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A note on strong edge choosability of toroidal subcubic graphs

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Abstract

Let G be a graph. A proper edge-coloring of G is called a strong edge-coloring if any two edges on a path of length at most three receive distinct colors. Given a list assignment $L = \{L(e) \mid e \in E(G)\}$ of G, if there exists a strong edge-coloring π of G such that $\pi(e) \in L(e)$ for all $e \in E(G)$, then we say that G is strongly L-edge-colorable. If G is strongly L-edge-colorable for any list assignment L with $|L(e)| \geq k$ for all $e \in E(G)$, then G is strongly k-edge-choosable. It is known that every planar subcubic graph is strongly 10-edge-choosable. In this paper, by applying the famous Combinatorial Nullstellensatz, we extend this result by showing that every toroidal subcubic graph is strongly 10-edge-choosable.

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1 Introduction

All graphs considered in this paper are finite and simple unless otherwise stated. For a graph G, we let $V(G)$, $E(G)$, and $\Delta(G)$, denote the vertex set, edge set, and the maximum degree of G, respectively. We say a graph G *subcubic* if $\Delta(G) \leq 3$.

A *proper k-edge-coloring* of G is a mapping $\pi : E(G) \to \{1, \ldots, k\}$ such that for any adjacent edges e_1 and e_2, $\pi(e_1) \neq \pi(e_2)$. A *strong k-edge-coloring* of G is a proper k-edge-coloring such that any two edges adjacent to a common edge have distinct colors. The *strong chromatic index* of G, denoted by $\chi'_s(G)$, is the smallest integer k such that there is a strong k-edge-coloring in G.

Given a list assignment $L = \{L(e) \mid e \in E(G)\}$ of G, if G has a strong edge-coloring π such that $\pi(e) \in L(e)$ for all $e \in E(G)$, then we say that G is *strongly L-edge-colorable*. Call such a strong edge coloring π a *strong L-edge-coloring* of G. If G is strongly L-edge-colorable for all list assignments L of G satisfying that $|L(e)| \geq k$ for all $e \in E(G)$, then G is called *strongly k-edge-choosable*. The smallest integer k for which G is strongly k-edge-choosable is the *strong edge choosability* of G, denoted by $ch'_s(G)$. Obviously, $\chi'_s(G) \leq ch'_s(G)$ for any graph G.

The strong edge-coloring of graphs was first studied by Fouquet and Jolivet [11, 12] who investigated the case of 3-regular graphs. In 1989, Erdős and Nešetřil [9, 10] put forward the following challenging conjecture:

Conjecture 1.1 [9, 10] *Let G be a graph with maximum degree Δ. Then*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{when } \Delta \text{ is even;} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1), & \text{when } \Delta \text{ is odd.} \end{cases}$$

They also gave a construction to show that if the conjecture is true, then the bound is tight. Let $\mathcal{G}_m$ denote the family of graphs with maximum degree m. Conjecture 1.1 is clearly true for both $\mathcal{G}_1$ and $\mathcal{G}_2$. Andersen [2], and independently Horák, Qing and Trotter [13] confirmed the conjecture for $\mathcal{G}_3$. Namely, every graph $G \in \mathcal{G}_3$ satisfies $\chi'_s(G) \leq 10$. Later, Dai et al. [7] showed that $ch'_s(G) \leq 11$ if $G \in \mathcal{G}_3$, and further $ch'_s(G) \leq 10$ if $G \in \mathcal{G}_3$ and G is planar. For every graph G in $\mathcal{G}_4$, Cranston [6] proved that $\chi'_s(G) \leq 22$. This was further strengthened by Zhang et al. in [18] who showed $ch'_s(G) \leq 22$. Recently, Huang, Santana and Yu [14] successfully showed that $\chi'_s(G) \leq 21$. Meanwhile, Wang et al. [17] showed that every planar graph $G \in \mathcal{G}_4$ satisfies $\chi'_s(G) \leq 19$. This was recently improved to be $ch'_s(G) \leq 19$ in [5]. Since Conjecture 2.1 is still widely open, it is natural to ask if there exists a positive real number k such that $\chi'_s(G) \leq k\Delta^2$ when Δ is sufficiently large. Molloy and Reed [16] proved that such a k exists and $k = 1.998$. This result was later improved to $k = 1.93$ by Bruhn and Joos [4], and to $k = 1.835$ by Bonamy, Perrett and Postle [3]. Recently, this has been improved to $k = 1.772$ by Hurley, de Joannis de Verclos and Kang in [8]. The reader may refer to [15, 19] for more results relating to strong (list) edge-colorings of graphs.

In this paper we study strong edge choosability of toroidal graphs, which are graphs that can be drawn on the torus without crossing edges. The main theorem is the following, which extends a result in [7] that states every planar subcubic graph is strongly 10-edge-choosable.

Theorem 1.1 *If G is a toroidal subcubic graph, then $ch'_s(G) \leq 10$.*

2 Preliminaries

Before proving our main result, we need to introduce some necessary notation and terminology. Suppose that $G = (V, E, F)$ is a toroidal graph embedded on the torus with the face set F. We use i^+ to denote an integer at least i. Similarly define i^- to be an integer at most i. A *k-vertex* (*k^+-vertex*, *k^--vertex*, respectively) is a vertex of degree k (at least k, at most k, respectively). The same notation can be applied to cycles and faces. Let π be a partial strong L-edge-coloring of G. Note that for each proper subgraph of G, even if two colored edges are adjacent to an uncolored edge, the colors must be different under π. For $e, e' \in E(G)$, we say that e can *see* e' (with respect to π) if e and e' are either adjacent to each other or adjacent to a common edge. Furthermore, if e can see an edge e' with a color c, then we say that e *sees* the color c.

Let $P(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables, where $n \geq 1$. Let $c_p(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n})$ denote the coefficient of the monomial $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in $P(x_1, x_2, \ldots, x_n)$, where for $1 \leq i \leq n$, k_i is a nonnegative integer. To derive our result, we need the following elegant formulation of the Combinatorial Nullstellensatz.

Lemma 2.1 ([1], Combinatorial Nullstellensatz) *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Suppose that the degree of P, denoted by deg(P), equals $\sum_{i=1}^{n} k_i$, where each k_i is a nonnegative integer, and suppose $c_p(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}) \neq 0$. If $S_1, S_2, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > k_i$, then there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that $P(s_1, s_2, \ldots, s_n) \neq 0$.*

3 Proof of Theorem 1.1

Suppose Theorem 1.1 is false. Let $G = (V, E)$ be a toroidal graph which is not strongly 10-edge-choosable but every proper subgraph of G is. Clearly, G is connected. Embedding G into the torus, we get a toroidal graph $G = (V, E, F)$, where F is the face set of G. First, we state the following Lemma 3.1, whose proof was provided in [7].

Lemma 3.1 [7] *G is a 3-regular graph without any 5^--cycles.*

In what follows, let L be a list assignment of G with $|L(e)| \geq 10$ for all $e \in E(G)$.

Lemma 3.2 *G has no 6-cycles.*

Proof. Suppose to the contrary that G contains a 6-cycle $C = v_1v_2v_3v_4v_5v_6v_1$. By Lemma 3.1, C is an induced 6-cycle. Namely, for each $i \in \{1,\dots,6\}$, the third neighbor of v_i, denoted by v_i', cannot be on the boundary of C. Let x_i and y_i denote the two neighbors of v_i' other than v_i, as depicted in Figure 1. Also let x_i' and x_i'' (respectively y_i' and y_i'') denote the two neighborhoods of x_i (respectively y_i) other than v_i'. Again, by Lemma 3.1, we see that neither x_i nor y_i can be located on C.

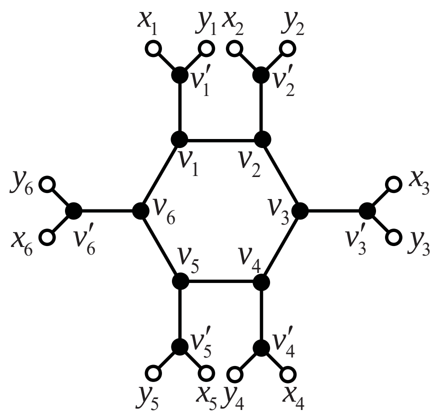

Figure 1: The configuration of the 6-cycle $C = v_1v_2v_3v_4v_5v_6v_1$.

Let $H = G - \{v_i : i \in \{1,\dots,5\}\}$. By the minimality of G, H admits a strong L-edge-coloring π. For our convenience, we denote $e_i = v_iv_{i+1}$, where $i \in \{1,\dots,6\}$ and i is taken modulo 6; and $e_j = v_{j-6}v'_{j-6}$, where $j \in \{7,8,\dots,11\}$. For each e_k, where $k \in \{1,\dots,11\}$, let $C_\pi(e_k)$ denote the set of colors seen by e_k, while let $S_k = L(e_k) \setminus C_\pi(e_k)$ for any $e_k \in E(G)$. Clearly, as there are no 5^--cycles in G, we note that the edge $v_1'v_i'$ does not exist for any $i \in \{2,3,5,6\}$. Similarly, the edge $v_2'v_j'$ does not exist for any $j \in \{3,4,6\}$. Associate with e_k a variable z_k. Next, by symmetry, we shall discuss three cases based on the existence of the edges $v_1'v_4'$ and $v_2'v_5'$.

Case 1: $v_1'v_4' \notin E(G)$ and $v_2'v_5' \notin E(G)$.

In what follows, if e_i and e_j are distinct edges in G, with $i,j \in \{1,\dots,11\}$, so that e_i and e_j see each other, then we represent this by using the binomial $(z_i - z_j)$. Thus we obtain the following polynomial Q_1:

$$\begin{aligned}
&Q_1(z_1,z_2,\dots,z_{11}) = \\
&(z_1-z_2)(z_1-z_3)(z_1-z_5)(z_1-z_6)(z_1-z_7)(z_1-z_8)(z_1-z_9)(z_2-z_3)(z_2-z_4)\\
&(z_2-z_6)(z_2-z_7)(z_2-z_8)(z_2-z_9)(z_2-z_{10})(z_3-z_4)(z_3-z_5)(z_3-z_8)(z_3-z_9)\\
&(z_3-z_{10})(z_3-z_{11})(z_4-z_5)(z_4-z_6)(z_4-z_9)(z_4-z_{10})(z_4-z_{11})(z_5-z_6)(z_5-z_7)\\
&(z_5-z_{10})(z_5-z_{11})(z_6-z_7)(z_6-z_8)(z_6-z_{11})(z_7-z_8)(z_8-z_9)(z_9-z_{10})(z_{10}-z_{11}).
\end{aligned}$$

Notice that $\deg(Q_1) = 36$. We observe the following:

$$\begin{aligned}
C_\pi(e_1) &= \{\pi(v_1'x_1), \pi(v_1'y_1), \pi(v_2'x_2), \pi(v_2'y_2), \pi(v_6v_6')\};\\
C_\pi(e_2) &= \{\pi(v_2'x_2), \pi(v_2'y_2), \pi(v_3'x_3), \pi(v_3'y_3)\};\\
C_\pi(e_3) &= \{\pi(v_3'x_3), \pi(v_3'y_3), \pi(v_4'x_4), \pi(v_4'y_4)\};\\
C_\pi(e_4) &= \{\pi(v_4'x_4), \pi(v_4'y_4), \pi(v_5'x_5), \pi(v_5'y_5), \pi(v_6v_6')\};\\
C_\pi(e_5) &= \{\pi(v_5'x_5), \pi(v_5'y_5), \pi(v_6v_6'), \pi(v_6'x_6), \pi(v_6'y_6)\};\\
C_\pi(e_6) &= \{\pi(v_1'x_1), \pi(v_1'y_1), \pi(v_6v_6'), \pi(v_6'x_6), \pi(v_6'y_6)\};\\
C_\pi(e_7) &= \{\pi(v_1'x_1), \pi(v_1'y_1), \pi(x_1x_1'), \pi(x_1x_1''), \pi(y_1y_1'), \pi(y_1y_1''), \pi(v_6v_6')\};\\
C_\pi(e_8) &= \{\pi(v_2'x_2), \pi(v_2'y_2), \pi(x_2x_2'), \pi(x_2x_2''), \pi(y_2y_2'), \pi(y_2y_2'')\};\\
C_\pi(e_9) &= \{\pi(v_3'x_3), \pi(v_3'y_3), \pi(x_3x_3'), \pi(x_3x_3''), \pi(y_3y_3'), \pi(y_3y_3'')\};\\
C_\pi(e_{10}) &= \{\pi(v_4'x_4), \pi(v_4'y_4), \pi(x_4x_4'), \pi(x_4x_4''), \pi(y_4y_4'), \pi(y_4y_4'')\};\\
C_\pi(e_{11}) &= \{\pi(v_5'x_5), \pi(v_5'y_5), \pi(x_5x_5'), \pi(x_5x_5''), \pi(y_5y_5'), \pi(y_5y_5''), \pi(v_6v_6')\}.
\end{aligned}$$

Since $|L(e_i)| \geq 10$, we deduce that $|S_i| \geq 3$ for $i \in \{7, 11\}$, $|S_i| \geq 4$ for $i \in \{8, 9, 10\}$, $|S_i| \geq 5$ for $i \in \{1, 4, 5, 6\}$, and $|S_i| \geq 6$ for $i \in \{2, 3\}$. By Python (the code is in the Appendix), we calculate that $c_{Q_1}(x_1^4x_2^5x_3^5x_4^4x_5^4x_6^4x_7^2x_8^2x_9^2x_{10}^2x_{11}^2) = -2$ and $\sum_{i=1}^{11} k_i = 36$. Since $k_i < |S_i|$ for each $i \in \{1, 2, \ldots, 11\}$, by Lemma 2.1, we get a desired strong L-edge-coloring of G, a contradiction.

Case 2: $v_1'v_4' \notin E(G)$ and $v_2'v_5' \in E(G)$.

Then e_8 and e_{11} are at distance exactly 2, as shown in Figure 2. We have the following polynomial Q_2:

$$\begin{aligned}
&Q_2(z_1, z_2, \ldots, z_{11}) =\\
&(z_1 - z_2)(z_1 - z_3)(z_1 - z_5)(z_1 - z_6)(z_1 - z_7)(z_1 - z_8)(z_1 - z_9)(z_2 - z_3)(z_2 - z_4)\\
&(z_2 - z_6)(z_2 - z_7)(z_2 - z_8)(z_2 - z_9)(z_2 - z_{10})(z_3 - z_4)(z_3 - z_5)(z_3 - z_8)(z_3 - z_9)\\
&(z_3 - z_{10})(z_3 - z_{11})(z_4 - z_5)(z_4 - z_6)(z_4 - z_9)(z_4 - z_{10})(z_4 - z_{11})(z_5 - z_6)\\
&(z_5 - z_7)(z_5 - z_{10})(z_5 - z_{11})(z_6 - z_7)(z_6 - z_8)(z_6 - z_{11})(z_7 - z_8)(z_8 - z_9)\\
&(z_8 - z_{11})(z_9 - z_{10})(z_{10} - z_{11}).
\end{aligned}$$

Observe that $\deg(Q_2) = 37$. We have the following:

$C_\pi(e_1) = \{\pi(v_1'x_1), \pi(v_1'y_1), \pi(v_2'x_2), \pi(v_2'v_5'), \pi(v_6v_6')\}$;

$C_\pi(e_2) = \{\pi(v_2'x_2), \pi(v_2'v_5'), \pi(v_3'x_3), \pi(v_3'y_3)\}$;

$C_\pi(e_3) = \{\pi(v_3'x_3), \pi(v_3'y_3), \pi(v_4'x_4), \pi(v_4'y_4)\}$;

$C_\pi(e_4) = \{\pi(v_4'x_4), \pi(v_4'y_4), \pi(v_5'x_5), \pi(v_5'v_2'), \pi(v_6v_6')\}$;

$C_\pi(e_5) = \{\pi(v_5'x_5), \pi(v_5'v_2'), \pi(v_6v_6'), \pi(v_6'x_6), \pi(v_6'y_6)\}$;

$C_\pi(e_6) = \{\pi(v_1'x_1), \pi(v_1'y_1), \pi(v_6v_6'), \pi(v_6'x_6), \pi(v_6'y_6)\}$;

$C_\pi(e_7) = \{\pi(v_1'x_1), \pi(v_1'y_1), \pi(x_1x_1'), \pi(x_1x_1''), \pi(y_1y_1'), \pi(y_1y_1''), \pi(v_6v_6')\}$;

$C_\pi(e_8) = \{\pi(v_2'x_2), \pi(v_2'v_5'), \pi(x_2x_2'), \pi(x_2x_2''), \pi(v_5'x_5)\}$;

$C_\pi(e_9) = \{\pi(v_3'x_3), \pi(v_3'y_3), \pi(x_3x_3'), \pi(x_3x_3''), \pi(y_3y_3'), \pi(y_3y_3'')\}$;

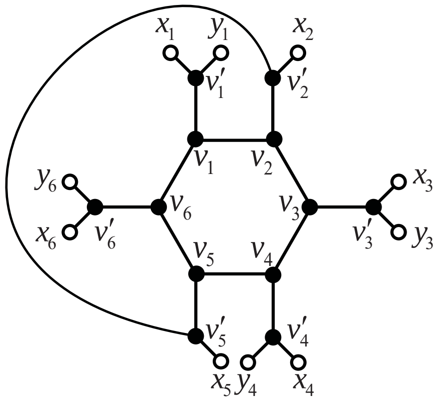

Figure 2: The configuration of Case 2.

$C_\pi(e_{10}) = \{\pi(v_4'x_4), \pi(v_4'y_4), \pi(x_4x_4'), \pi(x_4x_4''), \pi(y_4y_4'), \pi(y_4y_4'')\}$;
$C_\pi(e_{11}) = \{\pi(v_5'x_5), \pi(v_5'v_2'), \pi(x_5x_5'), \pi(x_5x_5''), \pi(v_2'x_2), \pi(v_6v_6')\}$.

Since $|L(e_i)| \geq 10$, we deduce that $|S_7| \geq 3$, $|S_i| \geq 4$ for $i \in \{9, 10, 11\}$, $|S_i| \geq 5$ for $i \in \{1, 4, 5, 6, 8\}$, and $|S_i| \geq 6$ for $i \in \{2, 3\}$. By Python (the code is in the Appendix), we calculate that $c_{Q_2}(x_1^4x_2^5x_3^5x_4^4x_5^4x_6^4x_7^2x_8^2\ x_9^2x_{10}^2x_{11}^3) = 2$ and $\sum_{i=1}^{11} k_i = 37$. Therefore, by Lemma 2.1, we obtain a desired strong L-edge-coloring of G, a contradiction.

Case 3: $v_1'v_4' \in E(G)$ and $v_2'v_5' \in E(G)$.

Then e_7 and e_{10} are at distance exactly 2, and also e_8 and e_{11} are at distance exactly 2; this is depicted in Figure 3. We have the following polynomial Q_3:

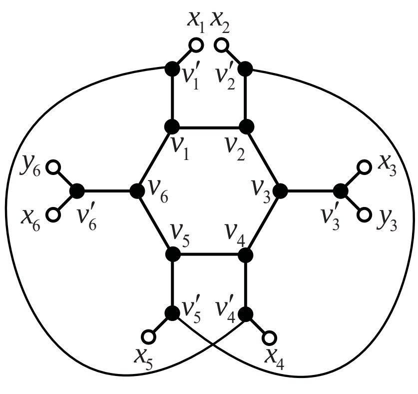

Figure 3: The configuration of Case 3.

$Q_3(z_1, z_2, \ldots, z_{11}) =$
$(z_1 - z_2)(z_1 - z_3)(z_1 - z_5)(z_1 - z_6)(z_1 - z_7)(z_1 - z_8)(z_1 - z_9)(z_2 - z_3)(z_2 - z_4)$
$(z_2 - z_6)(z_2 - z_7)(z_2 - z_8)(z_2 - z_9)(z_2 - z_{10})(z_3 - z_4)(z_3 - z_5)(z_3 - z_8)(z_3 - z_9)$
$(z_3 - z_{10})(z_3 - z_{11})(z_4 - z_5)(z_4 - z_6)(z_4 - z_9)(z_4 - z_{10})(z_4 - z_{11})(z_5 - z_6)$
$(z_5 - z_7)(z_5 - z_{10})(z_5 - z_{11})(z_6 - z_7)(z_6 - z_8)(z_6 - z_{11})(z_7 - z_8)(z_7 - z_{10})$
$(z_8 - z_9)(z_8 - z_{11})(z_9 - z_{10})(z_{10} - z_{11})$.

Note that $\deg(Q_3) = 38$. It is easy to observe the following:

$C_\pi(e_1) = \{\pi(v_1'x_1), \pi(v_1'v_4'), \pi(v_2'x_2), \pi(v_2'v_5'), \pi(v_6v_6')\}$;
$C_\pi(e_2) = \{\pi(v_2'x_2), \pi(v_2'v_5'), \pi(v_3'x_3), \pi(v_3'y_3)\}$;
$C_\pi(e_3) = \{\pi(v_3'x_3), \pi(v_3'y_3), \pi(v_4'x_4), \pi(v_4'v_1')\}$;
$C_\pi(e_4) = \{\pi(v_4'x_4), \pi(v_4'v_1'), \pi(v_5'x_5), \pi(v_5'v_2'), \pi(v_6v_6')\}$;
$C_\pi(e_5) = \{\pi(v_5'x_5), \pi(v_5'v_2'), \pi(v_6v_6'), \pi(v_6'x_6), \pi(v_6'y_6)\}$;
$C_\pi(e_6) = \{\pi(v_1'x_1), \pi(v_1'v_4'), \pi(v_6v_6'), \pi(v_6'x_6), \pi(v_6'y_6)\}$;
$C_\pi(e_7) = \{\pi(v_1'x_1), \pi(v_1'v_4'), \pi(x_1x_1'), \pi(x_1x_1''), \pi(v_4'x_4), \pi(v_6v_6')\}$;
$C_\pi(e_8) = \{\pi(v_2'x_2), \pi(v_2'v_5'), \pi(x_2x_2'), \pi(x_2x_2''), \pi(v_5'x_5)\}$;
$C_\pi(e_9) = \{\pi(v_3'x_3), \pi(v_3'y_3), \pi(x_3x_3'), \pi(x_3x_3''), \pi(y_3y_3'), \pi(y_3y_3'')\}$;
$C_\pi(e_{10}) = \{\pi(v_4'x_4), \pi(v_4'v_1'), \pi(x_4x_4'), \pi(x_4x_4''), \pi(v_1'x_1)\}$;
$C_\pi(e_{11}) = \{\pi(v_5'x_5), \pi(v_5'v_2'), \pi(x_5x_5'), \pi(x_5x_5''), \pi(v_2'x_2), \pi(v_6v_6')\}$.

Since $|L(e_i)| \geq 10$, we have that $|S_i| \geq 4$ for $i \in \{7, 9, 11\}$, $|S_i| \geq 5$ for $i \in \{1, 4, 5, 6, 8, 10\}$, and $|S_i| \geq 6$ for $i \in \{2, 3\}$. By Python (the code is in the Appendix), we calculate that $c_{Q_3}(x_1^4x_2^5x_3^5x_4^4x_5^4x_6^4x_7^3x_8^3x_9^2x_{10}^2x_{11}^2) = -2$. As $\sum_{i=1}^{11} k_i = 38$ and $k_i < |S_i|$ for each $i \in \{1, 2, \ldots, 11\}$, by Lemma 2.1, one may reach a strong L-edge-coloring of G, a contradiction. □

We now prove Theorem 1.1:

Euler's formula can be rewritten in the following identity:

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = 0.$$

By Lemmas 3.1 and 3.2, we confirm that there is no 6^--face in G, and therefore

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) > 0,$$

which leads to a contradiction and thus we complete the proof of Theorem 1.1. □

Appendix

% Case 1 of Lemma 3.2:

from sympy import Symbol, expand

$z1$ = Symbol('$z1$')
$z2$ = Symbol('$z2$')
$z3$ = Symbol('$z3$')
$z4$ = Symbol('$z4$')
$z5$ = Symbol('$z5$')
$z6$ = Symbol('$z6$')
$z7$ = Symbol('$z7$')
$z8$ = Symbol('$z8$')
$z9$ = Symbol('$z9$')
$z10$ = Symbol('$z10$')
$z11$ = Symbol('$z11$')

$p1 = (z_1 - z_2) * (z_1 - z_3) * (z_1 - z_5) * (z_1 - z_6) * (z_1 - z_7) * (z_1 - z_8) * (z_1 - z_9)$
$p2 = (z_2 - z_3) * (z_2 - z_4) * (z_2 - z_6) * (z_2 - z_7) * (z_2 - z_8) * (z_2 - z_9) * (z_2 - z_{10})$
$p3 = (z_3 - z_4) * (z_3 - z_5) * (z_3 - z_8) * (z_3 - z_9) * (z_3 - z_{10}) * (z_3 - z_{11})$
$p4 = (z_4 - z_5) * (z_4 - z_6) * (z_4 - z_9) * (z_4 - z_{10}) * (z_4 - z_{11})$
$p5 = (z_5 - z_6) * (z_5 - z_7) * (z_5 - z_{10}) * (z_5 - z_{11})$
$p6 = (z_6 - z_7) * (z_6 - z_8) * (z_6 - z_{11})$
$p7 = (z_7 - z_8)$
$p8 = (z_8 - z_9)$
$p9 = (z_9 - z_{10})$
$p10 = (z_{10} - z_{11})$

print((expand((expand((expand((expand((expand((expand((expand((expand
((expand((expand((expand($p1$).
coeff($z1$,4))*$p2$).coeff($z2$,5))*$p3$).coeff($z3$,5))*$p4$).coeff($z4$,4))*$p5$).coeff($z5$,4))*$p6$).
coeff($z6$,4))*$p7$). coeff($z7$,2))*$p8$).coeff($z8$,2))*$p9$).coeff($z9$,2))*$p10$).
coeff($z10$,2))).coeff($z11$,2)))

% Case 2 of Lemma 3.2:

from sympy import Symbol, expand

$z1$ = Symbol('$z1$')
$z2$ = Symbol('$z2$')
$z3$ = Symbol('$z3$')

$z4$ = Symbol('$z4$')
$z5$ = Symbol('$z5$')
$z6$ = Symbol('$z6$')
$z7$ = Symbol('$z7$')
$z8$ = Symbol('$z8$')
$z9$ = Symbol('$z9$')
$z10$ = Symbol('$z10$')
$z11$ = Symbol('$z11$')

$p1 = (z_1 - z_2) * (z_1 - z_3) * (z_1 - z_5) * (z_1 - z_6) * (z_1 - z_7) * (z_1 - z_8) * (z_1 - z_9)$
$p2 = (z_2 - z_3) * (z_2 - z_4) * (z_2 - z_6) * (z_2 - z_7) * (z_2 - z_8) * (z_2 - z_9) * (z_2 - z_{10})$
$p3 = (z_3 - z_4) * (z_3 - z_5) * (z_3 - z_8) * (z_3 - z_9) * (z_3 - z_{10}) * (z_3 - z_{11})$
$p4 = (z_4 - z_5) * (z_4 - z_6) * (z_4 - z_9) * (z_4 - z_{10}) * (z_4 - z_{11})$
$p5 = (z_5 - z_6) * (z_5 - z_7) * (z_5 - z_{10}) * (z_5 - z_{11})$
$p6 = (z_6 - z_7) * (z_6 - z_8) * (z_6 - z_{11})$
$p7 = (z_7 - z_8)$
$p8 = (z_8 - z_9) * (z_8 - z_{11})$
$p9 = (z_9 - z_{10})$
$p10 = (z_{10} - z_{11})$

print((expand((expand((expand((expand((expand((expand((expand((expand
((expand((expand((expand($p1$).
coeff($z1$,4))*$p2$).coeff($z2$,5))*$p3$).coeff($z3$,5))*$p4$).coeff($z4$,4))*$p5$).coeff($z5$,4))*$p6$).
coeff($z6$,4))*$p7$).coeff($z7$,2))*$p8$).coeff($z8$,2))*$p9$).coeff($z9$,2))*$p10$).
coeff($z10$,2))).coeff($z11$,3)))

% Case 3 of Lemma 3.2:

from sympy import Symbol, expand

$z1$ = Symbol('$z1$')
$z2$ = Symbol('$z2$')
$z3$ = Symbol('$z3$')
$z4$ = Symbol('$z4$')
$z5$ = Symbol('$z5$')
$z6$ = Symbol('$z6$')
$z7$ = Symbol('$z7$')
$z8$ = Symbol('$z8$')
$z9$ = Symbol('$z9$')
$z10$ = Symbol('$z10$')
$z11$ = Symbol('$z11$')

$p1 = (z_1 - z_2) * (z_1 - z_3) * (z_1 - z_5) * (z_1 - z_6) * (z_1 - z_7) * (z_1 - z_8) * (z_1 - z_9)$
$p2 = (z_2 - z_3) * (z_2 - z_4) * (z_2 - z_6) * (z_2 - z_7) * (z_2 - z_8) * (z_2 - z_9) * (z_2 - z_{10})$
$p3 = (z_3 - z_4) * (z_3 - z_5) * (z_3 - z_8) * (z_3 - z_9) * (z_3 - z_{10}) * (z_3 - z_{11})$
$p4 = (z_4 - z_5) * (z_4 - z_6) * (z_4 - z_9) * (z_4 - z_{10}) * (z_4 - z_{11})$
$p5 = (z_5 - z_6) * (z_5 - z_7) * (z_5 - z_{10}) * (z_5 - z_{11})$
$p6 = (z_6 - z_7) * (z_6 - z_8) * (z_6 - z_{11})$
$p7 = (z_7 - z_8) * (z_7 - z_{10})$
$p8 = (z_8 - z_9) * (z_8 - z_{11})$
$p9 = (z_9 - z_{10})$
$p10 = (z_{10} - z_{11})$

print((expand((expand((expand((expand((expand((expand((expand((expand((expand((expand((expand(*p*1).
coeff(*z*1,4))**p*2).coeff(*z*2,5))**p*3).coeff(*z*3,5))**p*4).coeff(*z*4,4)) **p*5).coeff(*z*5,4))**p*6).
coeff(*z*6,4))**p*7).
coeff(*z*7,3))**p*8).coeff(*z*8,3))**p*9).coeff(*z*9,2))**p*10).coeff(*z*10,2))).coeff(*z*11,2)))

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