}(T') \leq \lfloor\frac{11n'-\ell(T')+4s(T')}{10}\rfloor$ and $\gamma_{dR}(T'') \leq \lfloor\frac{11n''-\ell(T'')+4s(T'')}{10}\rfloor$. Moreover, since $\gamma_{dR}(T) \leq \gamma_{dR}(T')+\gamma_{dR}(T'')$, $\ell(T) = \ell(T')+\ell(T'')$ and $s(T) = s(T')+s(T'')$, we deduce that $\gamma_{dR}(T) \leq \lfloor\frac{11n-\ell(T)+4s(T)}{10}\rfloor$. Hence we can assume that T no two vertices of degree at least three are adjacent.

Let $v_1v_2\dots v_k$ $(k \geq 5)$ be a diametral path in T such that $\deg(v_2)$ is as large as possible. Root T at v_k. If $\deg(v_2) \geq 4$, then let $T' = T - v_1$ and f' be a $\gamma_{dR}(T')$-function. By Observation 1.2, $f'(v_2) = 3$ and so the function f' can be extended to a DRDF of T by assigning a 0 to v_1. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T')\\ &\leq \lfloor\frac{11(n-1)-\ell(T)+1+4s(T)}{10}\rfloor\\ &< \lfloor\frac{11n-\ell(T)+4s(T)}{10}\rfloor.\end{aligned}$$

Therefore, we will assume that $\deg(v_2) \in \{2,3\}$. We consider the following cases.

Case 1. $\deg_T(v_2) = 3$.
By assumption, $\deg_T(v_3) = 2$. Let $T' = T - T_{v_2}$ and f' be a $\gamma_{dR}(T')$-function. The function f defined on $V(T)$ by $f(v_2) = 3$, $f(x) = 0$ for all $x \in L(v_2)$ and $f(x) = f'(x)$ for all $x \in V(T) - V(T_{v_2})$ is a DRDF of T of weight $\omega(f') + 3$. It follows from the induction hypothesis and the fact $\ell(T') = \ell(T) - 1$ and $s(T') \leq s(T)$ that

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 3\\ &\leq \lfloor\frac{11(n-3)-\ell(T)+1+4s(T)}{10}\rfloor + 3\\ &\leq \lfloor\frac{11n-\ell(T)+4s(T)}{10}\rfloor.\end{aligned}$$

Case 2. $\deg_T(v_2) = 2$.
By the choice of diametral path, we may assume that all children of v_3 with depth 1, have degree 2. Assume first that $\deg_T(v_3) = 2$. Let $T' = T - T_{v_3}$ and f' is a $\gamma_{dR}(T')$-function. If T' has order 2, then $T = P_5$, and clearly the result holds. Hence we assume that $|V(T')| \geq 3$. Then the function f defined on $V(T)$ by $f(v_2) = 3$, $f(v_1) = f(v_3) = 0$ and $f(x) = f'(x)$ for all $x \in V(T')$ is a DRDF of T of weight $\omega(f') + 3$. Using the induction hypothesis and the fact $\ell(T') \geq \ell(T) - 1$ and $s(T') \leq s(T)$, the result follows. Now, let $\deg_T(v_3) = p \geq 3$. We distinguish the following.

Subcase 2.1. v_3 is a support vertex.
By assumption, $\deg_T(v_4) = 2$. Assume first that $T_{v_3} = DS_{1,p-2}$. Let $T' = T - T_{v_4}$ and let f' be a $\gamma_{dR}(T')$-function. If $|V(T')| = 1$, then $\gamma_{dR}(T) = 7 \leq \lfloor\frac{11n-\ell(T)+4s(T)}{10}\rfloor$. If $|V(T')| = 2$, then one can see that $\gamma_{dR}(T) \leq 8 \leq \lfloor\frac{11n-\ell(T)+4s(T)}{10}\rfloor$. Thus let

$|V(T')| \geq 3$. Then the function f defined on $V(T)$ by $f(v_3) = 3$, $f(v_1) = 2$, $f(x) = 0$ for $x \in N(v_3)$ and $f(x) = f'(x)$ for $x \in V(T') - \{v_4\}$ is a DRDF of T of weight $\omega(f') + 5$. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 5 \\ &\leq \lfloor \frac{11(n-p-2) - \ell(T) - p + 1 + 4s(T) - 4}{10} \rfloor + 5 \\ &< \lfloor \frac{11n - \ell(T) + 4s(T)}{10} \rfloor.\end{aligned}$$

Suppose now that $T_{v_3} \neq DS_{1,p-2}$, that is v_3 has at least two children of depth 1. Let $T' = T - \{v_1, v_2\}$ and f' be a $\gamma_{dR}(T')$-function. Assume that $u_2 \neq v_2$ is a child of v_3 with depth 1 and u_1 is the leaf neighbor of u_2. If $f'(v_3) = 0$, then $f'(x) = 2$ for every $x \in L(v_3)$ and $f'(u_1) + f'(u_2) = 3$. Then the function g defined on $V(T')$ by $g(v_3) = 3$, $g(u_1) = 2$, $g(x) = 0$ for $x \in L(v_3) \cup \{u_2\}$ and $g(x) = f'(x)$ elsewhere is a DRDF of T' with $g(v_3) = 3$. Hence, we may assume that $f'(v_3) \geq 2$ and thus the function f defined on $V(T)$ by $f(v_1) = 2, f(v_2) = 0$ and $f(x) = f'(x)$ for all $x \in V(T')$ is a DRDF of T of weight $\omega(f') + 2$. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 2 \\ &\leq \lfloor \frac{11(n-2) - \ell(T) + 1 + 4s(T) - 4}{10} \rfloor + 2 \\ &< \lfloor \frac{11n - \ell(T) + 4s(T)}{10} \rfloor.\end{aligned}$$

Subcase 2.2. v_3 is not a support vertex.
Hence T_{v_3} is a healthy spider centered at v_3. First let v_3 has at least three children. Suppose that $T' = T - \{v_1, v_2\}$ and f' be a $\gamma_{dR}(T')$-function. We may assume that $f'(v_3) \geq 2$. Then the function f defined on $V(T)$ by $f(v_1) = 2, f(v_2) = 0$ and $f(x) = f'(x)$ for $x \in V(T')$ is a DRDF of T of weight $\omega(f') + 2$. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 2 \\ &\leq \lfloor \frac{11(n-2) - \ell(T) + 1 + 4s(T) - 4}{10} \rfloor + 2 \\ &\leq \lfloor \frac{11n - \ell(T) + 4s(T)}{10} \rfloor.\end{aligned}$$

Now, let v_3 have exactly two children. By assumption, $\deg_T(v_4) = 2$. If $\deg_T(v_5) \geq 3$, then let $T' = T - T_{v_4}$ and f' be a $\gamma_{dR}(T')$-function. Then the function f defined on $V(T)$ by $f(v_3) = 3$, $f(x) = 2$ for $x \in L(T_{v_3})$, $f(x) = 0$ for $x \in N(v_3)$ and $f(x) = f'(x)$ for $x \in V(T')$ is a DRDF of T of weight $\omega(f') + 7$. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_{dR}(T) &\le \gamma_{dR}(T') + 7\\ &\le \lfloor\frac{11(n-6) - \ell(T) + 2 + 4s(T) - 8}{10}\rfloor + 7\\ &\le \lfloor\frac{11n - \ell(T) + 4s(T)}{10}\rfloor.\end{aligned}$$

Hence we assume that $\deg_T(v_5) = 2$. If $\deg_T(x) \le 2$ for $x \in V(T) - \{v_3\}$, then let $T' = T - T_{v_5}$. If $|V(T')| \le 2$, then T is a tree obtained from a path P_5 attached by its center to a leaf of a path $P_{2+|V(T')|}$. In this case, one can easily see that $\gamma_{dR}(T) \le \lfloor\frac{11n-\ell(T)+4s(T)}{10}\rfloor$. Hence we assume that $|V(T')| \ge 3$. Note that, $\ell(T) = 3$ and then, since $\gamma_{dR}(T_{v_3}) = 8$, we have $\gamma_{dR}(T) \le \gamma_{dR}(T') + 8$. It follows from the induction hypothesis and the fact $\ell(T') = \ell(T) - 1$ and $s(T') = s(T) - 1$ that

$$\begin{aligned}\gamma_{dR}(T) &\le \gamma_{dR}(T') + 8\\ &\le \lfloor\frac{11(n-7) - \ell(T) + 1 + 4s(T) - 4}{10}\rfloor + 8\\ &= \lfloor\frac{11n - \ell(T) + 4s(T)}{10}\rfloor.\end{aligned}$$

Now let $t \ge 5$ be the smallest integer such that $\deg(v_t) = 2$ and $\deg(v_{t+1}) \ge 3$. Suppose that $T' = T - T_{v_t}$. Clearly, $\gamma_{dR}(T) \le \gamma_{dR}(T') + t + 3$, $\ell(T') = \ell(T) - 2$ and $s(T') = s(T) - 2$. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_{dR}(T) &\le \gamma_{dR}(T') + t + 3\\ &\le \lfloor\frac{11(n-t-2) - \ell(T) + 2 + 4s(T) - 8}{10}\rfloor + t + 3\\ &\le \lfloor\frac{11n - \ell(T) + 4s(T)}{10}\rfloor.\end{aligned}$$

This completes the proof. □

Beeler et al. in [5] proved that for every graph G, $\gamma_{dR}(G) < 2\gamma_R(G)$. In the next theorem, we improve this bound for trees.

Theorem 2.2. *If T is a tree of order $n \ge 3$ with $s(T)$ support vertices, then*

$$\gamma_{dR}(T) \le 2\gamma_R(T) - \lceil\frac{s(T)}{\Delta(T)}\rceil.$$

This bound is sharp for $cor(P_{3k})$ with $k \ge 1$.

Proof. The proof is by induction on n. The statement holds for all trees of order $n \in \{3, 4\}$. Let $n \ge 5$ and assume that the result holds for all tree T' of order n' such that $3 \le n' < n$. Let T be a tree of order $n \ge 5$. If $\mathrm{diam}(T) = 2$, then T

is a star, where $\gamma_{dR}(T) = 3 = 4 - \lceil\frac{1}{n-1}\rceil$. If $\mathrm{diam}(T) = 3$, then $T = DS_{p,q}$ with $q \geq p \geq 1$. If $p = 1$, then $\gamma_{dR}(T) = 5 \leq 6 - \lceil\frac{2}{n-2}\rceil$. Suppose that $p \geq 2$. Then $\gamma_{dR}(T) = 6 < 8 - \lceil\frac{2}{\Delta(T)}\rceil$. Henceforth we may assume that $\mathrm{diam}(T) \geq 4$.

Let $v_1v_2\ldots v_k$ be a diametral path in T such that $\deg(v_2)$ is as large as possible. Root T at v_k. If T has a support vertex v with $L(v) \geq 3$, then let T' obtained from T by removing a leaf v' belonging to $L(v)$. Clearly, $\Delta(T) \geq \Delta(T')$, $s(T') = s(T)$. Moreover, it is easy to see that $\gamma_R(T') \leq \gamma_R(T)$ and $\gamma_{dR}(T) \leq \gamma_{dR}(T')$. By the induction hypothesis on T' we obtain that

$$\gamma_{dR}(T) \leq \gamma_{dR}(T') \leq 2\gamma_R(T') - \lceil\frac{s(T')}{\Delta(T')}\rceil \leq 2\gamma_R(T) - \lceil\frac{s(T)}{\Delta(T)}\rceil.$$

Hence, every support vertex of T has at most two leaves, in particular $\deg_T(v_2) \in \{2, 3\}$. Now, assume that $\deg_T(v_3) = 2$, and let $T' = T - T_{v_3}$. If $|V(T')| = 2$, then $\gamma_{dR}(T) = 6 < 8 - \lceil\frac{2}{\Delta(T)}\rceil$. Hence we assume that $|V(T')| \geq 3$. Clearly, $\Delta(T) \geq \Delta(T')$, $s(T) - 1 \leq s(T')$. Moreover, it is easy to see that $\gamma_R(T') \leq \gamma_R(T) - 2$ and $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 3$. By the induction hypothesis on T' we obtain

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 3\\ &\leq 2\gamma_R(T') - \lceil\frac{s(T')}{\Delta(T')}\rceil + 3\\ &\leq 2\gamma_R(T) - 4 - \lceil\frac{s(T)-1}{\Delta(T)}\rceil + 3\\ &\leq 2\gamma_R(T) - \lceil\frac{s(T)}{\Delta(T)}\rceil.\end{aligned}$$

Hence, let $\deg_T(v_3) \geq 3$, and consider the following two cases.

Case 1. $\deg_T(v_2) = 3$.
If v_3 is a strong support vertex or v_3 has a child of depth 1 different from v_2, then let $T' = T - T_{v_2}$. Clearly, $\Delta(T) \geq \Delta(T')$, $s(T') = s(T) - 1$. Moreover, it is easy to see that $\gamma_R(T') \leq \gamma_R(T) - 2$ and $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 3$. Using the induction on T', the result follows. Suppose that v_3 is a support vertex and $\deg_T(v_3) = 3$. Let $T' = T - T_{v_3}$. If $|V(T')| = 2$, then $\gamma_{dR}(T) = 8$, $\gamma_R(T) = 5$, $\Delta(T) = s(T) = 3$ and thus the result is valid. Hence we assume that $|V(T')| \geq 3$.Clearly, $\Delta(T) \geq \Delta(T')$, $s(T) - 2 \leq s(T')$. Moreover, it is easy to see that $\gamma_R(T') \leq \gamma_R(T) - 3$ and $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 5$. By the induction hypothesis on T' we obtain that

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 5\\ &\leq 2\gamma_R(T') - \lceil\frac{s(T')}{\Delta(T')}\rceil + 5\\ &\leq 2\gamma_R(T) - 6 - \lceil\frac{s(T)-2}{\Delta(T)}\rceil + 5\\ &\leq 2\gamma_R(T) - \lceil\frac{s(T)}{\Delta(T)}\rceil.\end{aligned}$$

Case 2. $\deg_T(v_2) = 2$.
By the choice of diametral path, we may assume that all children of v_3 with depth 1 have degree 2. In the sequel, let s_1 be the number of children of v_4 that are leaves and let $s_{\geq 2}$ be the number of children of v_4 of degree at least 2 having no grandchild. We distinguish the following subcases.

Subcase 2.1. v_3 is not a support vertex.
Let $T' = T - T_{v_3}$. If $|V(T')| = 2$, then T is a healthy spider, where $\gamma_{dR}(T) = 2\deg_T(v_3) + 2$, $\gamma_R(T) = 2 + \deg_T(v_3)$, $\Delta(T) = s(T) = \deg_T(v_3)$ and thus the result is valid. Hence let $|V(T')| \geq 3$. Clearly, $\Delta(T) \geq \Delta(T')$, $s(T) - \deg_T(v_3) + 1 \leq s(T')$. If $s_1 \geq 2$ or $s_{\geq 2} \geq 1$ or v_4 has a child of depth 2 different from v_2, then it is easy to see that $\gamma_R(T') \leq \gamma_R(T) - |C(v_3)| - 2$ and $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 2|C(v_3)| + 2$. By the induction hypothesis on T' we obtain

$$\begin{aligned}
\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 2|C(v_3)| + 2 \\
&\leq 2\gamma_R(T') - \lceil \frac{s(T')}{\Delta(T')} \rceil + 2|C(v_3)| + 2 \\
&\leq 2\gamma_R(T) - 2|C(v_3)| - 4 - \lceil \frac{s(T) - \deg_T(v_3) + 1}{\Delta(T)} \rceil + 2|C(v_3)| + 2 \\
&\leq 2\gamma_R(T) - \lceil \frac{s(T)}{\Delta(T)} \rceil.
\end{aligned}$$

Hence, let $s_1 \leq 1, s_{\geq 2} = 0$ and say v_4 has no child of depth 2 different from v_2. If $s_1 = 1$, then let $T' = T - T_{v_4}$. If $|V(T')| = 1$, then $\gamma_{dR}(T) = 2\deg_T(v_3) + 3$, $\gamma_R(T) = \deg_T(v_3) + 3$, $\Delta(T) = s(T) = \deg_T(v_3)$, and if $|V(T')| = 2$, then $\gamma_{dR}(T) = 2\deg_T(v_3) + 5$, $\gamma_R(T) = \deg_T(v_3) + 4$, $\Delta(T) = \deg_T(v_3)$, $s(T) = \deg_T(v_3) + 1$. In both cases, we have $\gamma_{dR}(T) < \gamma_R(T) - \lceil \frac{s(T)}{\Delta(T)} \rceil$. Hence let $|V(T')| \geq 3$. Clearly, $\Delta(T) \geq \Delta(T')$, $s(T) - \deg_T(v_3) \leq s(T')$. Moreover, one can see that $\gamma_R(T') \leq \gamma_R(T) - |C(v_3)| - 3$ and $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 2|C(v_3)| + 4$. By the induction hypothesis on T' we obtain that

$$\begin{aligned}
\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 2|C(v_3)| + 4 \\
&\leq 2\gamma_R(T') - \lceil \frac{s(T')}{\Delta(T')} \rceil + 2|C(v_3)| + 4 \\
&\leq 2\gamma_R(T) - 2|C(v_3)| - 6 - \lceil \frac{s(T) - \deg_T(v_3)}{\Delta(T)} \rceil + 2|C(v_3)| + 4 \\
&\leq 2\gamma_R(T) - \lceil \frac{s(T)}{\Delta(T)} \rceil.
\end{aligned}$$

Finally, assume that $s_1 = 0$, and let $T' = T - T_{v_4}$. If $|V(T')| = 2$, then $\gamma_{dR}(T) = 2\deg_T(v_3) + 3$, $\gamma_R(T) = \deg_T(v_3) + 3$, $\Delta(T) = s(T) = \deg_T(v_3)$, and the result holds. Hence let $|V(T')| \geq 3$. Clearly, $\Delta(T) \geq \Delta(T')$, $s(T) - \deg_T(v_3) + 1 \leq s(T')$. Moreover, $\gamma_R(T') \leq \gamma_R(T) - |C(v_3)| - 2$ and $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 2|C(v_3)| + 3$. By the induction hypothesis on T' we obtain

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 2|C(v_3)| + 3\\ &\leq 2\gamma_R(T') - \lceil\frac{s(T')}{\Delta(T')}\rceil + 2|C(v_3)| + 3\\ &\leq 2\gamma_R(T) - 2|C(v_3)| - 4 - \lceil\frac{s(T) - \deg_T(v_3) + 1}{\Delta(T)}\rceil + 2|C(v_3)| + 3\\ &\leq 2\gamma_R(T) - \lceil\frac{s(T)}{\Delta(T)}\rceil.\end{aligned}$$

Subcase 2.2. v_3 is a support vertex.
Let $T' = T - T_{v_3}$, and $t = |L(v_3)|$. Recall that we see that every support vertex has at most two leaves, and thus $t \in \{1,2\}$. If $|V(T')| = 2$, then $\gamma_{dR}(T) = 3+2(\deg_T(v_3)-t)$, $\gamma_R(T) = 2 + (\deg_T(v_3) - t)$, $\Delta(T) = \deg_T(v_3)$, $s(T) = \deg_T(v_3) - t$ and the result holds. Hence let $|V(T')| \geq 3$. Clearly, $\Delta(T) \geq \Delta(T')$, $s(T)-\deg_T(v_3)+t+1 \leq s(T')$. If $s_1 \geq 2$ or $s_{\geq 2} \geq 1$ or v_4 has a child of depth 2 different from v_2, it is easy to see that $\gamma_R(T') \leq \gamma_R(T) - |C(v_3)| - 2 + t$ and $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 2|C(v_3)| + 3 - 2t$. By the induction hypothesis on T' we obtain

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 2|C(v_3)| + 3 - 2t\\ &\leq 2\gamma_R(T') - \lceil\frac{s(T')}{\Delta(T')}\rceil + 2|C(v_3)| + 3 - 2t\\ &\leq 2\gamma_R(T) - 2|C(v_3)| - 4 + 2t - \lceil\frac{s(T)-\deg_T(v_3)+t+1}{\Delta(T)}\rceil + 2|C(v_3)| + 3 - 2t\\ &\leq 2\gamma_R(T) - \lceil\frac{s(T)}{\Delta(T)}\rceil.\end{aligned}$$

Hence, let $s_1 \leq 1, s_{\geq 2} = 0$ and v_4 has no a child of depth 2 different from v_2, Assume that $s_1 = 1$, and let $T' = T - T_{v_4}$. If $|V(T')| = 1$, then $\gamma_{dR}(T) = 2\deg_T(v_3) + 4 - 2t$, $\gamma_R(T) = \deg_T(v_3)+3-t$, $\Delta(T) = \deg_T(v_3)$, $s(T) = \deg_T(v_3)-t+1$ and if $|V(T')| = 2$, then $\gamma_{dR}(T) = 2\deg_T(v_3) + 6 - 2t$, $\gamma_R(T) = \deg_T(v_3) + 4 - t$, $\Delta(T) = \deg_T(v_3)$, $s(T) = \deg_T(v_3) + 2 - t$. In either case, we have $\gamma_{dR}(T) \leq \gamma_R(T) - \lceil\frac{s(T)}{\Delta(T)}\rceil$. Hence let $|V(T')| \geq 3$. Clearly, $\Delta(T) \geq \Delta(T')$, $s(T) - \deg_T(v_3) - 1 + t \leq s(T')$. Moreover, $\gamma_R(T') \leq \gamma_R(T) - |C(v_3)| - 3 + t$ and $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 2|C(v_3)| + 4 - 2t$. By the induction hypothesis on T' we obtain

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 2|C(v_3)| + 4 - 2t\\ &\leq 2\gamma_R(T') - \lceil\frac{s(T')}{\Delta(T')}\rceil + 2|C(v_3)| + 4 - 2t\\ &\leq 2\gamma_R(T) - 2|C(v_3)| - 6 + 2t - \lceil\frac{s(T)-\deg_T(v_3)-1+t}{\Delta(T)}\rceil + 2|C(v_3)| + 4 - 2t\\ &\leq 2\gamma_R(T) - \lceil\frac{s(T)}{\Delta(T)}\rceil.\end{aligned}$$

Finally, assume that $s_1 = 0$ and let $T' = T - T_{v_4}$. If $|V(T')| = 2$, then $\gamma_{dR}(T) = 4+2(\deg_T(v_3)-t)$, $\gamma_R(T) = 3+(\deg_T(v_3)-t)$, $\Delta(T) = \deg_T(v_3)$, $s(T) = \deg_T(v_3) - t+1$ and the result holds. Hence let $|V(T')| \geq 3$. Clearly, $\Delta(T) \geq \Delta(T')$, $s(T) - \deg_T(v_3) + t \leq s(T')$. Moreover, $\gamma_R(T') \leq \gamma_R(T) - |C(v_3)| - 2 + t$ and $\gamma_{dR}(T) \leq \gamma_{dR}(T') + 2|C(v_3)| + 3 - 2t$. By the induction hypothesis on T' we obtain

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T') + 2|C(v_3)| + 3 - 2t\\ &\leq 2\gamma_R(T') - \lceil\frac{s(T')}{\Delta(T')}\rceil + 2|C(v_3)| + 3 - 2t\\ &\leq 2\gamma_R(T) - 2|C(v_3)| - 4 + 2t - \lceil\frac{s(T) - \deg_T(v_3) + t}{\Delta(T)}\rceil + 2|C(v_3)| + 3 - 2t\\ &\leq 2\gamma_R(T) - \lceil\frac{s(T)}{\Delta(T)}\rceil.\end{aligned}$$

Note that if $T = cor(P_{3k})$ for with $k \geq 1$, then we have $\Delta(T) = 3$, $s(T) = 3k$, $\gamma_{dR}(T) = 7k$ and $\gamma_R(T) = 4k$. This completes the proof. □

3 Lower bound

Beeler et al. in [5] proved that for every graph G, $\gamma_{dR}(G) > \gamma_R(G)$. In the next theorem, we improve this bound for trees.

Theorem 3.1. *If T is a tree of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, then*

$$\gamma_{dR}(T) \geq \gamma_R(T) + \lceil\frac{\ell(T) - s(T)}{\Delta(T)}\rceil.$$

This bound is sharp for double stars $DS_{p,q}$ with $q \geq p \geq 4$.

Proof. The proof is by induction on n. The statement holds for all trees of order $n \in \{3, 4\}$. Let $n \geq 5$ and assume that the result holds for all tree T' of order n' such that $3 \leq n' < n$. Let T be a tree of order $n \geq 5$. If $\mathrm{diam}(T) = 2$, then T is a star, where $\gamma_{dR}(T) = 3 = 2 + \lceil\frac{n-2}{n-1}\rceil$. If $\mathrm{diam}(T) = 3$, then $T = DS_{p,q}$, with $q \geq p \geq 1$. If $p = 1$, then $\gamma_{dR}(T) = 5 > 3 + \lceil\frac{n-4}{n-2}\rceil$. If $p \geq 2$, then $\gamma_{dR}(T) = 6 \geq 4 + \lceil\frac{n-4}{\Delta(T)}\rceil$, and clearly the result is valid since $\lceil\frac{n-4}{\Delta(T)}\rceil \leq 2$. Henceforth we may assume that $\mathrm{diam}(T) \geq 4$.

Let $v_1v_2 \ldots v_k$ be a diametral path in T. Root T at v_k. Let $\deg_T(v_3) = 2$ and $T' = T - T_{v_3}$. Clearly, $|V(T')| \geq 2$. If $|V(T')| = 2$, then $\gamma_{dR}(T) = 5, \gamma_R(T) = 4, \Delta(T) = \ell(T) = \deg_T(v_2)$, $s(T) = 2$, and thus the result is valid. Hence we assume that $|V(T')| \geq 3$. Then $\Delta(T) \geq \Delta(T')$, $\ell(T) - |L(v_2)| \leq \ell(T'), s(T') \leq s(T)$. Moreover, it is easy to see that $\gamma_R(T) \leq \gamma_R(T') + 2$ and $\gamma_{dR}(T') \leq \gamma_{dR}(T) - 3$. By the induction hypothesis on T' we obtain

$$\begin{aligned}\gamma_{dR}(T) &\geq \gamma_{dR}(T') + 3\\ &\geq \gamma_R(T') + \lceil\frac{\ell(T') - s(T')}{\Delta(T')}\rceil + 3\\ &\geq \gamma_R(T) - 2 + \lceil\frac{\ell(T) - |L(v_2)| - s(T)}{\Delta(T)}\rceil + 3\\ &\geq \gamma_R(T) + \lceil\frac{\ell(T) - s(T)}{\Delta(T)}\rceil.\end{aligned}$$

Assume now that $\deg_T(v_3) \geq 3$. First, let v_3 be a support vertex and either has two children of depth 1 or v_3 is a strong support vertex. Let $T' = T - T_{v_2}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(v_2)|$, $s(T') = s(T) - 1$. If $\deg_T(v_2) \geq 3$, then $\gamma_R(T) \leq \gamma_R(T') + 2$ and $\gamma_{dR}(T') \leq \gamma_{dR}(T) - 3$. By the induction hypothesis on T' we obtain that

$$\begin{aligned}\gamma_{dR}(T) &\geq \gamma_{dR}(T') + 3\\ &\geq \gamma_R(T') + \lceil\frac{\ell(T') - s(T')}{\Delta(T')}\rceil + 3\\ &\geq \gamma_R(T) - 2 + \lceil\frac{\ell(T) - |L(v_2)| - s(T') + 1}{\Delta(T)}\rceil + 3\\ &\geq \gamma_R(T) + \lceil\frac{\ell(T) - s(T)}{\Delta(T)}\rceil.\end{aligned}$$

If $\deg_T(v_2) = 2$, then $\gamma_R(T) \leq \gamma_R(T')+2$ and $\gamma_{dR}(T') \leq \gamma_{dR}(T)-2$. By the induction hypothesis on T' we obtain

$$\begin{aligned}\gamma_{dR}(T) &\geq \gamma_{dR}(T') + 2\\ &\geq \gamma_R(T') + \lceil\frac{\ell(T') - s(T')}{\Delta(T')}\rceil + 2\\ &\geq \gamma_R(T) - 2 + \lceil\frac{\ell(T) - 1 - s(T') + 1}{\Delta(T)}\rceil + 2\\ &= \gamma_R(T) + \lceil\frac{\ell(T) - s(T)}{\Delta(T)}\rceil.\end{aligned}$$

We can now suppose that v_2 is the unique child of v_3 with depth 1 and $|L(v_3)| = 1$. If $\deg_T(v_2) \geq 3$, then let $T' = T - T_{v_3}$. Clearly, $|V(T')| \geq 2$. Assume that $|V(T')| = 2$. Then $\gamma_{dR}(T) = 8$, $\gamma_R(T) = 5$, $\Delta(T) = \deg_T(v_2)$, $\ell(T) = \deg_T(v_2) + 1$, $s(T) = 3$, and thus the result is valid. Hence we assume that $|V(T')| \geq 3$. Then $\Delta(T) \geq \Delta(T')$, $\ell(T) - |L(v_2)| - 1 \leq \ell(T')$ and $s(T') \leq s(T) - 1$. Moreover, it is easy to see that $\gamma_R(T) \leq \gamma_R(T') + 3$ and $\gamma_{dR}(T') \leq \gamma_{dR}(T) - 4$. By the induction hypothesis on T'

we obtain

$$\begin{aligned}\gamma_{dR}(T) &\geq \gamma_{dR}(T') + 4\\ &\geq \gamma_R(T') + \lceil \frac{\ell(T') - s(T')}{\Delta(T')} \rceil + 4\\ &\geq \gamma_R(T) - 3 + \lceil \frac{\ell(T) - |L(v_2)| - 1 - s(T) + 1}{\Delta(T)} \rceil + 4\\ &\geq \gamma_R(T) + \lceil \frac{\ell(T) - s(T)}{\Delta(T)} \rceil.\end{aligned}$$

Suppose that $\deg_T(v_2) = 2$. If $\deg_T(v_4) \geq 3$, then let $T'' = T - T_{v_3}$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - 2$ and $s(T'') = s(T) - 2$. Moreover, it is easy to see that $\gamma_R(T) \leq \gamma_R(T'') + 3$ and $\gamma_{dR}(T'') \leq \gamma_{dR}(T) - 3$. By the induction hypothesis on T'' we obtain

$$\begin{aligned}\gamma_{dR}(T) &\geq \gamma_{dR}(T'') + 3\\ &\geq \gamma_R(T'') + \lceil \frac{\ell(T'') - s(T'')}{\Delta(T'')} \rceil + 3\\ &\geq \gamma_R(T) - 3 + \lceil \frac{\ell(T) - 2 - s(T'') + 2}{\Delta(T)} \rceil + 3\\ &\geq \gamma_R(T) + \lceil \frac{\ell(T) - s(T)}{\Delta(T)} \rceil.\end{aligned}$$

If $\deg_T(v_4) = 2$, then let $T'' = T - T_{v_4}$. If $|V(T'')| \leq 2$, we can see that $\gamma_{dR}(T) \geq \gamma_R(T) + \lceil \frac{\ell(T)-s(T)}{\Delta(T)} \rceil$. Hence we assume that $|V(T'')| \geq 3$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T) - 2 \leq \ell(T'')$ and $s(T'') \leq s(T) - 1$. Moreover, it is easy to see that $\gamma_R(T) \leq \gamma_R(T'') + 3$ and $\gamma_{dR}(T'') \leq \gamma_{dR}(T) - 5$. By the induction hypothesis on T'' we obtain

$$\begin{aligned}\gamma_{dR}(T) &\geq \gamma_{dR}(T'') + 5\\ &\geq \gamma_R(T'') + \lceil \frac{\ell(T'') - s(T'')}{\Delta(T'')} \rceil + 5\\ &\geq \gamma_R(T) - 3 + \lceil \frac{\ell(T) - 2 - s(T) + 1}{\Delta(T)} \rceil + 5\\ &> \gamma_R(T) + \lceil \frac{\ell(T) - s(T)}{\Delta(T)} \rceil.\end{aligned}$$

Finally, assume that v_3 is not a support vertex, and let $T' = T - T_{v_2}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(v_2)|$ and $s(T') = s(T) - 1$. On the other hand, if $\deg_T(v_2) \geq 3$, then $\gamma_R(T) \leq \gamma_R(T') + 2$ and $\gamma_{dR}(T') \leq \gamma_{dR}(T) - 3$, and if $\deg_T(v_2) = 2$, then $\gamma_R(T) \leq \gamma_R(T') + 2$ and $\gamma_{dR}(T') \leq \gamma_{dR}(T) - 2$. Using the induction on T' and according to each situation, the result follows. This completes the proof. □

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