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Skew throttling

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Abstract

Zero forcing is a process that colors the vertices of a graph blue by starting with some vertices blue and applying a color change rule. Throttling minimizes the sum of the number of initial blue vertices and the time to color the graph. In this paper, we study throttling for skew zero forcing. We characterize the graphs of order n with skew throttling numbers $1, 2, n-1$, and n. We find the exact skew throttling numbers of paths, cycles, and balanced spiders with short legs. In addition, we find a lower bound on the skew throttling number in terms of the diameter of the graph for graphs of minimum degree at least two.

1 Introduction

Zero forcing is a process on graphs in which vertices have two possible colors, blue and white. In each *round* (also called a *time step*), each current blue vertex with only one white neighbor will *force* (or *color*) that neighbor blue (we use the words "color" and "force" interchangeably in this paper). We call the set of blue vertices at the beginning of the zero forcing process an *initial coloring*. An initial set S of blue vertices that eventually colors the whole graph blue is called a *zero forcing*

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set. For any graph G, the minimum possible size of a zero forcing set is called the *zero forcing number* $\mathrm{Z}(G)$. The zero forcing number of any graph G gives an upper bound for the maximum nullity of the family of symmetric matrices with off-diagonal nonzero pattern described by the edges of G [1]. Zero forcing has also been applied to quantum systems control [4, 11] and graph searching [12]. For any set S of blue vertices, the number of rounds for the whole graph G to be colored blue is denoted $\mathrm{pt}(G; S)$, the *propagation time of* S. We use the convention of letting $\mathrm{pt}(G; S) = \infty$ when S is not a zero forcing set. The *propagation time of* G, $\mathrm{pt}(G)$, is the minimum value of $\mathrm{pt}(G; S)$ over all minimum zero forcing sets S [8].

Several other variants of zero forcing have been studied, including positive semidefinite (PSD) zero forcing and skew zero forcing. In *PSD zero forcing*, each blue vertex colors any vertex that is its only white neighbor in a connected component obtained by removing all of the blue vertices. The *PSD zero forcing number* $\mathrm{Z}_+(G)$ [2] and *PSD propagation times* $\mathrm{pt}_+(G; S)$ and $\mathrm{pt}_+(G)$ [13] are defined analogously to $\mathrm{Z}(G)$, $\mathrm{pt}(G; S)$, and $\mathrm{pt}(G)$. Like $\mathrm{Z}(G)$, $\mathrm{Z}_+(G)$ gives an upper bound for the maximum nullity of the family of positive semidefinite matrices corresponding to G [2]. The PSD zero forcing number has also been applied to study the cop versus robber game on trees [3]. In *skew zero forcing*, every vertex with only one white neighbor colors that neighbor blue in each round. This differs from standard zero forcing in that a white vertex is allowed to color its neighbor. The *skew zero forcing number* $\mathrm{Z}_-(G)$ [9] and *skew propagation times* $\mathrm{pt}_-(G; S)$ and $\mathrm{pt}_-(G)$ [10] are defined in analogy with $\mathrm{Z}(G)$, $\mathrm{pt}(G; S)$, and $\mathrm{pt}(G)$. As in the case of $\mathrm{Z}(G)$ and $\mathrm{Z}_+(G)$, $\mathrm{Z}_-(G)$ gives an upper bound for the maximum nullity of the family of skew-symmetric matrices corresponding to G [9] and for the maximum nullity of zero-diagonal symmetric matrices corresponding to G (the maximum nullity of weighted adjacency matrices) [7].

For each variant of zero forcing, the propagation time of a graph G is defined using only minimum zero forcing sets, but it is natural to investigate the propagation time for larger sets and to minimize the sum of the number of initially blue vertices and the propagation time of that set. This is called *throttling*. Throttling minimizes the sum of the resources and the time needed to accomplish the task. For a graph G and set $S \subseteq V(G)$, define $\mathrm{th}(G; S) = |S| + \mathrm{pt}(G; S)$ and $\mathrm{th}(G) = \min_{S \subseteq V(G)} \mathrm{th}(G; S)$. The *zero forcing throttling number* $\mathrm{th}(G)$ was introduced in [5], where a tight lower bound was presented. Throttling numbers $\mathrm{th}_+(G)$ and $\mathrm{th}_c(G)$ have also been defined analogously for PSD zero forcing in [6] and the cop versus robber game in [3], where it was proved that $\mathrm{th}_+(T) = \mathrm{th}_c(T)$ for trees T (but not for all graphs).

In this paper, we introduce the study of throttling for skew zero forcing. For a graph G and set $S \subseteq V(G)$, define $\mathrm{th}_-(G; S) = |S| + \mathrm{pt}_-(G; S)$ and the *skew throttling number* $\mathrm{th}_-(G) = \min_{S \subseteq V(G)} \mathrm{th}_-(G; S)$. For $k \geq \mathrm{Z}_-(G)$, it is also convenient to define $\mathrm{th}_-(G, k) = \min_{|S|=k} \mathrm{th}_-(G; S)$; with this notation, $\mathrm{th}_-(G) = \min_k \mathrm{th}_-(G, k)$. In Section 2, we characterize the graphs of order n with skew throttling numbers of 1, 2, $n - 1$ and n. In Section 3, we determine skew throttling numbers for several families of graphs including paths, cycles, and some spiders. We also prove a lower bound $\mathrm{th}_-(G) = \Omega(\sqrt{d})$ for graphs G of diameter d and minimum degree at least two, and exhibit a family of graphs that achieve this bound.

We define some graph terminology that is used in our results. A *cograph* is a graph that can be generated from K_1 using only complementation and disjoint union. Equivalently, a cograph is a graph that does not contain P_4 (a path on four vertices) as an induced subgraph. The *corona* $G_1 \circ G_2$ of G_1 with G_2 is obtained by making one copy of G_1, $|V(G_1)|$ copies of G_2, and connecting every vertex in the i^{th} copy of G_2 to the i^{th} vertex of G_1. A *spider* is a tree with a single vertex of degree at least 3, which is called the *center*. The graph obtained by removing this vertex is a disjoint union of paths. Each of these paths is a *leg* of the spider, and the *length* of the leg is the number of vertices in the path. The spider is called *balanced* if all legs have the same length. A *universal vertex* of G is a vertex adjacent to every other vertex in G.

2 Extreme skew throttling numbers

In this section we characterize graphs with very low or very high skew throttling numbers. Butler and Young [5] show that $\lceil 2\sqrt{n} - 1\rceil \leq \text{th}(G)$ for all graphs G of order n. However, there are in general no useful bounds on the skew throttling number in terms of the order of the graph, since we exhibit graphs G of order n with $\text{th}_-(G) = 1$ and $\text{th}_-(G) = n$ and we show that there are connected graphs G of order $n \geq 3$ with $\text{th}_-(G) = 2$ and $\text{th}_-(G) = n - 1$.

2.1 Low skew throttling

In this section, we determine graphs having skew throttling number at most two. We use rK_2 to denote the graph consisting of r disjoint copies of K_2.

Proposition 2.1. *For a graph G, $\text{th}_-(G) = 1$ if and only if $G = K_1$ or $G = rK_2$ for $r \geq 1$.*

Proof. If $\text{th}_-(G) = 1$, then $\text{th}_-(G, 1) = 1$ or $\text{th}_-(G, 0) = 1$, which imply $G = K_1$ or $G = rK_2$, respectively. The converse is clear. □

Lemma 2.2. *A graph G has $\text{th}_-(G) = \text{th}_-(G, 0) = 2$ if and only if $G = \big(\widehat{G} \circ K_1\big) \,\dot\cup\, rK_2$ where $\widehat{G}$ is a graph of order at least two in which each component of $\widehat{G}$ has an edge and r is a nonnegative integer. In this case, the order of G is even.*

Proof. Suppose $G = \big(\widehat{G} \circ K_1\big) \,\dot\cup\, rK_2$, $\big|V\big(\widehat{G}\big)\big| \geq 2$, and each component of $\widehat{G}$ has an edge. Each leaf (vertex of degree one) forces its neighbor in the first round. Then each vertex in $\widehat{G}$ forces its one leaf neighbor, so $\text{th}_-\big(\big(\widehat{G} \circ K_1\big) \,\dot\cup\, rK_2, 0\big) = 2$. Since the order of a component of $\widehat{G} \circ K_1$ is at least four, $\text{th}_-\big(\big(\widehat{G} \circ K_1\big) \,\dot\cup\, rK_2\big) \neq 1$ by Proposition 2.1. Thus $\text{th}_-\big(\big(\widehat{G} \circ K_1\big) \,\dot\cup\, rK_2\big) = \text{th}_-\big(\big(\widehat{G} \circ K_1\big) \,\dot\cup\, rK_2, 0\big) = 2$.

Now assume $\text{th}_-(G) = \text{th}_-(G, 0) = 2$. Let L be the set of leaves of G. With $S = \emptyset$, the vertices in L are the only vertices that can force during the first round.

Any K_2 component of G is now all blue, but G is not. Define G' to be the subgraph of components of G that are not entirely blue, L' to be the set of leaves of G', and $U = \{u : u \in N(\ell) \text{ for some } \ell \in L'\}$. No vertex in L' is blue after the first round because $\deg u \geq 2$ for every $u \in U$. In the next round all vertices in L' must be colored blue. This means that the neighbor u of ℓ must force ℓ in the second round, so every other neighbor of u must be blue after the first round. Thus $G' = G[U] \circ K_1$ and $G = G' \dot\cup rK_2$. □

For nonnegative integers s and t, define $H(s,t)$ to be the graph with $V(H(s,t)) = \{b\} \dot\cup \{x_i, y_i : i = 1, \ldots, s\} \dot\cup \{z_i, w_i : i = 1, \ldots, t\}$ and $E(H(s,t)) = \{bx_i, x_iy_i : i = 1, \ldots, s\} \cup \{bz_i, bw_i, z_iw_i : i = 1, \ldots, t\}$. The graph $H(2,3)$ is shown in Figure 2.1.

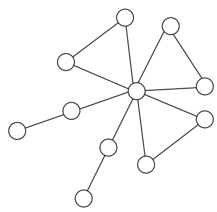

Figure 2.1: The graph $H(2,3)$

Lemma 2.3. *A graph G has $\text{th}_-(G) = \text{th}_-(G,1) = 2$ if and only if $G = H(s,t) \dot\cup rK_2$ for some $r, s, t \geq 0$ with $r + s + t \geq 1$. In this case, the order of G is odd.*

Proof. It is straightforward to verify that $\text{th}_-(H(s,t) \dot\cup rK_2, \{b\}) = 2$ for $r+s+t \geq 1$. By Proposition 2.1, $\text{th}_-(H(s,t) \dot\cup rK_2) \geq 2$ when $r + s + t \geq 1$.

Assume that $\text{th}_-(G) = 2$ and that G can be skew-throttled in one round with one initial blue vertex b. Let $\widetilde{G}$ be the connected component containing b. Since K_2 is the only connected graph that can force itself in one round with no blue vertices, $G = \widetilde{G} \dot\cup rK_2$ for some $r \geq 0$. If $|V(\widetilde{G})| = 1$, then $\text{th}_-(G) = 2$ implies $r \geq 1$ and $G = H(0,0) \dot\cup rK_2$. It is not possible to have $|V(\widetilde{G})| = 2$, because this would imply $G = (r+1)K_2$ and $\text{th}_-(G) = 1$. If $|V(\widetilde{G})| = 3$, then $\widetilde{G} = P_3 = H(1,0)$ or $\widetilde{G} = K_3 = H(0,1)$.

So assume $|V(\widetilde{G})| \geq 4$. Let v be a vertex at maximum distance from b in $\widetilde{G}$. If $\text{dist}(b,v) \geq 3$, then for any neighbor u of v, $\deg u \geq 2$ and b is not a neighbor of u. Thus, $\text{dist}(b,v) \geq 3$ implies v cannot be colored blue in the first round. So no vertex in $\widetilde{G}$ is at distance more than two from b. Since $|V(\widetilde{G})| \geq 4$, this implies $\deg b \geq 2$ and b cannot perform a force in the first round. Indeed, if we had $\deg b = 1$, then the only neighbor u of b is a universal vertex in $\widetilde{G}$. Since u has at least two neighbors besides b, no neighbor of u other than b would ever get colored, so we conclude that $\deg b \neq 1$. Since b cannot force, forcing in $\widetilde{G}$ is the same as forcing in $\widetilde{G} - b$. Thus $\text{th}_-(\widetilde{G} - b, 0) = 1$, so $\widetilde{G} - b = qK_2$. For a K_2 that has one edge between it and b, we designate its vertices as x_i, y_i whereas a K_2 that has two edges between it and b has its vertices designated as z_i, w_i; with the vertices labeled this way we have identified $\widetilde{G}$ as some $H(s,t)$ with $s + t \geq 2$. □

Since $\text{th}_-(G) = 2$ implies one of $\text{th}_-(G, 0) = 2$, $\text{th}_-(G, 1) = 2$, or $\text{th}_-(G, 2) = 2$, the next result follows from Lemmas 2.2 and 2.3 and the observation that $\text{th}_-(G, k) = k$ implies $|V(G)| = k$.

Theorem 2.4. *A graph G has $\text{th}_-(G)=2$ if and only if G is one of $2K_1$, $H(s,t) \,\dot\cup\, rK_2$ with $r + s + t \geq 1$, or $\left(\widehat{G} \circ K_1\right) \dot\cup\, rK_2$ where each component of $\widehat{G}$ has an edge and $\widehat{G}$ has order at least two.*

By considering $G = C_m \circ K_1$ we see that $\text{th}_-(G) = 2$ can be achieved by a graph of arbitrarily large order with maximum degree three, unlike the case of PSD throttling [6]. Lemma 2.3 also implies that graphs in a well-known family have skew throttling number equal to two: For $n \geq 1$, the *friendship graph* F_n is the planar graph with $2n + 1$ vertices and $3n$ edges constructed by joining a universal vertex to n disjoint copies of K_2. That is, $F_n = H(0, n)$, so $\text{th}_-(F_n) = 2$ by Lemma 2.3. The proof of Lemma 2.3 also established that $\text{Z}_-(F_n) = 1$ and $\text{pt}_-(F_n) = 1$.

2.2 High skew throttling

We now turn to graphs with high skew throttling number. For graphs G with all vertices isolated, it is clear that $\text{th}_-(G) = n$. In the next result, we establish an upper bound on the skew throttling number for graphs that have an edge.

Proposition 2.5. *Let G be a graph of order n. If G has an edge, then $\text{th}_-(G) \leq n-1$. Thus $\text{th}_-(G) = n$ if and only if $G = nK_1$.*

Proof. If G has an edge uv, then $\text{pt}_-(G; V(G) \setminus \{u, v\}) = 1$, which implies $\text{th}_-(G) \leq \text{th}_-(G; V(G) \setminus \{u, v\}) = n - 1$. Thus $\text{th}_-(G) = n$ if and only if $G = nK_1$. □

Remark 2.6. Let G be a graph of order n that has an edge, so $\text{th}_-(G) \leq n - 1$. This implies $\text{pt}_-(G; S) \geq 1$ for any set S such that $\text{th}_-(G; S) = \text{th}_-(G)$, which then implies $|S| \leq n - 2$ for any such S. Furthermore, it is straightforward to see that $\text{pt}_-(G) \leq 2$ implies $\text{th}_-(G) = \text{Z}_-(G) + \text{pt}_-(G)$: This is immediate for $\text{pt}_-(G) = 1$. In the case $\text{pt}_-(G) = 2$, it is not possible to improve throttling by adding one to the skew zero forcing set.

Let G be a cograph. The $\cup - \vee$ *decomposition tree* T_G of G is a rooted binary tree such that the vertices of G are the leaves of T_G and each non-leaf vertex is labeled either $\cup$ or $\vee$, where $\cup$ represents disjoint union and $\vee$ represents join. For a non-leaf vertex x of T_G, the branches at x are the two connected components of T_G induced by the descendants of x. If y is a vertex of G (and a leaf of T_G), define $G_y = G[\{y\}]$. For x a non-leaf vertex of T_G, define G_x to be the subgraph of G induced by the leaves of T_G that are descendants of x. Observe that G_x can be obtained by applying the operation in the label of x to G_y and G_z, where y and z denote the immediate descendants of x, and $G = G_r$ where r is the root of T_G.

If G is a cograph with no induced $2K_2$, then every $\cup$ vertex in the $\cup - \vee$ decomposition of G has a branch with no $\vee$, since otherwise each of the disjoint subgraphs of G induced by the descendants of the $\cup$ vertex would have a K_2.

Theorem 2.7. *For a graph G of order n, $\text{th}_-(G) = n - 1$ if and only if G is a cograph with no induced $2K_2$ and at least one edge.*

Proof. Let G be a graph of order n. We first establish that $\text{th}_-(G) \neq n-1$ if G has no edges, or if G has an induced P_4 or $2K_2$. If G has no edges, then $\text{th}_-(G) = n$. If G has an induced P_4, then let S consist of all vertices except those in an induced P_4, so $\text{pt}_-(G; S) = 2$ and $\text{th}_-(G) \leq n - 2$. If G has an induced $2K_2$, then let S' consist of all vertices except those in an induced $2K_2$, so $\text{pt}_-(G; S') = 1$ and $\text{th}_-(G) \leq n-3$.

Next, we prove by induction on the order of the graph that every cograph G of order n with no induced $2K_2$ has skew throttling number n if G has no edges and $n - 1$ if G has at least one edge. Clearly the statement is true for graphs of order 1, since there are no edges and the skew throttling number is 1. The induction hypothesis is that every cograph of order $k < n$ with no induced $2K_2$ must have skew throttling number $k-1$ if it has an edge and k if it has no edge. Let G be a cograph of order $n > 1$ with no induced $2K_2$.

Let x be the root of T_G (so $G_x = G$), and denote the immediate descendants of x by y and z. The induction hypothesis applies to G_y and G_z since each has order less than n.

Suppose first that x is labeled with $\cup$. Then at least one of the branches of x, say the one that contains y, has no $\vee$, so G_y consists of isolated vertices. If neither branch has a $\vee$, then G_x consists of isolated vertices and $\text{th}_-(G_x) = |G_x|$. Suppose the branch at z has a $\vee$, so G_z has an edge. Any zero forcing set of G_x must consist of a zero forcing set for G_z along with every vertex in G_y. Thus

$$\text{th}_-(G_x) = \text{th}_-(G_z) + |G_y| = |G_z| - 1 + |G_y| = |G_x| - 1.$$

Now suppose x is labeled with $\vee$. Let S be a set of vertices such that $\text{th}_-(G_x) = \text{th}_-(G_x, S)$. The number of white vertices (i.e., vertices not in S) is at least 2 by Remark 2.6. No vertex in G_y can force any other vertex in G_y until every vertex in G_z is blue, and vice versa. Moreover, no vertex in G_y can force any vertex in G_z until all but one vertex in G_z is blue, and vice versa. For the zero forcing process to start, G_y or G_z must initially have at most one white vertex. If each of G_y and G_z has exactly one white vertex at the start, then $\text{pt}_-(G_x, S) = 1$, and $\text{th}_-(G_x) = |G_x| - 1$.

If initially G_y has one white vertex and G_z has more than one white vertex, then no vertex in G_z can be colored blue in the first round: Every vertex in G_y has at least two white neighbors in G_z and a vertex in G_z that has a white neighbor in G_z has at least two white neighbors including one in G_y. So in the first round there is exactly one force $u \to w$ where $u \in G_z$ and $w \in G_y$, and all vertices of G_y are blue after the first round. Thus the initial set S has the same skew throttling number as $S' = S \cup \{w\}$, since throttling with S' adds one to the size of the zero forcing set but subtracts one from the propagation time. Thus we replace S by S' for the rest of the proof; observe that $S' \cap V(G_z) = S \cap V(G_z)$.

If all of the vertices in G_z were isolated, then all but one vertex in G_z would have to be in S', or else no vertex in G_y could color any vertex in G_z. However, we assumed that S' omits more than one white vertex of G_z, so G_z must have an edge. Since G_z has an edge, $\mathrm{th}_-(G_z) = |G_z| - 1$ by the induction hypothesis. Suppose that as S' colors all the vertices blue, a vertex v in G_y forces a vertex w' of G_z. Necessarily $v \to w'$ is the last force, and this force is the only force in the last round. Thus S' has the same skew throttling number as $S'' = S' \cup \{w'\}$, since throttling with S'' adds one to the size of the zero forcing set but subtracts one from the propagation time. In this case, we replace S' by S'' for the rest of the proof (if this case does not apply, then let $S'' = S'$). Define $Z = S'' \setminus V(G_y)$. Using S'', no vertex of G_y performs a force, so Z is a skew zero forcing set for G_z. Thus $\mathrm{pt}_-(G_x; S'') = \mathrm{pt}_-(G_z, Z)$ and

$$\mathrm{th}_-(G_x) = \mathrm{th}_-(G_x; S'') = |G_y| + |Z| + \mathrm{pt}_-(G_z, Z) = |G_x| - 1. \qquad \square$$

It follows from Theorem 2.7 that the complete multipartite graph $K_{n_1,n_2,\ldots,n_s}$ with $s \geq 2$ and $n := n_1+n_2+\cdots+n_s$ is an example of a graph with $\mathrm{th}_-(K_{n_1,n_2,\ldots,n_s}) = n-1$; this also follows from results of Kingsley, who showed in [10] that $\mathrm{Z}_-(K_{n_1,n_2,\ldots,n_s}) = n-2$ and $\mathrm{pt}_-(K_{n_1,n_2,\ldots,n_s}) = 1$.

3 Skew throttling numbers of families of graphs

In this section we determine the skew throttling numbers of hypercube graphs, paths, cycles, and balanced spiders with short legs. We also find the maximum and minimum skew throttling numbers of trees of order n, and we bound the skew throttling numbers of all balanced spiders.

Just as connected graphs of order n have skew throttling numbers ranging between 2 and $n-1$ for $n \geq 3$, the same is true of trees of order n. The maximum is at most $n-1$ by Proposition 2.5, and T achieves $\mathrm{th}_-(T) = n-1$ if T is a star. The minimum is at least 2 by Proposition 2.1, and T achieves $\mathrm{th}_-(T) = 2$ if $T = T' \circ K_1$ for some tree T' of order at least two. Even if we restrict the tree to have maximum degree d, there are still trees with $\Omega(n)$ skew throttling numbers, e.g., when T is obtained from a tree of maximum degree $d-2$ by adding two leaves to every vertex. Although the skew throttling number of the star of order n is close to the (standard) throttling number, which is n, the skew throttling numbers of paths and cycles behave more like the PSD throttling numbers of those graphs. We begin with cycles and paths.

Proposition 3.1. *For all $n \geq 3$, $\mathrm{th}_-(C_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$.*

Proof. The proof for the lower bound is the same as the proof of Proposition 2.5 in [6].

For the upper bound, we can start with the same construction as in the proof of Theorem 3.3 of [6]. We initially color blue every $(k+1)^{st}$ vertex around the cycle, where k is the largest even integer such that $n \geq \frac{k^2}{2}$. Let r be the remainder when n is divided by $k+1$. The initial blue coloring splits the white vertices into paths of size k when $r = 0$ and one short path containing $r-1$ white vertices when $r \geq 1$. Since k is even, the white paths of size k turn blue in $\frac{k}{2}$ rounds. If $r = 0$ we are done and $\text{th}_-(C_n) \leq \lceil \sqrt{2n} - \frac{1}{2} \rceil$. When $r \geq 1$ is odd, the number of white vertices in the short path is even, so the short path will also turn blue in at most $\frac{k}{2}$ rounds. We have used the same number of blue vertices as in the proof of [6, Theorem 3.3], so again $\text{th}_-(C_n) \leq \lceil \sqrt{2n} - \frac{1}{2} \rceil$.

If $r \geq 1$ is even and $r < k$, then we can modify the initial coloring by increasing the lengths of $\frac{r}{2}$ of the white paths of length k to $k+2$ and decreasing the length of the short path to 0. Note that in this case, there are at least $\frac{k}{2} - 1$ white paths of length k before the modification because $n \geq \frac{k^2}{2}$. Moreover, we removed a blue vertex. Thus again we have $\text{th}_-(C_n) \leq \lceil \sqrt{2n} - \frac{1}{2} \rceil$.

If $r = k$, then there are two possible values of n, namely $n = \left(\frac{k}{2} - 1\right)(k+1) + k$ and $n = \frac{k}{2}(k+1) + k$, since any other integer m of the form $q(k+1) + k$ with integer q has $m \geq \frac{(k+2)^2}{2}$ for $q > \frac{k}{2}$ and $m < \frac{k^2}{2}$ for $q < \frac{k}{2} - 1$. For $n = \frac{k}{2}(k+1) + k$, we can again modify the initial coloring by increasing the lengths of the $\frac{r}{2} = \frac{k}{2}$ white paths of length k on the cycle to $k+2$, thereby decreasing the length of the short path to 0. In this process we also removed a blue vertex, so we have $\text{th}_-(C_n) \leq \lceil \sqrt{2n} - \frac{1}{2} \rceil$. For $n = \left(\frac{k}{2} - 1\right)(k+1) + k$, we can modify the initial coloring by decreasing the lengths of the white paths of length k on the cycle to $k-2$, which decreases the propagation time by one. The number of white paths increases from $\frac{k}{2}$ (consisting of $\frac{k}{2} - 1$ paths with k white vertices and one short path with $k-1$ white vertices) to $\frac{k}{2}+1$ (with each path having $k-2$ white vertices). Thus again we have $\text{th}_-(C_n) \leq \lceil \sqrt{2n} - \frac{1}{2} \rceil$. □

Proposition 3.2. *For all* $n \geq 3$, $\text{th}_-(P_n) = \left\lceil \sqrt{2(n+1)} - \frac{3}{2} \right\rceil$.

Proof. The proof for the lower bound is almost the same as the proof of Proposition 2.5 in [6], except we use the inequality $s(2p+1) + 2p \geq n$ instead of $s(2p+1) \geq n$ since the leaves can color their neighbors, where s denotes the number of initial blue vertices and p denotes the propagation time.

For the upper bound, choose a set S of blue vertices for the cycle on $n+1$ vertices with $\text{th}_-(C_{n+1}; S) = \text{th}_-(C_{n+1}) = \left\lceil \sqrt{2(n+1)} - \frac{1}{2} \right\rceil$ that has an initial blue vertex v with no initial blue neighbor. Delete v from the graph and the set S. This results in the graph P_n and set of blue vertices $S' = S \setminus \{v\}$ such that

$$\text{th}_-(P_n; S') = \left\lceil \sqrt{2(n+1)} - \frac{1}{2} \right\rceil - 1 = \left\lceil \sqrt{2(n+1)} - \frac{3}{2} \right\rceil. \qquad \square$$

Remark 3.3. For a cycle C_n with $n \geq 4$,

- [6] $\text{th}(C_n) = \begin{cases} \lceil 2\sqrt{n} - 1 \rceil & \text{unless } n = (2k+1)^2 \\ 2\sqrt{n} & \text{if } n = (2k+1)^2 \end{cases}$.

- [6] $\text{th}_+(C_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$.
- $\text{th}_-(C_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$.

For a path P_n with $n \geq 3$,

- [5] $\text{th}(P_n) = \lceil 2\sqrt{n} - 1 \rceil$.
- [6] $\text{th}_+(P_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$.
- $\text{th}_-(P_n) = \left\lceil \sqrt{2(n+1)} - \frac{3}{2} \right\rceil$.

We use $T_{p,\ell}$ to denote the balanced spider with p legs, each of length ℓ; $T_{4,3}$ is shown in Figure 3.1. Observe that the order of $T_{p,\ell}$ is $n = p\ell + 1$. Note that $T_{p,1} = K_{1,p}$ and $\text{th}_-(K_{1,p}) = p$ by Theorem 2.7, so the discussion here focuses on $\ell \geq 2$.

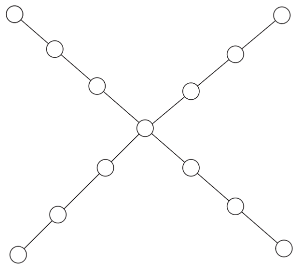

Figure 3.1: The graph $T_{4,3}$

Theorem 3.4. *Let $\ell \geq 2$ be fixed and let $p > \frac{\ell}{2} + 1$. Then*

$$\text{th}_-(T_{p,\ell}) = \begin{cases} 1 + \frac{\ell}{2} & \text{if } \ell \text{ is even} \\ 1 + p + \frac{\ell-1}{4} & \text{if } \ell = 4q+1 \text{ for some integer } q \\ 1 + p + \frac{\ell+1}{4} & \text{if } \ell = 4q+3 \text{ for some integer } q. \end{cases}$$

Proof. If ℓ is even, consider the skew zero forcing set consisting of one vertex, specifically the center. This partitions the graph into p disjoint copies of P_ℓ. Therefore, the graph will be colored in $\frac{\ell}{2}$ rounds. For the lower bound, note first that $\text{Z}_-(T_{p,\ell}) = 1$. If there is a leg with no initial blue vertex, then the propagation time is at least $\frac{\ell}{2}$ and $\text{th}_-(T_{p,\ell}) \geq \frac{\ell}{2} + 1$. If every leg has an initial blue vertex, then this would not be an optimal skew throttling set because $p > \frac{\ell}{2} + 1$. Thus $\text{th}_-(T_{p,\ell}) = \frac{\ell}{2} + 1$ when ℓ is even.

Now suppose ℓ is odd and let S denote the set of vertices that are blue initially. First note that if $T_{p,\ell}$ has two or more legs that contain no vertex from S, then S is not a skew zero forcing set. Thus $|S| \geq p-1$. We consider two cases, based on the parity of $\ell \mod 4$.

First, suppose that $\ell = 4q+1$ for some positive integer q. Consider the skew zero forcing set S consisting of the center c and each vertex at distance d from c where $d = \frac{\ell+1}{2}$. Thus, $G-S$ has been partitioned into $2p$ disjoint copies of $P_{\frac{\ell-1}{2}}$ and the graph will be colored in $\frac{\ell-1}{4}$ rounds. For the lower bound, if there is a leg that does not contain any vertex in S, then $\mathrm{pt}_-(T_{p,\ell}; S) \geq \frac{\ell+1}{2}$, which implies that

$$\mathrm{th}_-(T_{p,\ell}; S) \geq p-1+\frac{\ell+1}{2} \geq 1+p+\frac{\ell-1}{4} \tag{1}$$

since $\ell \geq 5$. Thus there must exist an optimal initial blue set S of size at least p with a blue vertex in every leg, and we assume we have chosen such a S. If there is a leg that has only one vertex in S, then the propagation time is at least $\frac{\ell-1}{4}$, and this is achieved only when the center vertex is also blue. If every leg has at least two vertices in S, then S would not be an optimal skew throttling set because

$$p > \frac{\ell}{2}+1 > 1+\frac{\ell-1}{4}. \tag{2}$$

Thus $\mathrm{th}_-(T_{p,\ell}) = 1+p+\frac{\ell-1}{4}$ when $\ell = 4q+1$ for some positive integer q.

Now suppose that $\ell = 4q+3$ for some integer $q \geq 0$. It is straightforward to verify that $\mathrm{th}_-(T_{p,3}) = p+2 = 1+p+\frac{3+1}{4}$, so we assume $\ell \geq 7$. The argument is similar to the case $\ell = 4q+1$. For the upper bound, the blue vertices in the legs are placed at distance $\frac{\ell-1}{2}$ from the center and $G-S$ is partitioned into p disjoint copies of $P_{\frac{\ell-3}{2}}$ and p disjoint copies of $P_{\frac{\ell+1}{2}}$. The lower bound argument is the same until (1), where $\frac{\ell-1}{4}$ is replaced by $\frac{\ell+1}{4}$, but the equation remains true because now $\ell \geq 7$. The statement (2) also remains valid with $\frac{\ell-1}{4}$ replaced by $\frac{\ell+1}{4}$. So we assume an optimal S in which each leg has at least one vertex in S and there is a leg with only one vertex in S. In $G-S$, a leg with exactly one vertex in S is best partitioned into one $P_{\frac{\ell-3}{2}}$ and one $P_{\frac{\ell+1}{2}}$ and the best possible propagation time is $\frac{\ell+1}{4}$. This can be achieved in two ways: When the center is in S, or when the $P_{\frac{\ell-3}{2}}$ is next to the center and there is a blue vertex at distance two from the center. In the latter case, there must be another vertex in S in the leg with the blue vertex at distance two from the center. Thus $|S| \geq p+1$ and $\mathrm{pt}_-(T_{p,\ell}; S) \geq \frac{\ell+1}{4}$, or $|S| \geq p$ and $\mathrm{pt}_-(T_{p,\ell}; S) \geq \frac{\ell+1}{4}+1$. So $\mathrm{th}_-(T_{p,\ell}) = 1+p+\frac{\ell+1}{4}$ when $\ell = 4q+3$ for some positive integer q. □

Theorem 3.5. *For all $\ell, p \geq 2$, $\frac{1}{2}f(p,\ell) \leq \mathrm{th}_-(T_{p,\ell}) \leq 3f(p,\ell)$, where*

$$f(p,\ell) = \begin{cases} \min(\ell, \sqrt{p\ell}) & \text{if } \ell \text{ is even} \\ \max(p, \sqrt{p\ell}) & \text{if } \ell \text{ is odd.} \end{cases}$$

Proof. We split the proof into cases depending on whether $p > \sqrt{p\ell}$ or $p \leq \sqrt{p\ell}$. In the case that $p > \sqrt{p\ell}$, then $p > \ell \geq 1+\frac{\ell}{2}$, so Theorem 3.4 applies. When ℓ is even,

$$f(p,\ell) = \ell \geq \mathrm{th}_-(T_{p,\ell}) = 1+\frac{\ell}{2} > \frac{\ell}{2} = \frac{1}{2}f(p,\ell).$$

When ℓ is odd,

$$2f(p,\ell) = 2p \geq 1 + p + \frac{\ell+1}{4} \geq \mathrm{th}_-(T_{p,\ell}) \geq 1 + p + \frac{\ell-1}{4} > \frac{1}{2}p = \frac{1}{2}f(p,\ell).$$

Now suppose $p \leq \sqrt{p\ell}$, so $\sqrt{p\ell} \leq \ell$ and $f(p,\ell) = \sqrt{p\ell}$. For the lower bound, define b to be the least number of initial blue vertices on any leg of the spider. If $b = 0$, then some leg has no initial blue vertices, so the propagation time is at least $\frac{\ell}{2} \geq \frac{1}{2}f(p,\ell)$. So assume $b > 0$. Then there is a leg which has an interval of white vertices between blue vertices of length at least $\frac{\ell-b}{b+1}$ by the pigeonhole principle, so the propagation time is at least $\frac{\ell-b}{4(b+1)}$. Since in this case there are at least pb initial blue vertices and $b \geq 1$, we have $\mathrm{th}_-(T_{p,\ell}) \geq pb + \frac{\ell}{4(b+1)} - \frac{1}{4} \geq \sqrt{\frac{p\ell}{2}} - \frac{1}{4} \geq \frac{\sqrt{p\ell}}{2}$.

For the upper bound, use $2p\left\lceil \frac{\ell+1}{2\lfloor\sqrt{p\ell}\rfloor+2} \right\rceil \leq 2p\left\lceil \frac{\ell}{2\sqrt{p\ell}} \right\rceil \leq \frac{2\ell p}{\sqrt{p\ell}} \leq 2\sqrt{p\ell}$ initial blue vertices arranged in adjacent pairs such that every white vertex is within distance $\sqrt{p\ell}$ of a blue vertex, for a propagation time of at most $\sqrt{p\ell}$. □

Remark 3.6. Note that the method of the last proof can be used to obtain similar bounds for balanced spiders under other variants of throttling. In particular, the bound for the odd ℓ case in the last theorem has the same bound up to a constant factor as standard zero forcing throttling, while the even ℓ case has the same bound up to a constant factor as PSD zero forcing throttling. Specifically, we have $\mathrm{th}_+(T_{p,\ell}) = \Theta(\min(\ell, \sqrt{p\ell}))$ for all positive ℓ and p, while $\mathrm{th}(T_{p,\ell}) = \Theta(\max(p, \sqrt{p\ell}))$.

Our next result on hypercube graphs is an immediate corollary of the results of Kingsley [10].

Proposition 3.7. [10] *For $n \geq 2$, the n^{th} hypercube Q_n has $\mathrm{Z}_-(Q_n) = 2^{n-1}$ and $\mathrm{pt}_-(Q_n) = 1$.*

Corollary 3.8. *For $n \geq 2$, the n^{th} hypercube Q_n has $\mathrm{th}_-(Q_n) = 2^{n-1} + 1$.*

Proposition 3.9. [10] *For a connected graph $G \neq K_1$, the skew zero forcing number of the corona $G \circ K_1$ is $\mathrm{Z}_-(G \circ K_1) = 0$ and $\mathrm{pt}_-(G \circ K_1) = 2$.*

Corollary 3.10. *For a connected graph $G \neq K_1$, the skew throttling number of the corona $G \circ K_1$ is $\mathrm{th}_-(G \circ K_1) = 2$.*

Proposition 3.11. *For any graph G, $\mathrm{th}_-(G \circ K_2) \leq |G| + 1$.*

Proof. Consider the skew zero forcing set consisting of all vertices in G. Then the remaining vertices, which are copies of K_2 attached to each vertex in G, are forced in one round. □

In general, $|G|+1$ does not serve as a lower bound for $\mathrm{th}_-(G \circ K_2)$. For example, suppose a connected graph G contains $\ell \geq 3$ leaves and no two leaves of G share a neighbor. The set of all vertices of graph G except for the leaves serves as a skew zero forcing set with the copies of K_2 attached to each initially blue vertex being forced in the first round, the leaves being forced in the second round, and finally the ℓ copies of K_2 are forced no later than the third and final round. Thus, $\text{th}_-(G \circ K_2) \leq |G| - \ell + 3 \leq |G| - 3 + 3 = |G|$.

Our next bound for graphs of fixed diameter is sharp up to a constant factor, as shown by paths and cycles. However, we also find a much larger family of graphs that achieve this bound. The *ball* $B(v, r)$ at vertex v of radius r in G is the set of all vertices at distance at most r from v.

Lemma 3.12. *Let G be a graph, let $L = \{y_1, \ldots, y_\ell\}$ denote the set of leaves of G, let $S = \{x_1, \ldots, x_k\} \subseteq V(G)$, and let $t = \text{pt}_-(G; S)$. Then*

$$V(G) = (\cup_{i=1}^{\ell} B(y_i, 2t)) \bigcup (\cup_{j=1}^{k} B(x_j, 2t)).$$

Proof. A vertex can perform a force in the first round if and only if it has at most one white neighbor. Thus in order to force, a vertex must be a leaf (so it has only one neighbor) or it is a neighbor of a blue vertex. Thus any vertex colored blue in the first round must be at distance at most two from a blue vertex or a leaf, i.e. in some $B(y_i, 2)$ or $B(x_j, 2)$. This process is iterated through the t rounds. □

Theorem 3.13. *For a connected graph G of diameter $d \geq 4$ with minimum degree at least two,* $\text{th}_-(G) \geq \sqrt{d} - \frac{1}{4}$.

Proof. Suppose that G has diameter d and minimum degree at least two, $S = \{x_1, \ldots, x_k\} \subseteq V(G)$ is a skew zero forcing set for G such that $\text{th}_-(G) = \text{th}_-(G; S)$, and $t = \text{pt}_-(G; S)$. By Lemma 3.12, $V(G) = \bigcup_{j=1}^{k} B(x_j, 2t)$ since G has no leaves. Let v_1 and v_{d+1} be vertices that have distance d in G and let $v_1, \ldots, v_{d+1}$ be a shortest path between these vertices. We may assume that $k \leq d$, or else $\text{th}_-(G) = k + t > d > \sqrt{d} - \frac{1}{4}$. Since there are only k balls $B(x_j, 2t)$ with $j = 1, \ldots, k$, there exists j such that $B(x_j, 2t)$ contains at least two of the $k+1$ vertices of the form $v_{1+i\lfloor \frac{d}{k} \rfloor}$ with $i = 0, \ldots, k$; denote two such vertices in $B(x_j, 2t)$ by v_a and v_b with $a \neq b$. The maximum distance between vertices in $B(x_j, 2t)$ is $4t$, and this is at least as large as the smallest possible distance $\lfloor \frac{d}{k} \rfloor$ between the vertices v_a and v_b. Since $\lfloor \frac{d}{k} \rfloor > \frac{d}{k} - 1$,

$$\text{th}_-(G) = \text{th}_-(G; S) = k + t \geq k + \frac{1}{4}\left(\frac{d}{k} - 1\right) \geq \sqrt{d} - \frac{1}{4}.$$ □

The bound in Theorem 3.13 is sharp up to a constant factor for C_n, P_n, $C_n \circ K_2$, and $P_n \circ K_2$. Using adjacent pairs of initial blue vertices on the cycle and path at intervals of approximately $\sqrt{n}$, the graphs $C_n \circ K_2$ and $P_n \circ K_2$ can be colored in $\Theta(\sqrt{n})$ rounds using $\Theta(\sqrt{n})$ initial blue vertices (this is established in Proposition 3.14). Figure 3.2 illustrates such a coloring for $C_{4r(4r+2)} \circ K_2$.

Let $\mathcal{G}$ be the family of graphs that can be constructed by starting with a *base graph* that is a path or cycle and then for each vertex v of the base graph, connecting

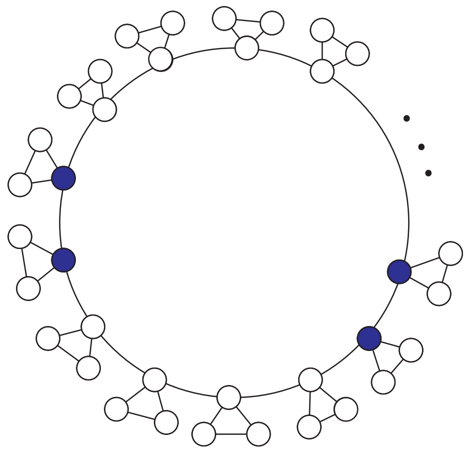

Figure 3.2: An initial coloring of $C_{4r(4r+2)} \circ K_2$ with $r = 1$ that shows $\mathrm{th}_-(C_{4r(4r+2)} \circ K_2) \leq 12r + 1$ with $4r$ pairs of blue vertices and intervals of exactly $4r$ white vertices between all consecutive pairs of blue vertices

any number of copies of K_2 to v (with both vertices of the K_2 adjacent to v) such that the resulting graph has minimum degree at least two.

Proposition 3.14. *For any graph $G \in \mathcal{G}$ of diameter d,* $\mathrm{th}_-(G) = \Theta(\sqrt{d})$.

Proof. Suppose $G \in \mathcal{G}$. Since the minimum degree of G is at least two, the lower bound follows from Theorem 3.13. For the upper bound, place adjacent pairs of initial blue vertices on the base graph in G at intervals of length $\left\lfloor \sqrt{d} \right\rfloor$, with at most one interval of lesser length. Also place one initial blue vertex at each of the ends of the base graph if the base graph is a path. In the first round, all of the copies of K_2 attached to the initial blue vertices turn blue. By the second round, the neighbors of the initial blue vertices on the base graph turn blue. By the $(2i+1)^{\mathrm{st}}$ round, copies of K_2 will turn blue if they are attached to the vertices on the base graph that turned blue in the $(2i)^{\mathrm{th}}$ round. By the $(2i+2)^{\mathrm{nd}}$ round, vertices on the base graph will turn blue if they are adjacent to vertices that turned blue in the $(2i)^{\mathrm{th}}$ round. Thus each interval of white vertices in the base graph takes at most $2\left\lceil \frac{\lfloor \sqrt{d} \rfloor}{2} \right\rceil$ rounds to be colored entirely blue, so G is colored entirely blue in at most $2\left\lceil \frac{\lfloor \sqrt{d} \rfloor}{2} \right\rceil + 1$ rounds. □

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References

[1] AIM Minimum Rank — Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen and A. Wangsness), Zero forcing sets and the minimum rank of graphs, *Linear Algebra Appl.* 428 (2008), 1628–1648.

[2] F. Barioli, W. Barrett, S. Fallat, H.i T. Hall, L. Hogben, B. Shader, P. van den Driessche and H. van der Holst, Zero forcing parameters and minimum rank problems, *Linear Algebra Appl.* 433 (2010), 401–411.

[3] J. Breen, B. Brimkov, J. Carlson, L. Hogben, K. E. Perry and C. Reinhart, Throttling for the game of Cops and Robbers on graphs, *Discrete Math.* 341 (2018), 2418–2430.

[4] D. Burgarth and V. Giovannetti, Full control by locally induced relaxation, *Phys. Rev. Lett.* 99 (2007), 100501.

[5] S. Butler and M. Young, Throttling zero forcing propagation speed on graphs, *Australas. J. Combin.* 57 (2013), 65–71.

[6] J. Carlson, L. Hogben, J. Kritschgau, K. Lorenzen, M. S. Ross, S. Selken and V. Valle Martinez, Throttling positive semidefinite zero forcing propagation time on graphs, *Discrete Appl. Math.* 254 (2019), 33–46.

[7] C. Grood, J. A. Harmse, L. Hogben, T. Hunter, B. Jacob, A. Klimas and S. McCathern, Minimum rank of zero-diagonal matrices described by a graph, *Electron. J. Linear Algebra* 27 (2014), 458–477.

[8] L. Hogben, M. Huynh, N. Kingsley, S. Meyer, S. Walker and M. Young, Propagation time for zero forcing on a graph, *Discrete Appl. Math.* 160 (2012), 1994–2005.

[9] IMA-ISU research group on minimum rank (M. Allison, E. Bodine, L. M. DeAlba, J. Debnath, L. DeLoss, C. Garnett, J. Grout, L. Hogben, B. Im, H. Kim, R. Nair, O. Pryporova, K. Savage, B. Shader and A. Wangsness Wehe), Minimum rank of skew-symmetric matrices described by a graph, *Linear Algebra Appl.* 432 (2010), 2457–2472.

[10] N. F. Kingsley, Skew propagation time, Ph.D. Thesis, Iowa State University (2015).

[11] S. Severini, Nondiscriminatory propagation on trees, *J. Physics A* 41 (2008), 482–002 (Fast Track Communication).

[12] B. Yang, Fast-mixed searching and related problems on graphs, *Theoret. Comput. Sci.* 507 (2013), 100–113.

[13] N. Warnberg, Positive semidefinite propagation time, *Discrete Appl. Math.* 198 (2016), 274–290.

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