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A note on the Turán number of a Berge odd cycle

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Abstract

In this note we obtain upper bounds on the number of hyperedges in 3-uniform hypergraphs not containing a Berge cycle of given odd length. We improve the bound given by Füredi and Özkahya in 2017. The result follows from a more general theorem. We also obtain some new results for Berge cliques.

1 Introduction

We say that a hypergraph $\mathcal{H}$ is a Berge copy of a graph F (in short: $\mathcal{H}$ is a Berge-F) if $V(F) \subset V(\mathcal{H})$ and there is a bijection $f : E(F) \to E(\mathcal{H})$ such that for any $e \in E(F)$ we have $e \subset f(e)$. This definition was introduced by Gerbner and Palmer [11], extending the well-established notion of Berge cycles and paths. Note that there are several non-uniform Berge copies of F, and a hypergraph $\mathcal{H}$ is a Berge copy of several graphs. A particular copy of F defining a Berge-F is called its *core*. Note that there can be multiple cores in a Berge-F.

We denote by $ex_r(n, \text{Berge-}F)$ the largest number of hyperedges in an r-uniform Berge-F-free hypergraph on n vertices. There are several papers dealing with $ex_r(n, \text{Berge-}C_k)$ (e.g. [8, 14, 15, 16]) or $ex_r(n, \text{Berge-}F)$ in general (e.g. [9, 10, 11, 12, 20]). For a short survey on this topic see Subsection 5.2.2 in [13].

In this note we consider $ex_3(n, \text{Berge-}C_k)$. In the case $k = 5$, this was first studied by Bollobás and Győri [2]. They showed $ex_3(n, \text{Berge-}C_5) \leq \sqrt{2}n^{3/2} + 4.5n$. This bound was improved to $(0.254 + o(1))n^{3/2}$ by Ergemlidze, Győri and Methuku [5]. For cycles of any length, Győri and Lemons [15, 16] proved $ex_r(n, \text{Berge-}C_k) = O(n^{1+1/\lfloor k/2 \rfloor})$. The constant factors were improved by Jiang and Ma [18], and in the case k is even by Gerbner, Methuku and Vizer [10]. In the 3-uniform case, Füredi and Özkahya [8] obtained better constant factors (depending on k). In the case k is even, further improvements were obtained by Gerbner, Methuku and Vizer [10] and by Gerbner, Methuku and Palmer [9].

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A closely related area is counting triangles in C_k-free graphs. More generally, let $ex(n,H,F)$ denote the maximum number of copies of H in an F-free graph on n vertices. After some sporadic results, the systematic study of these problems (often called *generalized Turán problems*) was initiated by Alon and Shikhelman [1]. Their connection to Berge hypergraphs was established by Gerbner and Palmer [12], who proved

$$ex(n,K_r,F) \leq ex_r(n,\text{Berge-}F) \leq ex(n,K_r,F) + ex(n,F)$$

for any r, n and F.

Counting triangles in C_k-free graphs and counting hyperedges in Berge-C_k-free 3-uniform hypergraphs was handled together already by Bollobás and Győri [2] for C_5, and by Füredi and Özkahya [8], who proved $ex(n,K_3,C_{2k}) \leq \frac{2k-3}{3}ex(n,C_{2k})$ and $ex_3(n,\text{Berge-}C_{2k}) \leq \frac{2k}{3}ex(n,C_{2k})$. Their upper bound for $ex(n,K_3,C_{2k})$ is still the best known bound, but their other upper bound was improved to $ex_3(n,\text{Berge-}C_{2k}) \leq \frac{2k-3}{3}ex(n,C_{2k})$ by Gerbner, Methuku and Vizer [10] in the case $k \geq 5$ and by Gerbner, Methuku and Palmer [9] in the case $k = 3,4$.

In the case of forbidden cycles of any odd length, the number of triangles was first studied by Győri and Li [17], who proved[1] $ex(n,K_3,C_{2k+1}) \leq \frac{(2k-2)(16k-1)}{3}ex(n,C_{2k})$. It was improved independently by Füredi and Özkahya [8] and by Alon and Shikhelman [1]. The latter had the stronger bound $ex(n,K_3,C_{2k+1}) \leq \frac{16(k-1)}{3}ex(\lceil n/2\rceil,C_{2k})$. In the case $k=2$, the current best bound $ex(n,K_3,C_5) \leq 0.231975n^{3/2}$ is due to Ergemlidze and Methuku [6].

Füredi and Özkahya [8] obtained the currently best upper bound on the Berge version by showing

$$ex_3(n,\text{Berge-}C_{2k+1}) \leq ex(n,K_3,C_{2k+1}) + 4ex(n,C_{2k}) + 12ex_3^{lin}(n,\text{Berge-}C_{2k+1}), \tag{1.1}$$

where $ex_r^{lin}(n,\text{Berge-}F)$ denotes the largest number of hyperedges in an r-uniform Berge-F-free linear hypergraph on n vertices. Recall that a linear hypergraph is one in which any two hyperedges share at most one vertex.

In this note we improve the bound (1.1). Recall that we have $ex_3(n,\text{Berge-}C_{2k+1}) \geq ex(n,K_3,C_{2k+1})$, thus we cannot hope for a huge improvement, especially as $ex(n,K_3,C_{2k+1})$ might be the largest of the three terms. Indeed, the best upper bound currently known is $O(n^{1+1/k})$ for all the three terms, but the dependence of the known upper bound in k is the largest for $ex(n,K_3,C_{2k+1})$ (we will state these bounds after Theorem 1.2).

Recall that in case of C_{2k}, the two upper bounds obtained by Füredi and Özkahya [8] were $ex(n,K_3,C_{2k}) \leq \frac{2k-3}{3}ex(n,C_{2k})$ and $ex_3(n,\text{Berge-}C_{2k}) \leq \frac{2k}{3}ex(n,C_{2k})$, and the Berge bound was improved in [10, 9] to match the generalized Turán bound. Our goal would be to do the same here and get rid of the terms $4ex(n,C_{2k+1}) + 12ex_3^{lin}(n,\text{Berge-}C_{2k+1})$ in (1.1). We cannot achieve that, but we decrease these additional terms. Recall that the currently best bound for the generalized Turán problem

[1]We note that the bound is incorrectly stated in their paper [17].

is $ex(n, K_3, C_{2k+1}) \le \frac{16(k-1)}{3} ex(\lceil n/2 \rceil, C_{2k})$ by Alon and Shikhelman [1]. Our new upper bound on $ex_3(n, \text{Berge-}C_{2k+1})$ is larger than that bound by $ex_3^{lin}(n, \text{Berge-}C_{2k+1})$. We wonder if it is an example of a more general phenomenon and whether similar bounds could be obtained for other graphs.

The way we use the linearity involves subdividing an edge uv, i.e. deleting it and adding uw and vw for a new vertex w. Our method uses only the following two properties of C_{2k+1}: it can be obtained from C_{2k} by subdividing an edge, and deleting a vertex from C_{2k+1} we obtain a path. In the next theorem we state our result in the most general form.

Theorem 1.1. *Let F be a connected graph obtained from F_0 by subdividing an edge and F' be obtained from F by deleting a vertex. Let $c = c(n)$ be such that $ex(n, K_{r-1}, F') \le cn$ for every n. Then we have*

(i) $ex_r(n, \text{Berge-}F) \le ex(n, K_r, F) + 2^{r-1} ex(n, F_0) + ex_r^{lin}(n, \text{Berge-}F)$,

(ii) $ex_r(n, \text{Berge-}F) \le \max\left\{1, \frac{2c}{r}\right\} 2^{r-1} ex(n, F_0) + ex_r^{lin}(n, \text{Berge-}F)$.

In the case $F = C_{2k+1}$ we have $F_0 = C_{2k}$ and $F' = P_{2k}$, the path on $2k$ vertices. A theorem of Luo [19] shows $ex(n, K_{r-1}, P_{2k}) \le \frac{n}{2k-1}\binom{2k-1}{r-1}$, but what we need for the 3-uniform case is the Erdős-Gallai theorem [4] showing $ex(n, P_{2k}) \le (k-1)n$. Using this, **(ii)** of Theorem 1.1 gives $ex_3(n, \text{Berge-}C_{2k+1}) \le \frac{8k-8}{3} ex(n, C_{2k}) + ex_3^{lin}(n, \text{Berge-}C_{2k+1})$ if $k > 2$. We can improve this a little bit.

Theorem 1.2. *If $k > 2$, then*

$$\begin{aligned} &ex_3(n, \text{Berge-}C_{2k+1}) \\ &\quad \le \frac{16k-16}{3} ex(\lceil n/2 \rceil, C_{2k}) + ex_3^{lin}(n, \text{Berge-}C_{2k+1}) \\ &\quad \le \left(\frac{1280k-1280}{3}\sqrt{k}\log k\right) \lceil n/2 \rceil^{1+1/k} + 2kn^{1+1/k} + 9kn + \frac{16k-16}{3} 10k^2 \lceil n/2 \rceil. \end{aligned}$$

The bound in Theorem 1.2 is currently stronger than the bound given by **(i)** of Theorem 1.1 for $F = C_{2k+1}$ and $r = 3$. However, an improvement on $ex(n, K_3, C_{2k+1})$ would immediately improve the bound in **(i)**. Any significant improvement would make **(i)** stronger than Theorem 1.2 for $F = C_{2k+1}$.

The second inequality in Theorem 1.2 follows from known results. Füredi and Özkahya [8] proved $ex_3^{lin}(n, \text{Berge-}C_{2k+1}) \le 2kn^{1+1/k} + 9kn$, and Bukh and Jiang [3] obtained the strongest bound on the Turán number of even cycles by showing $ex(n, C_{2k}) \le 80\sqrt{k}\log k n^{1+1/k} + 10k^2 n$. As we do not have good lower bounds on $ex(n, C_{2k})$, we cannot be sure that the first term is actually the larger term. However, if $ex_3^{lin}(n, \text{Berge-}C_{2k+1})$ is the larger term, then our improvement on the upper bound of $ex_3(n, \text{Berge-}C_{2k+1})$ is more significant, as we changed the constant factor of that term from 12 to 1. Obviously we have $ex_3^{lin}(n, \text{Berge-}C_{2k+1}) \le ex_3(n, \text{Berge-}C_{2k+1})$, hence further improvement is impossible here.

We prove Theorem 1.1 by combining the ideas of [8] and [1] with the methods developed in [9, 10]. In the next section we state some lemmas needed for the proof. We give a new proof of a lemma by Gerbner, Methuku and Palmer [9], and we strengthen the lemma a little bit. This strengthens results on $ex_r(n, \text{Berge-}K_k)$ for some values of r, k and n. In Section 3 we prove Theorems 1.1 and 1.2.

2 Lemmas

We say that a graph G is red-blue if each of its edges is colored with one of the colors red and blue. For a red-blue graph G, we denote by G_{red} the subgraph spanned by the red edges and G_{blue} the subgraph spanned by the blue edges. For two graphs H and G we denote by $N(H,G)$ the number of subgraphs of G that are isomorphic to H. Let $g_r(G) = |E(G_{red})| + N(K_r, G_{blue})$.

Lemma 2.1 (Gerbner, Methuku, Palmer [9])**.** *For any graph F and integers r and n, there is a red-blue F-free graph G on n vertices, such that $ex_r(n, \text{Berge-}F) \leq g_r(G)$.*

Note that an essentially equivalent version was obtained by Füredi, Kostochka and Luo [7]. The proof of Lemma 2.1 relies on a lemma about bipartite graphs (hidden in the proof of Lemma 2 in [9]). If M is a matching and ab is an edge in M, then with a slight abuse of notation we say $M(a) = b$ and $M(b) = a$.

Lemma 2.2. *Let Γ be a finite bipartite graph with parts A and B and let M be a largest matching in Γ. Let B' denote the set of vertices in B that are incident to M. Then we can partition A into A_1 and A_2 and partition B' into B_1 and B_2 such that for $a \in A_1$ we have $M(a) \in B_1$, and every neighbor of the vertices of A_2 is in B_2.*

Here we present a proof that is built on the same principle, but is somewhat simpler than the proof found in [9]. Before that, let us recall the well-known notion of alternating paths. Given a bipartite graph Γ and a matching M in it, a path P in Γ is called *alternating* if its first edge is not in M, and then it alternates between edges in M and edges not in M, finishing with an edge not in M. It is well-known and easy to see that deleting the edges of P from M and replacing them with the edges of P that were not in M, we obtain another matching, that is larger than M.

Proof. First we build a set $V' \subset V(\Gamma)$ in the following way. Let V_0 be the set of vertices in A that are not incident to any edges of M. Then in the first step we add to V_0 the set of vertices in B that are neighbors of a vertex in V_0, to obtain V_1. In the second step we add to V_1 the vertices in A that are connected to a vertex in V_1 by an edge in M, to obtain V_2. Similarly, in the ith step, if i is odd we add to V_{i-1} the set of vertices in B that are neighbors of a vertex in V_{i-1}, while if i is even, we add to V_{i-1} the vertices in A that are connected to a vertex in V_{i-1} by an edge in M (i.e. $M(b)$ for some $b \in B \cap V_{i-1}$), to obtain V_i. After finitely many steps, V_i does not increase anymore, let V' be the resulting set of vertices.

We claim that no vertex from $B \setminus B'$ can be in V'. Indeed, such a vertex could be reached by an alternating path from a vertex in A that is not incident to M, thus M is not a largest matching, a contradiction.

Then let $A_2 = A \cap V'$, $A_1 = A \setminus A_2$, $B_2 = B' \cap V'$ and $B_1 = B' \setminus B_2$. A vertex in A_2 cannot be connected to a vertex v not in B_2, as v could be added to V' then. Similarly, for a vertex $u \in A_1$, $M(u)$ has to be in B_1, otherwise $M(u)$ is in B_2 and then u can be added to V'. □

Let us briefly describe how we can apply this lemma to obtain Lemma 2.1. We take a Berge-F-free r-uniform hypergraph $\mathcal{H}$ on n vertices. Let A be the set of hyperedges in $\mathcal{H}$ and B be the set of sub-edges of these hyperedges (by edge and sub-edge we always mean an edge of size two, i.e. a pair of vertices). We connect $a \in A$ to $b \in B$ if $a \supset b$. Let Γ denote this auxiliary bipartite graph. Let M be an arbitrary largest matching and B' be the vertices of B incident to the edges in M. It is easy to see that the elements of B' form an F-free graph which we call G. Indeed, otherwise M defines the bijection between a copy of F and hyperedges in $\mathcal{H}$ to form a Berge-F.

Now we apply Lemma 2.2 to Γ and M. We define a red-blue coloring of G by taking the edges of G in B_1 to be the red edges, and the edges of G in B_2 to be the blue edges. We have $|\mathcal{H}| = |A_1|+|A_2| = |B_1|+|A_2| = |E(G_{red})|+|A_2|$. As hyperedges in A_2 have all their neighbors in B_2, they each contain a blue K_r, which is distinct from the other blue r-cliques obtained this way, showing $|A_2| \le N(K_r, G_{blue})$.

Let us remark here that Lemma 2.2 also gives some information on the structure of G. If there is $a \in A_1$ that has a neighbor $b \in B \setminus B'$, then we could obtain another matching M' by changing the neighbor of a to b, i.e. $M'(a) = b$ and if $a' \neq a$, then $M'(a') = M(a')$. Then B' is replaced by $B'' = B' \setminus \{M(a)\} \cup \{b\}$. In this case the same partition of A into A_1 and A_2, and the partition of B'' into B_2 and $B'' \setminus B_2$ satisfies Lemma 2.2. This means for G that we can delete the (red) edge $M(a)$ and replace it with the edge b, to obtain another F-free graph.

If on the other hand the vertices in A_1 have all their neighbors in B', then we could recolor the red edges to blue. Therefore, in G we can delete an edge and add another edge so that the resulting graph is still F-free. Let $\alpha = \alpha_{F,n}$ be the largest value of $g_r(G')$, where G' is an n-vertex F-free blue-red graph. Assume that each n-vertex F-free blue-red graph G' with $g_r(G') = \alpha$ is not monoblue and we cannot delete an edge and add another edge to G' so that the resulting graph is still F-free. Then by the above, G cannot be one of these graphs, thus $ex_r(n, \text{Berge-}F) \le g_r(G) < \alpha$. This is usually a negligible improvement, as we often do not even know the order of magnitude.

However, if $F = K_k$, Gerbner, Methuku and Palmer [9] proved that $\alpha_{K_k,n} = \max\{g_r(T_B(n, k-1)), g_r(T_R(n, k-1))\}$, where $T_B(n, k-1)$ is the monoblue Turán graph $T(n, k-1)$ and $T_R(n, k-1)$ is the monored Turán graph $T(n, k-1)$. We mention without going into the details that their proof also shows that for any other graphs G we have $g_r(G) < \alpha_{K_k,n}$. As we cannot delete an edge from $T(n, k-1)$ and add another edge to obtain a K_k-free graphs, we do have an improvement. For example, if $r = 4$ and $k = 5$, then the result in [9] determines $ex_4(n, \text{Berge-}K_k)$ for $n \ge 11$. For $n = 10$, $T(10, 4)$ has 36 copies of K_4 and 37 edges. Therefore, (as $ex(n, K_r, F)$ is a lower bound on $ex_r(n, \text{Berge-}F)$), we have $36 \le ex_4(n, \text{Berge-}K_k) \le 37$. With our new observation, we know $ex_4(n, \text{Berge-}K_k) = 36$.

3 Proof of Theorems 1.1 and 1.2

Let $\mathcal{H}$ be a Berge-F-free r-graph on n vertices. We say that an edge uv with $u, v \in V(\mathcal{H})$ is t-heavy if u, v are contained together in exactly t hyperedges. First we will build a linear subhypergraph $\mathcal{H}_1$ in a greedy way: if we can find a hyperedge H that does not share an edge with any hyperedge in $\mathcal{H}_1$, we add H to $\mathcal{H}_1$, and then repeat this procedure. By definition, $\mathcal{H}_1$ is linear. Let $\mathcal{H}_2$ consist of the remaining hyperedges. Note that $|\mathcal{H}| = |\mathcal{H}_1| + |\mathcal{H}_2| \leq ex_r^{lin}(n, \text{Berge-}F) + |\mathcal{H}_2|$, and the remainder of the proof is for proving the needed upper bound on $|\mathcal{H}_2|$.

We build an auxiliary bipartite graph Γ in the usual way: let A be the set of hyperedges in $\mathcal{H}_2$ and B be the set of sub-edges of these hyperedges. We connect $a \in A$ to $b \in B$ if $a \supset b$. We will let M be a largest matching in Γ, however, we do not choose M arbitrarily. Let M_0 be an arbitrary largest matching in Γ. Let B' be the set of vertices in B that are incident to some edge of M_0 and A_0 denote the set of vertices in A that are incident to some edge of M_0. Now a hyperedge $a \in A_0$ contains a sub-edge $M_0(a)$, at least one sub-edge b_0 shared with a hyperedge in $\mathcal{H}_1$, maybe some sub-edges that are matched to some other $a' \in A$, and maybe some other sub-edges $b \in B \setminus B'$. We have the option to replace in M_0 the edge between a and $M_0(a)$ with any of the edges of Γ between a and an unused sub-edge of a, to obtain another largest matching. We will build a largest matching M, that contains the same vertices (A_0) from A as M_0.

For $a \in A_0$, we pick $M(a)$ to be one of the sub-edges $b \in B$ of a (potentially we let $M(a) = M_0(a)$) in the following way: $M(a)$ should share exactly one vertex with b_0 (where b_0 is a sub-edge that is also a sub-edge of a hyperedge in $\mathcal{H}_1$) if possible. We go through the hyperedges greedily; as long as there is a hyperedge $a \in A_0$ such that $M_0(a)$ can be changed in this way, we execute the change (it is possible that $M_0(a)$ cannot be changed originally, but later a sub-edge of a that is $M_0(a')$ becomes free to use, when $M(a')$ is chosen to be different from $M_0(a')$). This process finishes after finitely many (at most $|A_0|$) steps, as we change $M_0(a)$ to $M(a)$ at most once for every $a \in A_0$. After this, we rename the unchanged $M_0(a)$ to $M(a)$.

The resulting matching M has the following property: for every $a \in A_0$, a shares a sub-edge b_0 with a hyperedge in $\mathcal{H}_1$, such that that either $M(a)$ shares exactly one vertex with b_0, or all the sub-edges of a sharing exactly one vertex with b_0 are $M(a')$ for some $a' \in A_0$.

Now we can apply Lemma 2.2 to Γ and M to obtain A_1, A_2, B_1, B_2. Let us call the elements of B_1 *red* edges and the elements of B_2 *blue* edges. Let G be the graph consisting of all the red and blue edges. Then G is obviously F-free.

Let us now take a random partition of $V(\mathcal{H})$ into V_1 and V_2. For every $a \in A_0$, we look at $b = M(a)$. If the two vertices of b are in one part, and all the other vertices of a are in the other part, we keep a, otherwise we delete it. Let A^* denote the set of elements in A that are not deleted (note that elements in $A \setminus A_0$ are never deleted, thus are in A^*). Let G' be the graph consisting of the elements of B' that are connected by an edge in M to an element of A^*. Then G' is obviously F-free, as it is a subgraph of G.

Claim 3.1. *G' is F_0-free, where F_0 is any graph for which F can be obtained from F_0 by subdividing an edge of F_0.*

Proof. Let us assume we are given a copy Q of F_0 in G' such that uv is the edge that needs to be subdivided to obtain F. Observe that there is no edge between V_1 and V_2 in G', thus Q is in one of them, say V_1. Let w be a vertex of $M(uv)$ with $u \neq w \neq v$, then $w \in V_2$, thus w is not in Q.

We say that a hyperedge H in $\mathcal{H}$ is *good* if H contains u and w for some $w \in M(uv) \setminus \{u, v\}$ and H is not $M(e)$ for any edge e of Q. If there is a good hyperedge, then we build a Berge-F with the following core: we subdivide uv with w. For each edge e of this core we assign $M(e)$ except for uw (where we assign H) and vw (where we assign $M(uv)$). This way we obtain a Berge-F, a contradiction.

$M(uv)$ shares at least one sub-edge with a hyperedge $H \in \mathcal{H}_1$. If the sub-edge shares exactly one vertex with uv, then H is good and we are done. Thus every sub-edge of $M(uv)$ shared with a hyperedge in $\mathcal{H}_1$ has to contain none or both of u and v. In both cases, when we tried to change $M_0(M(uv))$ when constructing M, we failed, because all such edges are matched to some other hyperedges of $\mathcal{H}_2$. In particular, uw is $M(a)$ for some $a \in A_0$ and for some $w \in M(u, v) \setminus \{u, v\}$. Observe that w is in V_2, thus $M(a)$ has vertices from both parts V_1 and V_2, hence a cannot be in A^* by the definition of A^*. This implies a is good, finishing the proof. □

The above claim implies G' has at most $ex(n, F_0)$ edges. For an arbitrary $a \in A$, the probability that a is in A^* is at least $1/2^{r-1}$. Let S be any subset of A, then we have that the expected value of the number of hyperedges in $A^* \cap S$ is at least $|S|/2^{r-1}$, thus there is a partition with $|A^* \cap S| \geq |S|/2^{r-1}$.

There are $|B_1| = |A_1|$ red edges in G, and there is a random partition where at least $|A_1|/2^{r-1}$ elements of A_1 are undeleted, hence there are at least $|A_1|/2^{r-1}$ red edges in G'. This implies $|A_1|/2^{r-1} \leq ex(n, F_0)$. Hence there are at most $2^{r-1}ex(n, F_0)$ red edges altogether. For the total number of edges in G we can use the same argument: there is a random partition where at least $|A_0|/2^{r-1}$ hyperedges in A_0 are undeleted, thus for the G' defined by that partition, we have $|A_0| = |E(G)| \leq 2^{r-1}|E(G')| \leq 2^{r-1}ex(n, F_0)$.

Observe that we have $|\mathcal{H}_2| = |A_1| + |A_2| \leq |A_1| + N(K_r, G_{blue}) \leq |A_1| + ex(n, K_r, F)$, hence we are done with the proof of **(i)**.

Note that G is not necessarily F_0-free, but it is F-free. Let m be the number of blue edges in G, then G has at most $2^{r-1}ex(n, F_0) - m$ red edges. An argument of Gerbner, Methuku and Vizer [10] bounds the number of r-cliques in F-free graphs with the given number of vertices and edges. For sake of completeness, we include the argument here.

Let $d(v)$ be the degree of v in G_{blue}. Obviously the neighborhood of every vertex in G_{blue} is F'-free. An F'-free graph on $d(v)$ vertices contains at most $ex(d(v), K_{r-1}, F') \leq cd(v)$ copies of K_{r-1}. Thus v is contained in at most $cd(v)$

copies of K_r in $G_{\rm blue}$. If we sum, for each vertex, the number of K_r's containing a vertex, then each K_r is counted r times. On the other hand as $\sum_{v\in V(G_{\rm blue})} d(v) = 2|E(G_{blue})| = 2m$, we have $\sum_{v\in V(G_{\rm blue})} cd(v) = 2cm$. This gives that the number of blue K_r's is at most $2cm/r$. Thus we have

$$\begin{aligned} g_r(G) &\leq 2^{r-1}ex(n,F_0) - m + 2cm/r \\ &\leq \max\left\{1, \frac{2c}{r}\right\}(2^{r-1}ex(n,F_0) - m + m) \\ &= \max\left\{1, \frac{2c}{r}\right\}2^{r-1}ex(n,F_0). \end{aligned}$$

The above inequality, together with Lemma 2.1, implies that

$$|\mathcal{H}_2| \leq \max\left\{1, \frac{2c}{r}\right\}2^{r-1}ex(n,F_0),$$

finishing the proof of **(ii)**.

Now we show how to obtain the small improvement needed to prove Theorem 1.2. It is based on the proof of the upper bound on $ex(n,K_3,C_{2k+1})$ in [1]. If n is odd, replace it by $n+1$. As the stated upper bound is the same in both cases, obvious mononicity conditions show we can do this. Thus we can assume n is even. When we take the random partition into V_1 and V_2, first we take a random partition into $n/2$ sets $U_1,\dots,U_{n/2}$ of size 2, and then randomly put one vertex into V_1 and the other into V_2. The obtained graph G' will be C_{2k}-free, and it is divided into two components, hence it has at most $ex(|V_1|,C_{2k}) + ex(|V_2|,C_{2k})$ edges. The way we chose V_1 ensures the above sum is $2ex(\lceil n/2\rceil,C_{2k})$. Then we can go through every step of the remaining part of the proof to obtain the result we need, if for an arbitrary $a\in A$, the probability that a is in A^* is still at least $1/2^{r-1} = 1/4$. We will separate into cases according to the intersection of a with the parts U_i. In case the three vertices of a are in three different U_i's, the probability is 1/4. In case a contains U_i for some i, there are two cases. If $M(a) = U_i$, then the probability is 0, otherwise it is 1/2. As $M(a) = U_i$ happens with probability 1/3 (having the condition that a contains U_i), for every i we have that the probability of a being in A^* if a contains U_i is $\frac{2}{3}\cdot\frac{1}{2} \geq 1/4$.

This gives the first inequality of Theorem 1.2. As we have mentioned after the statement, the second inequality follows from earlier results, stated there.

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