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An index-preserving bijection between marked tableaux and $P_{n,2}$-tableaux

Samuel Fulton

Delta, CO
U.S.A.
samjfulton11@gmail.com

Stephen O'Connor-Seville

Durango, CO
U.S.A.
smoconnorseville@gmail.com

Matthew Welz

Department of Mathematics, Fort Lewis College
1000 Rim Drive, Durango, CO 81301
U.S.A.
mwelz@fortlewis.edu

Abstract

In 1992, Stembridge introduced marked tableaux and showed that the number of admissible marked tableaux of shape $\lambda \vdash n$ is equal to the multiplicity of the irreducible Specht module S^λ in a certain representation of S_n. Through their seemingly unrelated work with chromatic quasisymmetric functions, Shareshian and Wachs established in 2012 that this multiplicity of S^λ is also equal to the number of $P_{n,2}$-tableaux of shape λ. Shareshian and Wachs went on to observe indirectly that the number of marked tableaux of shape λ and index j equals the number of $P_{n,2}$-tableaux of shape λ and index j, while suggesting it might be interesting to find a bijective proof of this fact. In this paper, we present such a bijection. In particular, we develop an index-preserving bijection from the set of all marked tableaux of shape λ to the set of all $P_{n,2}$-tableaux of shape λ.

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1 Definitions and Introduction

In [11], Stembridge investigates a representation of the cohomology of the toric variety X_n associated with the Coxeter complex of the symmetric group, S_n. In particular, Stembridge establishes that the multiplicity of the irreducible Specht module S^λ for the $2i$-th component of the cohomology is the number of marked tableaux on shape $\lambda \vdash n$ with index i. Through their seemingly unrelated work with chromatic quasisymmetric functions, Shareshian and Wachs establish in [5] that this multiplicity of S^λ is also equal to the number of $P_{n,2}$-tableaux of shape λ' and index i, where λ' is the transpose of λ. We point out that the product of the sign character with the inverse Frobenius characteristic applied to the chromatic quasi-symmetric function of Shareshian and Wachs gives the same character as that of the S_n-representation on the cohomology of X_n. Consequently, on account of the sign character, to count the multiplicty of the irreducible associated to λ, we count $P_{n,2}$-tableaux of shape λ'. To eliminate the need to transpose and thereby ease the description of our algorithm, we shall take as our definition of marked tableau the transpose of what is found in [11]. This equality established by Shareshian and Wachs is indirect and relies on q-Eulerian polynomials, chromatic quasisymmetric functions, and Smirnov words. Therefore, they ask for a direct index-preserving combinatorial bijection between marked tableaux and $P_{n,2}$-tableaux. The algorithm we present has some elements in common with the well-known Robinson-Schensted algorithm ([2] and [4]) as well as Stembridge's cryptomorphism found in [11]. For background and notation, we refer the reader to Sagan ([3]), Stanley ([9] and [8]), and Stembridge ([11]).

Given $\lambda \vdash n$ and the Young diagram corresponding to λ, we define a *tableau* T to be a filling of the Young diagram with integers. Given a tableau T, we denote the positive content of T by

$$S^+(T) = \{T(i,j) \mid T(i,j) > 0\}.$$

We say that T is k-admissible if $S^+(T) = [k] = \{1, 2, \ldots, k\}$ for some $k > 0$. For example, the following are tableau of shape $(4, 2, 1)$:

$$T_1 = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline 1 & 3 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \quad \text{and} \quad T_2 = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 3 & 4 \\ \hline 1 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

where T_1 is 3-admissible and T_2 is not admissible. Let T be an admissible tableau with increasing rows and nondecreasing columns. A *marked tableau* is a pair (T, f) where f is a function $f : S^+(T) \to \mathbb{N}$ such that

$$1 \leq f(i) < m_i(T) \quad \text{for all} \quad i \in S^+(T)$$

where $m_i(T)$ is the number of $i's$ occurring in T. Note that the definition implies that each $i \in S^+(T)$ occurs at least twice in T. To visually indicate the values of $f(i)$ for all $i \in S^+(T)$, we will replace one occurrence of i in T by $\hat{\imath}$ such that there are $f(i)$ occurrences of i *above* $\hat{i}$ in T. For example, if

$$(T,f) = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline \hat{1} & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} \hat{2} & \hat{3} & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

then it follows that $f(1) = 1, f(2) = 1$, and $f(3) = 2$.

We define the *index* of a k-admissible marked tableau (T,f), denoted $\text{ind}(T,f)$, to be:

$$\text{ind}(T,f) = \sum_{i=1}^{k} f(i).$$

So for the marked tableau above, we have $\text{ind}(T,f) = f(1)+f(2)+f(3) = 1+1+2 = 4$.

A *Young tableau* T of shape $\lambda \vdash n$ is a tableau filled bijectively with the integers in $[n]$. To define $P_{n,2}$-tableaux, we first define the following partial order on $[n]$. Let $i,j \in [n]$, then $i <_2 j$ if $j - i \geq 2$. With this partial order in hand, we define a $P_{n,2}$-tableau to be a Young tableau P such that

(i) if j appears immediately to the right of i in the same row of P, then $i <_2 j$, and

(ii) if j appears immediately below i in the same column of P, then $i \not>_2 j$.

In other words, the rows of P are increasing and the columns of P are nondecreasing with respect to $<_2$. For example, the following are valid $P_{n,2}$-tableaux of shape $(4,3,3,1)$:

$$P_1 = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 8 & 10 \\ \hline 3 & 5 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & 6 & 9 & \multicolumn{1}{c}{} \\ \cline{1-3} 11 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \text{and} \quad P_2 = \begin{array}{|c|c|c|c|} \hline 3 & 5 & 9 & 11 \\ \hline 2 & 6 & 8 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & 7 & 10 & \multicolumn{1}{c}{} \\ \cline{1-3} 4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

As with marked tableaux, $P_{n,2}$-tableaux carry a notion of index. Let R_i denote the row containing i in P. The index of a $P_{n,2}$-tableau P is

$$\text{ind}(P) = |\{i \mid R_i > R_{i+1}\}|$$

i.e., the number of $i \in P$ that appear south of $i+1$ in P. In the examples above, we have that $\text{ind}(P_1) = 5$ and $\text{ind}(P_2) = 6$. It is worth noting that discussion of $P_{n,2}$-tableaux and the corresponding partial order can take place in the broader context of $(\mathbf{3}+\mathbf{1})$-free posets and incomparability graphs. Analogues of $P_{n,2}$-tableaux are first defined by Gasharov in [1] for a larger class of partial orders called natural unit-interval orders. He uses them to prove the Schur-positivity of the Stanley chromatic symmetric function associated with the natural unit-interval orders. For more on this, see [1], [6], [7], or [10].

Before we present our index-preserving bijection between k-admissible marked tableaux and $P_{n,2}$-tableaux of shape λ, we introduce the notion of $[i,k]$-admissible *skew* marked tableaux which will play a crucial role in the algorithm which defines our bijection. Given a partition $\lambda \vdash n$ and $\mu \subseteq \lambda$ as Young diagrams, then the *skew shape* λ/μ is the set of cells

$$\lambda/\mu = \{c \mid c \in \lambda \quad \text{and} \quad c \notin \mu\}.$$

A *skew tableau* is therefore a filling of a skew shape and an $[i,k]$*-admissible skew tableau* T of shape λ/μ satisfies the condition $S^+(T) = [i,k]$ where $0 < i \leq k$ for integers i, k. If $\lambda = (5,3,2,1)$ and $\mu = (2,2,1)$ then the following is an example of a $[2,4]$-admissible skew tableau of shape λ/μ:

$$\begin{array}{ccccc} & & 2 & 3 & 4 \\ & & 3 & & \\ & 2 & & & \\ 4 & & & & \end{array}$$

A *skew marked tableau* is a pair (T, f) where T is an $[i,k]$-admissible skew tableau in which the following hold:

(i) if j appears to the right of i in the same row of T, then $j > i$,

(ii) if j appears below i in the same column of T, then $j \geq i$, and

(iii) f is a function $f : S^+(T) \to \mathbb{N}$ such that $1 \leq f(j) < m_j(T)$ for all $j \in S^+(T)$.

We indicate the value of $f(j)$ using the same system as with marked tableaux above. The index of a skew marked tableau (T, f) is defined as expected:

$$\operatorname{ind}(T, f) = \sum_{j=i}^{k} f(j).$$

For example, the following is a skew marked tableau of index 4 and shape λ/μ where $\lambda = (4,4,3,3,1)$ and $\mu = (2,2,2,1)$:

$$\begin{array}{cccc} & & 3 & 5 \\ & & 4 & 5 \\ & & \hat{4} & \\ & \hat{3} & 4 & \\ \hat{5} & & & \end{array}$$

We are now able to describe our algorithm which establishes a direct, index-preserving bijective correspondence between marked tableaux of shape λ and $P_{n,2}$-tableaux of shape λ.

2 The bijection between marked tableaux and $P_{n,2}$-tableaux of shape λ

The main result of this paper is the following.

Theorem. *There is a one-to-one correspondence* $(T, f) \overset{\varphi}{\mapsto} P$ *between marked tableaux* (T, f) *of shape* $\lambda \vdash n$ *and* $P_{n,2}$*-tableaux of shape* λ*. Furthermore, we have* $\mathrm{ind}(T, f) = \mathrm{ind}(\varphi(T, f))$.

We define the map φ by iterating a *chain deletion step.* Given a skew marked tableau (T, f) of shape λ/μ which is $[i, k]$-admissible, and a $P_{N,2}$-tableau P of shape μ, the chain deletion step returns an $[i+1, k]$-admissible skew marked tableau (T', f') and a $P_{N+s,2}$-tableau P' which contains P.

Chain Deletion

Let R_l denote the row of P containing the integer l, and let $m = f(i)$. To begin the step, we let $C = \{(x_1, y_1), \ldots, (x_s, y_s)\}$ be the set of positions in T occupied by i's such that $x_1 < x_2 < \cdots < x_s$. Delete the cells in C from T to produce T' and let f' be the restriction of f to $[i + 1, k]$. We now add cells to P in order to construct P'.

Case I: $x_{m+1} > R_N$

To construct P', add cells $(x_{m+1}, y_{m+1}), (x_m, y_m), \ldots, (x_1, y_1)$ to P and fill them with $N + 1, \ldots, N + m + 1$, respectively. We call this an *up fill.* To complete the step, add cells $(x_{m+2}, y_{m+2}), (x_{m+3}, y_{m+3}), \ldots, (x_s, y_s)$ to P' and fill them with $N + m + 2, \ldots, N + s$, we call this a *down fill.*

Case II: $x_{m+1} \le R_N$

As in Case I, we begin with an up fill: to construct P', add cells (x_m, y_m), $(x_{m-1}, y_{m-1}), \ldots, (x_1, y_1)$ to P and fill them with $N+1, \ldots, N+m$, respectively. We now complete the step with a down fill by adding cells $(x_{m+1}, y_{m+1}), (x_{m+2}, y_{m+2})$, $\ldots, (x_s, y_s)$ to P' and filling them with $N + m + 1, \ldots, N + s$.

In all examples that follow, we include empty gray cells of shape μ with each skew marked tableau (T, f) to emphasize that (T, f) is of shape λ/μ. These gray cells are not actually part of the skew marked tableau (T, f) and are only included to aid visualization. We point out that the gray cells are necessarily of the same shape as the $P_{n,2}$-tableaux with which (T, f) is paired in our algorithm.

For example, if $\lambda = (4, 4, 3)$ and

$$(T,f) = \begin{array}{|c|c|c|c|} \hline \blacksquare & \blacksquare & 2 & 3 \\ \hline \blacksquare & \hat{2} & \hat{3} & 4 \\ \hline \blacksquare & 2 & \hat{4} & \\ \cline{1-3} \end{array} \quad \text{and} \quad P = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 & \\ \cline{1-1} 2 & \\ \cline{1-1} \end{array}$$

then $\mu = (2, 1, 1)$ so $C = \{(1, 3), (2, 2), (3, 2)\}$ where $i = 2$ and $N = 4$. Notice that $x_2 > R_4$ so that Case I applies here. In this case, chain deletion returns the following:

$$(T', f') = \begin{ytableau} *(gray) & *(gray) & *(gray) & 3 \\ *(gray) & *(gray) & \hat{3} & 4 \\ *(gray) & *(gray) & \hat{4} \end{ytableau} \quad \text{and} \quad P' = \begin{ytableau} 1 & 4 & \mathbf{6} \\ 3 & \mathbf{5} \\ 2 & \mathit{7} \end{ytableau}$$

where the bold integers in P' were filled in as part of the up fill and the italicized integer was filled in as part of the down fill.

We now apply chain deletion to (T', f') and P' to exhibit an example in which Case II applies. In this situation, $\mu = (3, 2, 2)$ so $C = \{(1, 4), (2, 3)\}$ where $i = 3$ and $N = 7$. Notice that here $x_2 < R_7$ so that Case II does indeed apply. In this case, chain deletion returns the following:

$$(T'', f'') = \begin{ytableau} *(gray) & *(gray) & *(gray) & *(gray) \\ *(gray) & *(gray) & *(gray) & 4 \\ *(gray) & *(gray) & \hat{4} \end{ytableau} \quad \text{and} \quad P'' = \begin{ytableau} 1 & 4 & 6 & \mathbf{8} \\ 3 & 5 & \mathit{9} \\ 2 & 7 \end{ytableau}$$

where the bold integer in P'' was filled in as part of the up fill and the italicized integer was filled in as part of the down fill. Note that in Case I it can happen that no down fill occurs if $m + 1 = s$ (though an up fill always occurs).

A key property of the chain deletion step is that given a $P_{N,2}$-tableau P, it does indeed return a valid $P_{N+s,2}$-tableau P' of appropriate index. To show this, we maintain the notation used in the definition of chain deletion. That P' is an actual tableau at all follows directly from the defining conditions of marked tableaux, and we omit the details here. We shall show that P' satisfies the necessary conditions on rows and columns, and is of appropriate index.

Since each iteration of the chain deletion step places integers in distinct rows, to show that $i <_2 j$ whenever j appears immediately to the right of i in the same row of P', we only need to show that $N+1$ is not placed in the same row as N. If Case I occurs, then $N+1$ is placed below N. If Case II occurs, then $N+1$ is placed above N which establishes that the row condition holds. To check the column condition, we assume that j appears immediately below i in the same column of P'. Suppose on the contrary that $i >_2 j$. If j and i were both contained in P, then P would not be a valid $P_{N,2}$-tableau. If P contains j but not i, then P again would not be a valid tableau (as it would have a "hole" above j). So we conclude that i and j were both added to P in the chain deletion step that produces P'. Since $i \geq j + 2$ by assumption, it follows that i and j were not placed as part of the same up fill or down fill as these fills place integers sequentially. Since $i > j$, we conclude that i was placed in the down fill and j was placed in the up fill. We arrive at a contradiction as j is below i in P', but all integers placed in an up fill are placed in rows above all integers placed in a down fill. Therefore it follows that $i \not>_2 j$. Finally, we must check that

$$\operatorname{ind}(T, f) + \operatorname{ind}(P) = \operatorname{ind}(T', f') + \operatorname{ind}(P').$$

By the definition of chain deletion we have that $\operatorname{ind}(T', f') = \operatorname{ind}(T, f) - m$, so we must show that $\operatorname{ind}(P') = \operatorname{ind}(P) + m$. We note that since the positions of $1, \ldots, N$ in P' are the same as in P, we only need to consider the relative positions of $N, N+1, \ldots, N+s$. If Case I occurs we have

$$R_N < R_{N+1} > R_{N+2} > R_{N+3} > \cdots > R_{N+m+1} < R_{N+m+2} < R_{N+m+3} < \cdots < R_{N+s}$$

which shows that the index of P' is exactly m greater than that of P. If Case II occurs we have

$$R_N > R_{N+1} > \cdots > R_{N+m} < R_{N+m+1} < R_{N+m+2} < \cdots < R_{N+2}$$

which again shows that the index of P' is exactly m greater than that of P as desired.

The Map φ

Now we define the map φ that maps all admissible marked tableaux of shape $\lambda \vdash n$ to $P_{n,2}$-tableaux of shape λ. Let (T, f) be a k-admissible marked tableau of shape λ. If any $0's$ occupy T, their positions must be $(1,1), (2,1), \ldots, (l,1)$ for some $l > 0$ and we remove those from T to produce T_0 and we define P_0 to be the $P_{l,2}$ tableau of shape (1^l) with cell $(x, 1)$ occupied by x. For example if

$$T = \begin{ytableau} 0 & 1 & 2 \\ 0 \\ \hat{1} \\ \hat{2} \end{ytableau}$$

then

$$T_0 = \begin{ytableau} *(gray) & 1 & 2 \\ *(gray) \\ \hat{1} \\ \hat{2} \end{ytableau} \quad \text{and} \quad P_0 = \begin{ytableau} 1 \\ 2 \end{ytableau}.$$

If no 0's occupy T, then set $T_0 = T$ and $P_0 = \emptyset$. We now proceed by iteratively applying chain deletion to (T_0, f) and P_0 (where $N = 0$ if $P_0 = \emptyset$) which yields a sequence of pairs (T_i, f_i) and P_i for $i = 1, 2, \ldots, k$. Observe that this process terminates in k steps, at which point $T_k = \emptyset$, $f_k = \emptyset$, and P_k is a valid $P_{n,2}$ tableaux. We have seen that chain deletion ensures that $\operatorname{ind}(T_i, f_i) + \operatorname{ind}(P_i) = \operatorname{ind}(T_{i+1}, f_{i+1}) + \operatorname{ind}(P_{i+1})$ and since the initialization step does not impact index, it follows that $\operatorname{ind}(T, f) = \operatorname{ind}(\varphi(T, f))$. We therefore define $\varphi(T, f) = P_k$. For example, φ maps the 4-admissible marked tableau

$$(T,f) = \begin{array}{|c|c|c|c|}\hline 0&1&2&3\\\hline 1&\hat{2}&\hat{3}&4\\\hline \hat{1}&2&\hat{4}\\\cline{1-3}\end{array}$$

to the $P_{11,2}$-tableau

$$P_4 = \begin{array}{|c|c|c|c|}\hline 1&4&6&8\\\hline 3&5&9&11\\\hline 2&7&10\\\cline{1-3}\end{array}$$

As an additional example, we apply φ to the set of all 8 marked tableaux of shape $\lambda = (3,2,1)$:

$$\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 0&\hat{1}\\\cline{1-2} \hat{2}\\\cline{1-1}\end{array} \overset{\varphi}{\longmapsto} \begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&4\\\cline{1-2} 5\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 0&1&2\\\hline 0&\hat{2}\\\cline{1-2} \hat{1}\\\cline{1-1}\end{array} \overset{\varphi}{\longmapsto} \begin{array}{|c|c|c|}\hline 1&4&6\\\hline 2&5\\\cline{1-2} 3\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 1&\hat{2}\\\cline{1-2} \hat{1}\\\cline{1-1}\end{array} \overset{\varphi}{\longmapsto} \begin{array}{|c|c|c|}\hline 1&4&6\\\hline 3&5\\\cline{1-2} 2\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 0&1&2\\\hline \hat{1}&\hat{2}\\\cline{1-2} 1\\\cline{1-1}\end{array} \overset{\varphi}{\longmapsto} \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 0&1&2\\\hline \hat{1}&2\\\cline{1-2} \hat{2}\\\cline{1-1}\end{array} \overset{\varphi}{\longmapsto} \begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&5\\\cline{1-2} 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 0&1&2\\\hline \hat{1}&\hat{2}\\\cline{1-2} 2\\\cline{1-1}\end{array} \overset{\varphi}{\longmapsto} \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline \hat{1}&\hat{2}\\\cline{1-2} \hat{3}\\\cline{1-1}\end{array} \overset{\varphi}{\longmapsto} \begin{array}{|c|c|c|}\hline 2&4&6\\\hline 1&3\\\cline{1-2} 5\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline \hat{1}&\hat{3}\\\cline{1-2} \hat{2}\\\cline{1-1}\end{array} \overset{\varphi}{\longmapsto} \begin{array}{|c|c|c|}\hline 2&4&6\\\hline 1&5\\\cline{1-2} 3\\\cline{1-1}\end{array}.$$

3 The Inverse of φ

In this section we construct a map π which we claim is the inverse of φ, thereby establishing that φ is indeed a one-to-one correspondence. We define the map π by iterating a *chain insertion step*. Given a skew marked tableau (T,f) of shape λ/μ which is $[i,k]$-admissible, and a $P_{n,2}$-tableau P of shape μ, the chain insertion step returns an $[i-1,k]$-admissible skew marked tableau (T',f') and $P_{n,2}$-tableaux P' which is a sub-tableau of P.

Chain Insertion

As in the previous section, let N be the largest integer in P and let R_l denote the row of P containing the integer l.

(Up-Phase) Determine the maximal sequence such that

$$R_N > R_{N-1} > R_{N-2} > \cdots > R_M.$$

Now remove the cells containing $N, N-1, \ldots, M$ from P and fill the corresponding cells in T with $i-1$.

(Down-Phase)

Case A: $N = M$

Determine the maximal sequence such that

$$R_{N-1} < R_{N-2} < \cdots < R_K.$$

Remove the cells containing $N, N-1, \ldots, K$ from P and fill the corresponding cells in T with $i-1$ and set $f'(j) = f(j)$ for $j \in [i, k]$, and $f'(i-1) = N - K$.

Case B: $N > M$

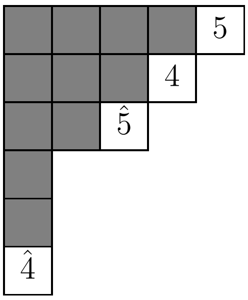
Determine the maximal sequence

$$R_{M-1} < R_{M-2} < \cdots < R_K \quad \text{and} \quad R_K < R_{M+1}.$$

Remove the cells containing $M, M-1, \ldots, K$ from P and fill the corresponding cells in T with $i-1$ and set $f'(j) = f(j)$ for $j \in [i, k]$, and

$$f'(i-1) = \begin{cases} M - K & R_K > R_{K-1} \\ M - K + 1 & R_K < R_{K-1}. \end{cases}$$

For example, if $\lambda = (5, 4, 3, 1, 1, 1)$ and

$(T, f) =$  and $P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 10 \\ \hline 2 & 5 & 9 \\ \cline{1-3} 4 & 7 \\ \cline{1-2} 8 \\ \cline{1-1} 11 \\ \cline{1-1} \end{array}$

then $\mu = (4, 3, 2, 1, 1)$ and (T, f) is $[4, 5]$-admissible. Applying chain insertion to this pair yields the new pair

$$(T', f') = \begin{array}{|c|c|c|c|c|}\hline \cellcolor{gray} & \cellcolor{gray} & \cellcolor{gray} & \mathbf{3} & 5 \\ \hline \cellcolor{gray} & \cellcolor{gray} & \mathit{3} & 4 \\ \cline{1-4} \cellcolor{gray} & \cellcolor{gray} & \hat{\mathit{5}} \\ \cline{1-3} \hat{\mathit{3}} \\ \cline{1-1} \mathbf{3} \\ \cline{1-1} \hat{4} \\ \cline{1-1} \end{array} \quad \text{and} \quad P' = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2} \end{array}$$

where the bold integers in (T', f') were filled in as part of the up-phase and the italicized integers as part of the down-phase. Notice that since $R_8 > R_7$, we have that $f'(3) = 10 - 8 = 2$.

As another example, we apply chain insertion to (T', f') and P' which yields the pair

$$(T'', f'') = \begin{array}{|c|c|c|c|c|}\hline \cellcolor{gray} & \cellcolor{gray} & \mathbf{2} & 3 & 5 \\ \hline \cellcolor{gray} & \mathit{2} & 3 & 4 \\ \cline{1-4} \cellcolor{gray} & \hat{\mathbf{2}} & \hat{5} \\ \cline{1-3} \hat{3} \\ \cline{1-1} 3 \\ \cline{1-1} \hat{4} \\ \cline{1-1} \end{array} \quad \text{and} \quad P'' = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

where the bold integers in (T'', f'') were filled as part of the up-phase and the italicized integer as part of the down-phase. Notice that since $R_5 < R_4$, we have that $f''(2) = 6 - 5 + 1 = 2$.

The Map π

Now we define the map π that maps all $P_{n,2}$-tableaux of shape $\lambda \vdash n$ to marked tableaux of shape λ. Let be a $P_{n,2}$-tableau of shape λ. We proceed by iteratively applying chain insertion beginning with the empty skew tableau $(\emptyset, \emptyset)$ and P which yields a sequence of pairs $(T^{(i)}, f^{(i)})$ and $P^{(i)}$ for $i = 1, 2, 3 \ldots$ This inverse process carries with it a technical difficulty: to start the algorithm, we do not yet know the largest integer that should occur in $\pi(P)$. To remedy this, we initially place the indeterminate l into $T^{(1)}$ during the first chain insertion and adjust values in our output when the algorithm terminates. We halt iteration after the kth step if either

(a) $P^{(k)} = \emptyset$, or

(b) the remaining integers $1, \ldots, N$ in $P^{(k)}$ are arranged such that $R_N > R_{N-1} > \cdots > R_1$.

If (a) occurs, then $(T^{(k)}, f^{(k)})$ is a $[l-k+1, l]$-admissible marked tableaux of shape λ. By replacing l by the integer k in $T^{(k)}$ we obtain T, a k-admissible marked tableaux.

We then set $f(j) = f^{(k)}(l-k+j)$ for $j \in [k]$ thereby producing a valid marked tableau (T, f). We then define $\pi(P) = (T, f)$.

If (b) occurs, then it has to be the case that $P^{(k)}$ is of shape (1^N) and the $(i, 1)$ cell of $P^{(k)}$ is occupied by i. Here we form a $[l-k+1, l]$-admissible marked tableaux $(T', f^{(k)})$ of shape λ by adding the cells $(1, 1), (2, 1), \ldots, (N, 1)$ to $T^{(k)}$ and filling them with 0's. By replacing l by the integer k in T' we obtain T, a k-admissible marked tableaux. We then set $f(j) = f^{(k)}(l-k+j)$ for $j \in [k]$ thereby producing a valid marked tableau (T, f). We then define $\pi(P) = (T, f)$.

For example, if P is the following $P_{n,2}$-tableau of shape $\lambda = (5, 4, 3, 1, 1, 1)$:

$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 6 & 10 & 15 \\ \hline 2 & 5 & 9 & 13 \\ \cline{1-4} 4 & 7 & 14 \\ \cline{1-3} 8 \\ \cline{1-1} 11 \\ \cline{1-1} 12 \\ \cline{1-1} \end{array}$$

then the iterations of chain insertion in our algorithm terminate with

$$(T^{(5)}, f^{(5)}) = \begin{array}{|c|c|c|c|c|}\hline \blacksquare & l-4 & l-3 & l-2 & l \\ \hline \widehat{l-4} & l-3 & l-2 & l-1 \\ \cline{1-4} l-4 & \widehat{l-3} & \hat{l} \\ \cline{1-3} \widehat{l-2} \\ \cline{1-1} l-2 \\ \cline{1-1} \widehat{l-1} \\ \cline{1-1} \end{array} \quad \text{and} \quad P^{(5)} = \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

so that after applying case (b) above, the algorithm returns

$$\pi(P) = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 5 \\ \hline \hat{1} & 2 & 3 & 4 \\ \cline{1-4} 1 & \hat{2} & \hat{5} \\ \cline{1-3} \hat{3} \\ \cline{1-1} 3 \\ \cline{1-1} \hat{4} \\ \cline{1-1} \end{array}.$$

Observe that the two cases for the termination of the algorithm defining π correspond to the two cases in the initialization of the algorithm defining φ. In particular, φ begins with a down fill if and only if π ends with the up-phase. As above, we remark that the defining conditions on skew marked and $P_{n,2}$ tableaux ensure that at every stage the algorithm returns a valid pair of skew marked and $P_{n,2}$-tableaux. Finally, we note that by the definition of chain insertion, we have that

$$\mathrm{ind}(T^{(i)}, f^{(i)}) + \mathrm{ind}(P^{(i)}) = \mathrm{ind}(T^{(i+1)}, f^{(i+1)}) + \mathrm{ind}(P^{(i+1)})$$

and since the algorithm termination step does not impact index, we have that $\mathrm{ind}(P) = \mathrm{ind}(\pi(P))$.

4 Proof of Theorem

In this final section, we establish that π is indeed the inverse of φ, and that we have therefore produced an index-preserving bijection between the sets of marked tableaux and $P_{n,2}$-tableaux of a given shape λ. We shall show that application of chain deletion followed by chain insertion, and vice versa, leave pairs of tableaux $(T, f), P$ unchanged.

To begin, let (T, f) be a skew marked tableau of shape λ/μ which is $[i, k]$-admissible with $f(i) = m$, and P a $P_{n,2}$-tableau of shape μ with largest entry N. We first establish that if we apply chain deletion and then chain insertion to $(T, f), P$, the result is $(T, f), P$.

Carrying all notation from the description of the chain deletion step, we apply chain deletion to produce $(T', f'), P'$. In the instance of Case I, we have that $x_{m+1} > R_N$. We note that the product of chain deletion, (T', f'), is $[i+1, k]$-admissible and P' has largest entry $N + s$. We now apply chain insertion to produce $(T'', f''), P''$ by first considering the maximal sequence such that

$$R_{N+s} > R_{N+s-1} > R_{N+s-2} > \cdots > R_M.$$

From the definition of chain deletion, it follows that $M = N + m + 1$. Remove the cells containing $N + s, N + s - 1, \ldots, N + m + 1$ from P' and fill those cells in T'' with i. In the event that Case A of chain insertion occurs, i.e. $N + s = N + m + 1$, we have that $s = m + 1$. We now consider the maximal sequence

$$R_{N+m} < R_{N+m-1} < \cdots < R_K$$

and observe that $K = N + 1$ as $R_N < x_{m+1}$ and the integers $N + m + 1, N + m, \ldots, N + 1$ were placed in rows $x_1 < x_2 < \cdots x_{m+1}$, respectively. We now remove the cells containing $N+m, N+m-1, \ldots, N+1$ from P' and fill those cells in T'' with i and set $f''(j) = f'(j)$ for $j \in [i+1, k]$ and $f''(i) = N + m + 1 - (N + 1) = m$ which forces $f'' = f$. Since we have added the same cells to T' that were removed from T and filled them with $i's$ to produce T'', it follows that $(T'', f'') = (T, f)$. Similarly, since the cells removed from P' are those that were added to P in producing P'', we have that $P'' = P$, and therefore $(T'', f''), P'' = (T, f), P$ as desired. In the event that Case B of chain insertion occurs, we have that $N + s > N + m + 1$. We now consider the maximal sequence

$$R_{N+m} < R_{N+m-1} < \cdots < R_K \quad \text{and} \quad R_K < R_{N+m+2}$$

and observe that $K = N+1$ as $R_N < x_{m+1} = R_{N+1}$ and the integers $N+m+1, N+m, \ldots, N+1$ were placed in rows $x_1 < x_2 < \cdots x_{m+1}$, respectively. We now remove the cells containing $N+m, N+m-1, \ldots, N+1$ from P' and fill those cells in T'' with i and set $f''(j) = f'(j)$ for $j \in [i+1, k]$ and $f''(i) = N+m+1-(N+1) = m$ which forces $f'' = f$. Since we have added the same cells to T' that were removed from T and filled them with $i's$ to produce T'', it follows that $(T'', f'') = (T, f)$. Similarly, since the cells removed from P' are those that were added to P in producing P'', we have that $P'' = P$, and therefore $(T'', f''), P'' = (T, f), P$ as desired.

We now consider Case II of chain deletion and mention that Case A of chain insertion cannot follow as a down fill must occur. In other words, in this situation we have that

$$R_{N+1} > R_{N+2} > \cdots > R_{N+m} < R_{n+m+1}.$$

The arguments for Case B are similar to those above with a small difference: cells containing $N+s, \ldots, N+m$ are first removed from P' and the corresponding cells are added to T' and filled with i's. Note that $M = N+m$ so then the cells containing $N+m-1, \ldots, N+1$ are removed from P' because $R_{N+1} < R_{M+1}$ but $R_N \geq R_{M+1}$, and the corresponding cells are added to T' and filled with $i's$. Since $R_{N+1} < R_N$ in this case, we have that $f''(i) = N+m-(N+1)+1 = m$ and it follows that $(T'', f''), P'' = (T, f), P$ as above. This establishes that chain deletion followed by chain insertion leaves $(T, f), P$ unchanged. It remains to show that chain insertion followed by chain deletion leaves $(T, f), P$ unchanged.

Carrying all notation from the description of the chain insertion step, we apply chain insertion to $(T, f), P$ to produce $(T', f'), P'$. In the instance of Case A, we have that $N = M$, and denote the set of cells containing $N, N-1, \ldots, K$ by $C = \{(x_1, y_1), (x_2, y_2), \ldots, (x_{N-K+1}, y_{N-K+1})\}$ with $x_1 < x_2 < \cdots < x_{N-K+1}$. The cells in C are removed from P with those same cells added to T and filled with $i-1$ and $f'(i-1) = N-K$. We note that $R_K > R_{K-1}$ in P, and now apply chain deletion where it follows that Case I must apply as $x_{N-K+1} > R_{K-1}$ where we point out that $K-1$ is the largest value in P'. The chain deletion step removes all cells containing $i-1$ from T' to produce T'' and consequently $f'' = f$, i.e. $(T'', f'') = (T, f)$. The step also adds the cells $(x_{N-K+1}, y_{N-K+1}), (x_{N-K}, y_{N-K}), \ldots, (x_1, y_1)$ back to P' and fills them with $K, \ldots, N$, respectively (no down fill occurs in this case), thereby producing $P'' = P$. We have therefore established that $(T'', f''), P'' = (T, f), P$. To complete the proof, we must show that the same holds when Case B of chain insertion occurs followed by chain deletion.

In the instance of Case B of chain insertion, we have that $N > M$. We denote the cells containing $N, N-1, \ldots, K$, by $C = \{(x_1, y_1), (x_2, y_2), \ldots, (x_{N-K+1}, y_{N-K+1}\}$ with $x_1 < x_2 < \cdots < x_{N-K+1}$ so that $M, M-1, \ldots, K$ occupy cells $(x_1, y_1), (x_2, y_2), \ldots, (x_{M-K+1}, y_{M-K+1})$, respectively and $M+1$, $M+2$, $\ldots$, N occupy cells $(x_{M-K+2}, y_{M-K+2}), (x_{M-K+3}, y_{M-K+3}), \ldots, (x_{N-K+1}, y_{N-K+1})$, respectively. The cells in C are removed from P with those same cells added to T and filled with $i-1$ to produce $(T', f'), P'$ where $f'(i-1) = M-K$ or $f'(i-1) = M-K+1$ as $R_K > R_{K-1}$ or $R_K < R_{K-1}$. When chain deletion is applied to $(T', f'), P'$ all cells in T' containing $i-1$, i.e. the cells in C, are removed from T' to produce T'' and consequently $f'' = f$

which forces $(T, f) = (T'', f'')$. Those same cells in C are added back to P' to produce P'', but how they are filled depends on the value of $f'(i-1)$. If $f'(i-1) = M - K$, then Case I of chain deletion is applied to $(T', f'), P'$ as $x_{M-K+1} = R_K > R_{K-1}$ (in P) where $K-1$ is the largest integer in P'. In this situation, cells (x_{M-K+1}, y_{M-K+1}), $(x_{M-K}, y_{M-K}), \ldots, (x_1, y_1)$ are filled with $K, K+1, \ldots, M$ in the up fill and then cells

$$(x_{M-K+2}, y_{M-K+2}), (x_{M-K+3}, y_{M-K+3}), \ldots, (x_{N-K+1}, y_{N-K+1})$$

are filled with $M+1, M+2, \ldots, N$ in the down fill which gives us that $P'' = P$. If $f'(i-1) = M - K + 1$, then Case II of chain deletion is applied to $(T', f'), P'$ as $x_{M-K+1} = R_K < R_{K-1}$ (in P). In this situation, cells

$$(x_{M-K+1}, y_{M-K+1}), (x_{M-K}, y_{M-K}), \ldots, (x_1, y_1)$$

are filled with $K, K+1, \ldots, M$ in the up fill, and then cells

$$(x_{M-K+2}, y_{M-K+2}), (x_{M-K+3}, y_{M-K+3}), \ldots, (x_{N-K+1}, y_{N-K+1})$$

are filled with $M+1, M+2, \ldots, N$ in the downfill which again gives us that $P'' = P$.

The above, along with the fact that the initialization step in φ and the termination step in π are inverse processes, establishes that π is indeed the inverse of φ, thereby establishing the desired result.

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References

[1] V. Gasharov, Incomparability graphs of (3+1)-free posets are s-positive, In: *Proc. 6th Conf. on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*, vol. 157 (1996), 193–197.

[2] G. de B. Robinson, On the Representations of the Symmetric Group, *Amer. J. Math.* 60(3) (1938), 745–760.

[3] B. E. Sagan, *The symmetric group*, *Graduate Texts in Mathematics* Vol. 203, Springer-Verlag, New York, second ed., 2001, Representations, combinatorial algorithms, and symmetric functions.

[4] C. Schensted, Longest increasing and decreasing subsequences, *Canadian J. Math.* 13 (1961), 179–191.

[5] J. Shareshian and M. L. Wachs, Chromatic quasisymmetric functions and Hessenberg varieties, in: *Configuration spaces*, *CRM Series* Vol. 14 (2012), 433–460. Ed. Norm., Pisa.

[6] R. P. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, *Adv. Math.* 111(1) (1995), 166–194.

[7] R. P. Stanley, Graph colorings and related symmetric functions: ideas and applications: a description of results, interesting applications, & notable open problems, Selected papers in honor of Adriano Garsia (Taormina, 1994), *Discrete Math.* 193 (1-3) (1998), 267–286.

[8] R. P. Stanley, *Enumerative combinatorics Vol. 2*, *Cambridge Studies in Advanced Mathematics* Vol. 62 (1999), Cambridge University Press, Cambridge.

[9] R. P. Stanley, *Enumerative combinatorics Vol. 1*, *Cambridge Studies in Advanced Mathematics* Vol. 49 (2012) Cambridge University Press, Cambridge, second ed.

[10] R. P. Stanley and J. R. Stembridge, On immanants of Jacobi-Trudi matrices and permutations with restricted position, *J. Combin. Theory Ser. A* 62(2) (1993), 261–279.

[11] J. R. Stembridge, Eulerian numbers, tableaux, and the Betti numbers of a toric variety, *Discrete Math.* 99 (1992), 307–320.

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