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Lower bounds for rainbow Turán numbers of paths and other trees

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Abstract

For a fixed graph F, we would like to determine the maximum number of edges in a properly edge-colored graph on n vertices which does not contain a rainbow copy of F, that is, a copy of F all of whose edges receive a different color. This maximum, denoted by $\mathrm{ex}^*(n, F)$, is the *rainbow Turán number* of F. We show that $\mathrm{ex}^*(n, P_k) \geq \frac{k}{2}n + O(1)$ where P_k is a path on $k \geq 3$ edges, generalizing a result by Maamoun and Meyniel and by Johnston, Palmer and Sarkar. We show similar bounds for brooms on $2^s - 1$ edges and diameter at most 10 and a few other caterpillars of small diameter.

1 Introduction

Keevash, Mubayi, Sudakov, and Verstraëte introduced rainbow Turán numbers in [11], motivated by a direct application in additive number theory [14], as well as a desire to study a natural meeting point of Turán and Ramsey type problems, along the lines of [1]. The latter paper describes the problem of finding a rainbow copy of a graph F in a colouring of K_n in which each colour appears at most m times at every vertex. According to [11], the rainbow Turán problem is a natural Turán-type extension. For a fixed graph F, the Turán number of F, denoted $\mathrm{ex}(n, F)$, is the maximum number of edges in a graph on n vertices that contains no copy of F. The

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rainbow Turán number of F, denoted $\mathrm{ex}^*(n, F)$, is the maximum number of edges in a properly edge-colored graph on n vertices that contains no rainbow copy of F, that is, a copy of F whose edges all receive a different color. In [11], the authors showed that, when F is not bipartite,

$$\mathrm{ex}^*(n, F) = (1 + o(1))\,\mathrm{ex}(n, F).$$

Many open questions remain for bipartite graphs. In [11], the authors showed that, when F is bipartite,

$$\mathrm{ex}^*(n, K_{s,t}) = O(n^{2-\frac{1}{s}}),$$

where $K_{s,t}$ is the complete bipartite graph with partition classes of size s and t such that $s \le t$. For even cycles, the authors prove a lower bound of

$$\mathrm{ex}^*(n, C_{2k}) = \Omega(n^{1+\frac{1}{k}})$$

and find a matching upper bound in the case of $k = 3$. Das, Lee and Sudakov [6] showed that for every fixed integer $k \ge 2$,

$$\mathrm{ex}^*(n, C_{2k}) = O\left(n^{1+\frac{(1+\epsilon_k)\ln k}{k}}\right),$$

where $\epsilon_k \to 0$ as $k \to \infty$.

In [10], Johnston, Palmer and Sarkar showed that when F is a forest of k stars, $\mathrm{ex}^*(n, F)$ is the maximum value of $(k-1)n + O(1)$ or $\frac{1}{2}(|e(F)| - 1)n + O(1)$. They also showed that $\mathrm{ex}^*(n, P_k) = \frac{k}{2}n + O(1)$ for $k \in \{3, 4\}$. Here, we generalize this result to all values $k \ge 3$. In [10], the authors also showed an upper bound of $\mathrm{ex}^*(n, P_k) \le \lceil \frac{3k-2}{2} n \rceil$. This was improved to

$$\mathrm{ex}^*(n, P_k) < \left(\frac{9k-5}{7}\right) n$$

by Ergemlidze, Győri and Methuku [7], and this is currently the best known upper bound.

In [3], Alon and Shikhelman introduced the following generalized Turán problem: for fixed graphs H and F, what is the maximum number of copies of H, denoted by $\mathrm{ex}(n, H, F)$, that can appear in an n-vertex F-free graph? The special case $\mathrm{ex}(n, C_3, C_5)$ was studied earlier in [5]. This problem has applications in query complexity of testing graph properties [8]. This problem extends naturally to a rainbow Turán version, which is suggested in [9].

The rest of this paper is organized as follows. In Section 2, we give a few basic definitions, notation, and facts that will be used throughout the paper. In particular, we describe the two constructions that are the basis for the new lower bounds on $\mathrm{ex}^*(n, F)$ for several bipartite graphs F. In Section 3, we give new lower bounds on $\mathrm{ex}^*(n, P_k)$. Section 4, we give new lower bounds, and upper bounds, on $\mathrm{ex}^*(n, F)$ for some broom graphs, other caterpillars and a few other small trees. Finally, in Section 6, we list a few of the many open questions that remain.

2 Definitions, notation and basic results

Let $G = (V, E)$ be a *graph* on vertex set V and edge set $E \subseteq \binom{V}{2}$. For a vertex $v \in V(G)$ let $\Gamma_G(v) = \{w \in V(G) | \{v, w\} \in E(G)\}$ be the neighborhood of v and $d(v) = |\Gamma(v)|$ the degree of v. We let $d(G) = \frac{1}{n} \sum_V d(v)$ be the average degree of G. We will use the following fact about average vertex degrees.

Proposition 2.1. *If $d(v) < \frac{d(G)}{2}$ for some $v \in V(G)$, then $d(G - v) > d(G)$.* □

An edge-colored graph $G^* = (V, E, c)$ is a graph with an *edge coloring* $c : E \to \mathbb{N}$. We will only consider *proper* edge colorings, *i.e.* colorings such that $c(e) \neq c(f)$ if $e \cap f \neq \emptyset$. An edge coloring is *rainbow* if the function c is injective. Many of the lower-bound proofs in the remainder of this paper are based on two extremal edge-colored graphs: $K^*_{2^s}$ and $D^*_{2^s}$. The edge-colored graph $K^*_{2^s}$ is the complete graph on 2^s vertices, identified with the vectors in $\mathbb{F}_2^s$. The edge-coloring $c : E(K_{2^s}) \to \mathbb{F}_2^s$ is given by $c(vw) = v - w$. The graph $D^*_{2^s}$ is a spanning edge-colored subgraph of $K^*_{2^s}$. An edge vw with color $c(vw)$ is in $D^*_{2^s}$ if and only if $d_H(v, w) \in \{1, s\}$, where $d_H(v, w)$ is the Hamming distance between binary vectors v and w. Note that $K^*_{2^s}$ is $(2^s - 1)$-regular and $D^*_{2^s}$ is $(s+1)$-regular. The latter can be thought of as hypercubes with added "diagonals". We show examples of $K^*_{2^2} \sim D^*_{2^2}$ and $D^*_{2^3}$ in Figure 1. We

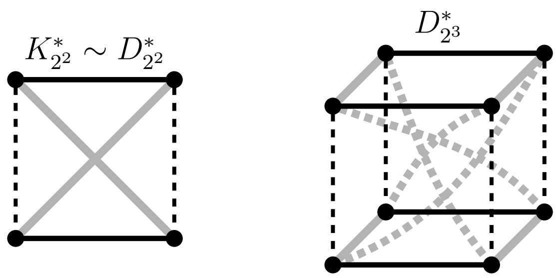

Figure 1: Examples of edge colored-graphs $K^*_{2^2} \sim D^*_{2^2}$ and $D^*_{2^3}$.

let P_k be the *path* on k edges (and $k+1$ vertices), and C_k the *cycle* on k edges (and k vertices). The *girth* $g(G)$ of a graph is the minimum k such that C_k is a subgraph of G. We define the *broom* $B_{k,l}$ as a tree on k edges that consists of a union of a P_{l-1} and a $K_{1,k-l}$, with an edge between an endpoint of the path and the centre of the star. We let $CP_{(s_1,s_2,\ldots,s_t)}$ be a *caterpillar* that consists of a *central path* P_{t-1} with s_i leaves added to the ith vertex on the central path. A broom is a special case of a caterpillar. We show examples of a broom and a caterpillar in Figure 2.

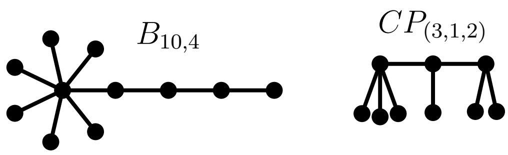

Figure 2: Example of a broom $B_{10,4}$ and a caterpillar $CP_{(3,1,2)}$.

We will use the following fact about rainbow paths in an edge-colored graph.

Proposition 2.2. *If $v \in V(G)$ is the endpoint of a rainbow path P of length k in a properly edge-colored graph G, and P cannot be extended at v to a longer rainbow path, then $d(v) \leq 2k-1$.*

Proof. This is true because if an edge that is incident to v cannot be added to P to create a longer rainbow path, then this edge either has a color that already appears on the path (including the edge on P incident to v), or the other endpoint of the edge is already on the path, creating a cycle. There can be at most $2k-1$ such edges. □

In this paper we are predominantly interested in the behavior of $\mathrm{ex}^*(n, F)$ as $n \to \infty$. A graph G is *balanced* if $d(H) \leq d(G)$ for all subgraphs H of G. The following proposition implies that we need only consider balanced graphs as lower-bound constructions to (rainbow) Turán numbers.

Proposition 2.3. *Suppose that G is an edge colored graph with no rainbow copy of some graph F, and that*

$$\mathrm{ex}^*(n, F) = \frac{d(G)}{2} n + O(1).$$

Then, G is balanced.

Proof. Suppose that G has a subgraph H such that $d(H) > d(G)$. Then, we can construct rainbow F-free graphs on n vertices, for n large enough, with average degree $d(H) + O(1)$ by taking disjoint copies of H (and a few isolated vertices). This implies $\mathrm{ex}^*(n, F) \geq \frac{d(H)}{2} n + O(1)$; a contradiction. □

3 Lower bound for P_k

In [12], Maamoun and Meyniel showed that $\mathrm{ex}^*(n, P_k) \geq \frac{k}{2} n + O(1)$, when $k+1 = 2^s$ for some $s \in \mathbb{N}$. We show that this is true for any $k \geq 3$. In [11], Keevash, Mubayi, Sudakov and Verstraëte conjectured that the extremal example for avoiding rainbow P_ks is a disjoint union of cliques of size $c(k)$, where $c(k)$ is chosen as large as possible so that $K_{c(k)}$ can be properly edge-colored with no rainbow P_k. This conjecture was proven false in [10], by providing a non-complete 4-regular edge-colored graph that does not have a P_4 and showing that any proper edge-coloring of K_5 yields a rainbow copy of P_4. This construction is $D^*_{2^3}$ as defined in the previous section. Hence, we generalize the construction to give a properly edge-colored k-regular graph that does not have a P_k for any $k \geq 2$. This construction is not the complete graph when $k > 3$.

Theorem 3.1. *Let P_k be the path of length k, then*

$$\mathrm{ex}^*(n, P_k) \geq \frac{k}{2}n + O(1).$$

Proof. Consider the edge-colored graph $D^*_{2^s}$. Suppose that P is a rainbow path of length $k = s+1$ in $D^*_{2^s}$ with endpoints v and w. Then,

$$v - w = \sum_{e \in E(P)} c(e).$$

However, if P is rainbow, then $c(e(P)) = \{c_1, \ldots, c_{s+1}\}$. This implies that $v - w = 0$, which contradicts P being a path.

The graph $D^*_{2^s}$ is $s+1$-regular, and therefore has $\frac{1}{2}n(s+1) = \frac{1}{2}kn$ edges. When n is a multiple of 2^{k-1}, we can therefore create a rainbow P_k-free k-regular graph by taking disjoint copies of $D^*_{2^s}$. □

We make an observation here about the edge-colored graph $D^*_{2^s}$ that will be useful in later sections. Let $\{c_1, c_2, \ldots, c_s\}$ be the standard basis of $\mathbb{F}_2^s$ and let c_{s+1} be the vector of all 1s of length s. It is easy to see that $D^*_{2^s}$ does not contain a rainbow cycle of length $< s+1$, by noting that there is no $S \subset \{c_1, \ldots, c_{s+1}\}$ such that $\sum_S c = 0$. Thus for any graph F with girth $g(F) < k$, we can obtain a properly edge-colored graph containing no rainbow copy of F having $\frac{k}{2}n + O(1)$ edges. Note that this construction does not improve the lower bound of $\mathrm{ex}^*(n, F)$ obtained from known bounds for $\mathrm{ex}^*(n, C_k)$.

In [15], it is shown that, for $k \leq 10$, each properly k-edge-colored k-regular graph contains a rainbow path of length $k-1$. Theorem 3.1 implies that this result is tight. If it is true that $\mathrm{ex}^*(n, P_k) > \frac{k}{2}n$ for $k \leq 10$, then there is no construction similar to $D^*_{2^s}$ that produces extremal graphs: those would be irregular or not $\Delta(G)$-edge-colored.

4 Caterpillars and other trees

We will start this section by focusing on broom graphs, since they are a natural tree to consider between stars and paths.

Lemma 4.1. *We have*

$$\mathrm{ex}^*(n, B_{k,2}) = \begin{cases} \frac{k}{2}n + O(1), & \text{for } k \text{ odd,} \\ \frac{k^2}{2(k+1)}n + O(1), & \text{for } k \text{ even.} \end{cases}$$

Proof. If k is odd, then we claim that no K_{k+1} with a k-edge-coloring contains a rainbow $B_{k,2}$. Suppose that we have a K_{k+1} with a k-edge-coloring that contains a rainbow $B_{k,2}$. Let v_0 be the vertex of degree $k-1$ in $B_{k,2}$, with edges of colors $1, \ldots, k-1$ incident to v_0 in $B_{k,2}$, and let w be the vertex such that $v_0w \notin E(B_{k,2})$. Then w has an edge of color k to a vertex other than v_0 in $B_{k,2}$. This is a contradiction, since we must have that edge v_0w has color k in K_{k+1}. Therefore,

$$\mathrm{ex}^*(n, B_{k,2}) \geq \frac{k}{2}n + O(1)$$

when k is odd. Let G be a graph with a proper edge coloring, and no rainbow copy of $B_{k,2}$. Suppose that G has a vertex v_0 with $d(v_0) \geq k$. If any neighbor of v_0 has an edge to a non-neighbor of v_0, this gives rise to a copy of $B_{k,2}$. If $d(v_0) > k$, there cannot be any edges in $G[\Gamma(v_0)]$, for the same reason. Therefore,

$$\mathrm{ex}^*(n, B_{k,2}) \leq \frac{k}{2}n.$$

This implies that, when k is odd,

$$\mathrm{ex}^*(n, B_{k,2}) = \frac{k}{2}n + O(1).$$

If k is even, suppose that G has no rainbow copy of $B_{k,2}$ and that G has vertex v_0 with $d(v_0) = k$, and edges of colors $1, \ldots, k$ incident to v_0. As before, this implies that there are no other vertices in the component of v_0, so we can suppose that $V(G) = \{v_0\} \cup \Gamma(v_0)$. There cannot be an edge of color $> k$ in G, as this would give rise to a rainbow copy of $B_{k,2}$ in G. For every color $1, \ldots, k$, there are at most $(k-2)/2$ edges of that color in $G[\Gamma(v_0)]$, since $|\Gamma(v_0)| = k$ is even and one neighbor of v_0 already uses this color on the edge to v_0. This implies that

$$|E(G)| \leq k + \frac{k(k-2)}{2} = \frac{k^2}{2} = \frac{k^2}{2(k+1)}n.$$

We can construct such a G: Take a properly $(k+1)$-edge-colored copy of K_{k+1} and remove all edges of color $k+1$. Now, take edge-disjoint unions of this graph to obtain

$$\mathrm{ex}^*(n, B_{k,2}) = \frac{k^2}{2(k+1)}n + O(1).$$

□

Lemma 4.2. *When* $3l-4 \leq k$,

$$\mathrm{ex}^*(n, B_{k,l}) \leq \frac{k+l-2}{2}n + O(1).$$

Proof. Let G be a graph with average degree $d(G) > k+l-2$ and let c be a proper edge-coloring of G. There must exist a vertex $w \in V(G)$ such that $d(w) \geq k+l-1$. If w is the endpoint of a rainbow path P of length l, then w is incident to at most l edges that have colors that occur on the path, and at most $l-1$ further edges that intersect with P. This implies that w is incident to at least $k+l-1-(l+l-1) = k-l$ edges that neither intersect P nor have colors in common with P. This gives a rainbow copy of $B_{k,l}$.

In order to show that w is indeed the endpoint of a rainbow path P of length l, we will show that every vertex of G is an endpoint of a rainbow path of length l. We may suppose that G is balanced, by Proposition 2.3. We must have that G has minimum degree

$$\delta(G) \geq \frac{d(G)}{2} > \frac{k+l-2}{2} \geq 2l-3.$$

If this was not the case, then, by Proposition 2.1, we would have $d(G-u) > d(G)$ whenever $d(u) = \delta(G)$, which contradicts G being balanced. Since $\delta(G) > 2l-3$, and by Proposition 2.2, we can start a rainbow path at any vertex, and extend it greedily until it we reach length l. Therefore, every vertex is an endpoint of some rainbow path of length l. This completes the proof. □

Lemma 4.3. *When* $k = 2^s - 2$ *for* $3 \leq s$, *we have*

$$\mathrm{ex}^*(n, B_{k,3}) = \frac{k+1}{2}n + O(1).$$

Proof. Consider the edge-colored graph $K^*_{2^s}$. Suppose that this edge-colored graph contains a rainbow copy of $B_{k,3}$, where v is the center of the star, and v, x, y, z is the broom stick of length 3. Note that $B_{k,3}$ has 2^s-1 vertices, and let u be the vertex not in the copy of $B_{k,3}$. The edges from v of colors $c(xy)$ and $c(yz)$ must go to the set u, y, z, and the only possibility is that $c(vu) = c(yz)$ and $c(vz) = c(xy)$. However, due to the definition of $K^*_{2^s}$, $c(vz) = c(xy)$ implies that $c(vx) = c(yz)$, a contradiction. The upper bound is given by Lemma 4.2. □

Remark 4.4. We claim that, for $4 \leq d \leq 9$ and $k = 2^s - 1$ for some $2 \leq s$, we have

$$\mathrm{ex}^*(n, B_{k,d}) \geq \frac{k}{2}n + O(1).$$

Consider the edge-colored graph $K^*_{2^s}$. Suppose, for the sake of contradiction, that this edge-colored graph contains a rainbow copy of $B_{k,d}$, this implies that we have a set of distinct vectors $W = \{w_1, w_2, \ldots, w_d\}$, which indicate the colors of the edges on the path along the broom stick. We claim that we have $\sum_{i=1}^{a} w_i \in W$ for all

$1 \le a \le d$. The vertex v of degree $k-d+1$ in the broom is incident to $k-d$ leaf-edges in the rainbow copy of $B_{k,d}$, which must use the remaining $k-d$ colors that are not in W. Therefore, all edges from v to other vertices on the broom stick must have colors in W. This implies that $\sum_{i=1}^{a} w_i \in W$ for all $1 \le a \le d$. It can be verified (by brute force) that such a sequence does not exist for $2 \le d \le 9$, for vectors of any length. Such a sequence does exist for $d = 10$, which shows that $K^*_{2^s}$ contains a rainbow $B_{k,10}$ when $k = 2^s - 1$ for $s \ge 4$.

The construction $K^*_{2^s}$ provides lower bounds for a few other caterpillars on $2^s - 1$ edges with short central paths, which we list in the following theorem.

Theorem 4.5. *Let F be a caterpillar on $k = 2^s - 1$, $s \ge 2$, edges, and suppose that F is of the form*

(a) $CP_{(1,t,1)}$, for $t \ge 2$,

(b) $CP_{(t,q)}$ for $t, q \ge 2$ odd,

(c) $CP_{(t,0,q)}$, for $t, q \ge 2$,

(d) $CP_{(t,0,0,q)}$, for $t, q \ge 2$,

(e) $CP_{(t,1,q)}$ for $t, q \ge 2$ odd.

Then

$$\mathrm{ex}^*(n, F) \ge \frac{k}{2}n + O(1).$$

Proof. We separate the cases (a), (b), (c,d) and (e). For all cases, consider the edge-colored graph $K^*_{2^s}$.

(a) Suppose that this graph has a rainbow copy of F. Let x be the center of the star, and let v and w be the vertices at distance 2 from x in F, with edges of colors c_v and c_w to vertices y_v and y_w, respectively, in F. Then $c(xv) = c_w$ and $c(xw) = c_v$. However, this implies that $y_v = y_w$: a contradiction.

(b) Suppose that $K^*_{2^s}$ has a copy of F. Let x and y be the vertices of degree t and q, respectively. Then, for all colors other than $c(xy)$, we have a bijection $f(c) = c + c(xy)$, such that pairs of edges in F on colors $c, f(c)$ must both be incident to x or both to y (as we cannot have a path $c, c(xy), f(c)$). This implies that q and t are even.

(c,d) In any copy of $CP_{(t,0,q)}$ or $CP_{(t,0,0,q)}$ in this graph, with x and y the endpoints of the central path, no edge can have color $c(xy)$. Therefore, it cannot be rainbow.

(e) In a rainbow copy of $CP_{(t,1,q)}$, let x, y, z be the vertices of the central path. Then the leaf-edge incident to y must have color $c(xz)$, or else this color does not appear in the rainbow copy. The remainder of the argument is similar to the proof of (b). □

Remark 4.6. Let F be a tree on 7 edges that is not isomorphic to one of the three trees in Figure 3. Then it can be verified (by computer) that there is no rainbow copy of F in $K^*_{2^3}$. Therefore

$$\mathrm{ex}^*(n, F) \geq \frac{7}{2}n + O(1).$$

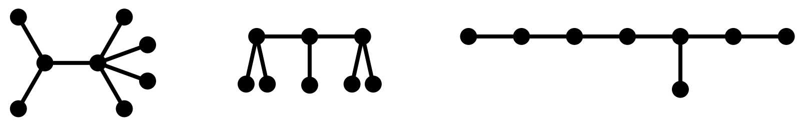

Figure 3: The only three trees on 7 edges that have rainbow copies in $K^*_{2^3}$.

5 Generalized Turán numbers

Here we consider a rainbow version of the generalized Turán problem suggested in [3]. For fixed graphs H and F, let the maximum number of rainbow copies of H in a graph with no rainbow copy of F be the generalized Turán number of H and F, denoted $\mathrm{ex}^*(n, H, F)$. First, consider our graphs that avoid long rainbow paths. Halfpap and Palmer use our construction $D^*_{2^s}$ to show that

$$\mathrm{ex}^*(n, C_k, P_k) \geq \frac{(k-1)!}{2}n + O(1).$$

They also show that $\mathrm{ex}^*(n, C_k, P_k) = \Theta(n)$ (Halfpap and Palmer, personal communication, March 2019). We note a few more similar bounds obtained from this construction in the following corollary.

Corollary 5.1. *For $k \geq 3$ we have*

$$\mathrm{ex}^*(n, P_\ell, P_k) \geq \frac{k!}{2(k-\ell)!}n + O(1), \quad \ell \leq k.,$$

and

$$\mathrm{ex}^*(n, P_\ell, C_k) \geq \frac{(\lfloor \log_2 n \rfloor + 1)!}{2(\lfloor \log_2 n \rfloor + 1 - \ell)!}n + O(1), \quad \ell \leq k,$$

and

$$\mathrm{ex}^*(n, C_k, \{C_3, \ldots, C_{k-1}\}) \geq \frac{(k-1)!}{2}n + O(1).$$

Proof. Consider $D^*_{2^s}$ with $k = s+1$. For any vertex v of $D^*_{2^s}$, and any $x_1, \ldots, x_\ell$ of ℓ distinct colors from the set $\{c_1, \ldots, c_k\}$, there is a unique path in $D^*_{2^s}$ of length ℓ that starts at v and whose edges have colors $x_1, \ldots, x_\ell$ in order along the path. Since $D^*_{2^s}$ is k-regular and properly k-edge colored, such a walk must exist, and the structure of $D^*_{2^s}$ prohibits such a walk from intersecting itself. Therefore, correcting for counting each path for both endpoints, this graph contains $\frac{k!}{2(k-\ell)!}n$ rainbow copies of P_ℓ.

For the second inequality, we count rainbow copies of P_ℓ in $D^*_{2^s}$ for $s \geq k$, which is rainbow C_k-free. A similar counting argument holds for C_k. □

The third inequality in Corollary 5.1 can be restated as follows: the highest number of rainbow copies of C_k in a graph of girth k is at least $n(k-1)!/2 + O(1)$.

For the next corollary, we consider the edge-colored graph $K^*_{2^s}$, and note that small cycles are easy to count.

Corollary 5.2. *For $k = 2^s - 1$, $s \geq 2$, and F a graph on k edges isomorphic to P_k or one of the caterpillars listed in Theorem 4.5, we have*

$$\begin{aligned}
\mathrm{ex}^*(n, C_3, F) &\geq \frac{k(k-1)}{6}n + O(1),\\
\mathrm{ex}^*(n, C_4, F) &\geq \frac{k(k-1)(k-2)}{8}n + O(1),\\
\mathrm{ex}^*(n, C_5, F) &\geq \frac{k(k-1)(k-3)(k-7)}{10}n + O(1),\\
\mathrm{ex}^*(n, C_\ell, F) &= \Omega(k^{\ell-1}n), \quad \ell \ll k.
\end{aligned}$$

6 Open questions

Question 6.1. In [11], Keevash, Mubayi, Sudakov and Verstraëte conjectured that the extremal example for avoiding rainbow P_ks is a disjoint union of cliques. This conjecture was proven false in [10], by providing a non-complete 4-regular edge-colored graph that does not have a P_4 and showing that any proper edge-coloring of K_5 yields a rainbow copy of P_4. The generalization of this construction, $D^*_{2^{k-1}}$, given here, is not a complete graph for $k > 3$. However, when $k = 5$, there is an equivalently dense union of cliques. The geometric construction [16] of a proper edge-coloring of K_6, shown in Figure 4, does not have a rainbow copy of P_5. (This geometric construction does not work for K_8 and avoiding a rainbow P_7.) The construction by Maamoun and Meyniel shows that there are proper colorings of K_n that avoid a rainbow P_{n-1} when $n = 2^s$ for $s \geq 2$. This leads to two natural questions: does every proper edge coloring of K_n have a rainbow copy of P_{n-1} when n is odd? Is there a proper edge coloring of K_n that avoids a rainbow copy of P_{n-1} for every even $n \geq 4$? In [2], Alon, Pokrovskiy and Sudakov show that every properly edge-colored K_n has a rainbow cycle of length $n - O(n^{3/4})$. This is currently the best we know for general n.

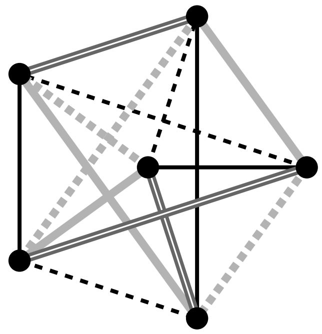

Figure 4: The geometric proper 5-edge-coloring of K_6 [16]. This construction avoids a rainbow P_5.

Question 6.2. In [15], it was shown that, for $k \leq 10$, each properly k-edge-colored k-regular graph contains a rainbow path of length $k-1$. Theorem 3.1 implies that this result is tight, because the construction $D^*_{2^s}$ for $k = s+1$ is a properly k-edge-colored k-regular graph with no P_k. This question of whether every properly k-edge-colored k-regular graph must have a rainbow P_{k-1} is open for $k > 10$. A theorem of Babu, Sunil Chandran, and Rajendraprasad implies that every properly k-edge-colored k-regular graph contains a rainbow path of length $\frac{2}{3}k$ [4].

Question 6.3. In [13], Pokrovskiy and Sudakov define a t-spider as a radius 2 tree with t degree 2 vertices (or equivalently a tree obtained from a star by subdividing t of its edges once), and show that every properly edge-colored K_n contains a spanning rainbow t-spider for any $0.0007n \leq t \leq 0.2n$. In Theorem 4.5 we showed that this does not hold for $t = 2$. For other values of t, must every properly edge-colored K_n have a rainbow t-spider?

Question 6.4. How many rainbow copies of C_k does $K^*_{2^s}$, for $k = 2^s - 1$, have? It is easy to see that for large enough n, using disjoint copies of $D^*_{2^s}$ is much better than using copies of $K^*_{2^s}$ in terms of maximizing the number of rainbow C_ks while avoiding P_k. Enumerating rainbow copies of C_k in $K^*_{2^s}$ would tell us more about $\mathrm{ex}^*(n, C_k, F)$ when F is another tree, such as one of the caterpillars listed in Theorem 4.5.

Question 6.5. There are still plenty of caterpillars and other trees that are not covered by Theorem 4.5. Are there other trees that we missed that are not in $K^*_{2^s}$? Are there other subgraphs of $K^*_{2^s}$, along the lines of $D^*_{2^s}$, that efficiently avoid other trees?

Question 6.6. For a given number of edges k, which tree is the “easiest” to avoid? In other words, which tree has the highest $\mathrm{ex}^*(n, T)$ over all trees on k edges? So far, the highest value found is for $T = B_{k,3}$, for certain values of k, in Lemma 4.3.

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