AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **79(3)** (2021), Pages 424–436

The number of s-separated k-sets in various circles

Emiliano J.J. Estrugo   Adrián Pastine

Instituto de Matemática Aplicada San Luis (IMASL)
Universidad Nacional de San Luis, CONICET
Ejército de los Andes 950 (D5700HHW)
San Luis
Argentina

juan.estrugo.tag@gmail.com   adrian.pastine.tag@gmail.com

Abstract

This article studies the number of ways of selecting k objects arranged in p circles of sizes $n_0, \dots, n_{p-1}$ such that no two selected ones have less than s objects between them. If $n_i \geq sk+1$ for all $0 \leq i \leq p-1$, this number is shown to be $\frac{n_0+\dots+n_{p-2}}{k}\binom{n_0+\dots+n_{p-2}-sk-1}{k-1}$. A combinatorial proof of this claim is provided, and two convolution formulas due to Rothe are obtained as corollaries.

1 Introduction

The study of separated sets in circles and lines has been of great interest in combinatorics throughout the years. The first important result was in 1943, when Kaplansky [19] proved that the number of ways to choose k elements arranged in a circle (respectively, a line) of n elements such that no two are consecutive is $\frac{n}{n-k}\binom{n-k}{k}$ (respectively, $\binom{n-k+1}{k}$). Later Konvalina [20] showed that the number of ways of selecting k elements arranged in a circle of n elements, with no two selected elements having unit separation, i.e. having exactly one object between them, is $\binom{n-k}{k} + \binom{n-k-1}{k-1}$ if $n \geq 2k+1$.

Afterwards Mansour and Sun [21] found that the number of ways of selecting k out of n elements arrayed in a circle such that no two selected ones are separated by $m-1, 2m-1, \dots, qm-1$ elements is given by $\frac{n}{n-qk}\binom{n-qk}{k}$, where $m, q \geq 1$ and $n \geq mqk+1$. New proofs of this result were given in [5] and [9]. In particular [5] provides a combinatorial proof, using partitions of $\mathbb{Z}_n$. The technique using partitions was further explored in [11].

Recently, in [6] the problem was generalized to the study of sets where no two elements are at even distance, or no two elements are at odd distance.

In [14, 15] Holroyd and Johnson introduced a problem concerning k-element sets in a circle of size n, with the additional condition that no two elements can be at

ISSN: 2202-3518

distance less than or equal to s. Such sets are said to be *s-separated*. This problem was solved in 2003 by Talbot [23]. The main result in [23] is an Erdős-Ko-Rado-type theorem, which characterizes the maximum-sized families of s-separated sets satisfying the condition that no two are disjoint. These sort of results are called Erdős-Ko-Rado-type because of [7], where Erdős, Ko and Rado bounded the size of any family of pairwise intersecting k-element sets of a base set of size n and characterized the families of maximum size. Nevertheless, we are interested in one of the lemmas from said work, which states that the number of s-separated k-element subsets of $[n] = \{1, \dots, n\}$ containing a fixed element is $\binom{n-sk-1}{k-1}$.

Another way to look at this type of problem, which is on the rise, is the study of independent sets of graphs; see [1, 2, 3, 4, 8, 12, 13, 16, 17, 18, 24]. In this sense, the results in [19], restated into a graph-theoretical language, say that the number of independent sets of size k on a cycle (respectively, a path) with n vertices is $\frac{n}{n-k}\binom{n-k}{k}$ (respectively, $\binom{n-k+1}{k}$). In the same way, the results in [5, 9, 11, 20, 21] can be restated in terms of the k-element independent sets of the corresponding graph. Nevertheless, most of the results are geared towards studying Erdős-Ko-Rado-type theorems. In the case of [23], the result could be restated as a characterization of maximum-sized families of pairwise intersecting independent k-element sets of the sth power of a cycle.

Given a graph G and a vertex v, the k-star at v is the family of independent k-element sets that contain the vertex v. A graph is said to be *k-starred* if the size of a maximum intersecting family of k-element independent sets coincides with the maximum size of a k-star. In [13], Hilton and Spencer proved that the union of p vertex-disjoint powers of cycles is k-starred if a condition related to the clique numbers of the different components is satisfied. Afterwards, in [12], Hilton, Holroyd and Spencer proved that the union of two powers of cycles is k-starred, without needing the condition on the clique numbers. Nevertheless, the size of the stars, and thus the maximum size of an intersecting family, was not provided. Stars have been studied on their own on trees [3].

Our aim is to study the number of k-element s-separated sets in p circles of sizes $n_0, \dots, n_{p-1}$. In Section 2 we study the number of k-element s-separated sets having a fixed element in two circles of sizes n_0 and n_1. In Section 3 we generalize this result to p circles of sizes $n_0, \dots, n_{p-1}$. Then, in Section 4 we use the results obtained with a fixed element to count the number of k-element s-separated sets in p circles of sizes $n_0, \dots, n_{p-1}$. Finally, in Section 5, we write our results in graph-theoretical notation, providing the sizes of k-stars and the number of k-element independent sets in unions of *sth*-powers of cycles.

2 s-separated k-sets in two circles with a fixed element

We begin this section by introducing notation. The set of the first n positive integers is denoted by $[n] := \{1, 2, \dots, n\}$ and the set of all s-separated k-element sets in a circle of size n is denoted by $[n]^s_k$. In order to distinguish between the elements of different circles, we introduce the following.

Definition 2.1. Let $n_0, n_1, \dots n_{p-1}$ be positive integers, with $p \geq 2$, and denote the set

$$[n_0, n_1, \dots, n_{p-1}] = ([n_0] \times \{0\}) \cup ([n_1] \times \{1\}) \cup \cdots ([n_{p-1}] \times \{p-1\}).$$

Thus the elements of the ith circle are those whose second coordinate is i.

Definition 2.2. Given $p \geq 2$ and a positive integer s we say that a k-element subset of $[n_0, \dots, n_{p-1}]$ is *s-separated* if it does not contain two elements in a circle with less than s elements between them. The set of all s-separated k-subsets of $[n_0, \dots, n_{p-1}]$ is denoted by $[n_0, \dots, n_{p-1}]^s_k$.

When we are considering the elements of an s-separated set, we sometimes say that the elements are s-separated.

Using the notation we just introduced, the lemma from [23] mentioned in the introduction can be written as follows.

Lemma 2.3. *[23] If $\mathcal{A}^{s,k}_1(n) = \{A \in [n]^s_k : 1 \in A\}$ then*

$$\left|\mathcal{A}^{s,k}_1(n)\right| = \binom{n - sk - 1}{k - 1}.$$

On the other hand, the following is a weaker version of the main result from [21].

Theorem 2.4. *[21]*

$$|[n]^s_k| = \frac{n}{n - sk}\binom{n - sk}{k}$$

In this section we establish a bijection between the set

$$\mathcal{A}^{s,k}_{(1,0)}(n_0, n_1) = \{A \in [n_0, n_1]^s_k : (1, 0) \in A\}$$

and the set in $\mathcal{A}^{s,k}_1(n_0 + n_1) = \{A \in [n_0 + n_1]^s_k : 1 \in A\}$. Together with Lemma 2.3 this yields the equality $\left|\mathcal{A}^{s,k}_{(1,0)}(n_0, n_1)\right| = \binom{n_0+n_1-ks-1}{k-1}$.

First, we define a function from $[n_0, n_1]$ to $[n_0+n_1]$ and its inverse. These functions will not be the required bijection as some s-separated sets will turn into sets that are not s-separated. To obtain the bijection we will introduce a process that will move the problematic elements (the ones that are not s-separated) in order to obtain s-separated sets.

Definition 2.5. Let $f : [n_0, n_1] \to [n_0 + n_1]$ be defined by

$$f(i, j) = \begin{cases} i & \text{if } j = 0, \\ n_0 + i & \text{if } j = 1. \end{cases}$$

So f transforms the two circles $[n_0] \times \{0\}$ and $[n_1] \times \{1\}$ into one circle $[n_0 + n_1]$ by relabeling the elements and making $(1, 1)$ come right after $(n_0, 0)$ in the cyclic order.

Definition 2.6. Let $g : [n_0 + n_1] \to [n_0, n_1]$ be defined by

$$g(i) = \begin{cases} (i, 0) & \text{if } 1 \le i \le n_0, \\ (i - n_0, 1) & \text{if } n_0 + 1 \le i \le n_0 + n_1. \end{cases}$$

Note that the function g as defined above is the inverse of the function f. The image of some s-separated sets (in their respective circles) under the functions f and g will not be s-separated sets although those sets will remain k-sets. Notice that if $A \in \mathcal{A}^{s,k}_{(1,0)}(n_0, n_1)$, then $f(A) \in \mathcal{A}^{s,k}_1(n_0 + n_1)$ if and only if $A \cap (\{n_1 - s + 1, \ldots, n_1\} \times \{1\}) = \emptyset$. If $f(A) \notin \mathcal{A}^{s,k}_1(n_0 + n_1)$, then $A \cap (\{n_1 - s + 1, \ldots, n_1\} \times \{1\})$ consists of a single element $(n_1 - d, 1)$. But as A is s-separated and $(1, 0) \in A$, we have $(n_0 - d, 0) \notin A$. Then we can switch the element $(n_1 - d, 1)$ by the element $(n_0 - d, 0)$, which through f will yield a set in $\mathcal{A}^{s,k}_1(n_0 + n_1)$ unless $A \cap (\{n_0 - d - s + 1, \ldots, n_0 - d\} \times \{0\}) = (n_0 - d - d', 0)$. But as A is s-separated and $(n_1 - d, 1) \in A$, $(n_1 - d - d', 1) \notin A$. Thus we can switch again and keep going. Notice that in order to do these switches we are identifying the elements in the circles in descending order.

$$\begin{aligned} (n_0, 0) &\longleftrightarrow (n_1, 1) \\ (n_0 - 1, 0) &\longleftrightarrow (n_1 - 1, 1) \\ &\vdots \\ (n_0 - s, 0) &\longleftrightarrow (n_1 - s, 1) \\ &\vdots \\ (n_0 - s(k-1), 0) &\longleftrightarrow (n_1 - s(k-1), 1). \end{aligned}$$

We are now ready to introduce the processes that yield the bijection. In the following definitions the first coordinate of an ordered pair is of importance; thus we use the projection notation: $\pi_1(a, b) = a$.

Definition 2.7 (zig)**.** Let $A \in \mathcal{A}^{s,k}_{(1,0)}(n_0, n_1)$, $n_0 \ge sk + 1$, $n_1 \ge sk$, and

$$Z_0 = A, \quad a_{-1} = n_0 + 1, \quad z_{-1} = n_1 + 1.$$

For $i = 0, 1, \ldots, k - 2$, do the following:

Let

$$B_i = \{z_{i-1} - 1, \ldots, z_{i-1} - s\} \times \{(i + 1) \pmod 2\}.$$

If $Z_i \cap B_i = \emptyset$, then the *zig* of A is $\mathcal{Z}(A) = Z_i$, the *z-order* of A is i and the process stops; else

$$\begin{aligned} a_i &= \pi_1(Z_i \cap B_i), \\ d_i &= z_{i-1} - a_i, \\ z_i &= a_{i-1} - d_i. \end{aligned}$$

$$Z_{i+1} = (Z_i - \{(a_i, (i+1) \pmod 2)\}) \cup \{(z_i, i \pmod 2)\}.$$

If the process does not stop before defining Z_{k-1}, then the *zig* of A is $\mathcal{Z}(A) = Z_{k-1}$, and the *z-order* of A is $k-1$.

Remark 1. Note that $1 \le d_i \le s$. Furthermore, if $f(Z_i)$ is not s-separated in $[n_0 + n_1]$ and $i \le k-2$, then $Z_i \cap B_i \neq \emptyset$ and removing either $(z_{i-1}, (i-1) \pmod 2)$ or $(a_i, (i+1) \pmod 2)$ yields an s-separated set. Hence, if the z-order of A is at most $k-2$, $f(\mathcal{Z}(A))$ is s-separated in $[n_0 + n_1]$.

Definition 2.8 (zag)**.** Let $\aleph \in \mathcal{A}_1^{s,k}(n_0+n_1)$, $n_0 \ge sk+1$, $n_1 \ge sk$, and

$$\widehat{Z}_0 = g(\aleph), \quad \bar{a}_{-1} = n_1 + 1, \quad \bar{z}_{-1} = n_0 + 1.$$

For $i = 0, 1, \ldots, k-2$, do the following:

Let

$$C_i = \{\bar{z}_{i-1} - 1, \ldots, \bar{z}_{i-1} - s\} \times \{i \pmod 2\}.$$

If $\widehat{Z}_i \cap C_i = \emptyset$, then the *zag* of $\aleph$ is $\widehat{\mathcal{Z}}(\aleph) = \widehat{Z}_i$ the $\widehat{z}$*-order* of $\aleph$ is i and the process stops; else

$$\begin{aligned} \bar{a}_i &= \pi_1\left(\widehat{Z}_i \cap C_i\right), \\ \bar{d}_i &= \bar{z}_{i-1} - \bar{a}_i, \\ \bar{z}_i &= \bar{a}_{i-1} - \bar{d}_i. \end{aligned}$$

$$\widehat{Z}_{i+1} = \left(\widehat{Z}_i - \{(\bar{a}_i, i \pmod 2)\}\right) \cup \{(\bar{z}_i, (i+1) \pmod 2)\}.$$

If the process does not stop before defining $\widehat{Z}_{k-1}$, then the *zig* of $\aleph$ is $\widehat{\mathcal{Z}}(\aleph) = \widehat{Z}_{k-1}$, and the $\widehat{z}$*-order* of $\aleph$ is $k-1$.

Remark 2. Again, note that $1 \le \bar{d}_i \le s$. Furthermore, if $\widehat{Z}_i$ is not s-separated in $[n_0, n_1]$ and $i \le k-2$, then $\widehat{Z}_i \cap C_i \neq \emptyset$ and removing either $(\bar{z}_{i-1}, i \pmod 2)$ or $(\bar{a}_i, i \pmod 2)$ yields an s-separated set. Hence, if the $\widehat{z}$*-order* of $\aleph$ is at most $k-2$, $\widehat{\mathcal{Z}}(\aleph)$ is s-separated in $[n_0, n_1]$.

We want $f(\mathcal{Z}(A))$ (respectively $\widehat{\mathcal{Z}}(\aleph)$) to be an s-separated k-set. Notice that for every i, $z_{i-1} \ge a_i$, $\sum_{j=0}^{i} d_j \le s(i+1)$, and $a_i = n_{(i+1) \pmod 2} + 1 - \sum_{j=0}^{i} d_j$. Hence, $a_{k-2} \le n_0 + 1 - s(i+1)$ if $i \equiv 1 \pmod 2$ and $a_{k-2} \le n_1 + 1 - s(i+1)$ if $i \equiv 0 \pmod 2$. By Remark 1, if the z-order of A is less than or equal to $k-2$, then $f(\mathcal{Z}(A))$ is s-separated in $[n_0 + n_1]$. Hence, we may assume that the z-order of A is $k-1$. Then $Z_{k-2} \cap B_{k-2} \neq \emptyset$, $k-1$ elements in total had to be moved around and $Z_{k-1} = \{(1,0), (z_0, 0), (z_1, 1), \ldots, (z_{k-3}, (k+1) \pmod 2), (z_{k-2}, k \pmod 2)\}$. But

$$\begin{aligned} z_{k-2} &\ge n_0 + 1 - s(k-1) \ge s+1 && \text{if } k \text{ is even} \\ z_{k-2} &\ge n_1 + 1 - s(k-1) \ge s && \text{if } k \text{ is odd.} \end{aligned}$$

Then, if k is even, $(1,0) \in Z_{k-1}$ is s-separated from $(z_{k-2},0)$. On the other hand, if k is odd, then $(1,1) \notin A$, as A is s-separated and the z-order of A is $k-1$ ($a_0 = n_1 + 1 - d_0 \in A$). Hence, $(z_{k-2},1)$ is s-separated from the other elements of Z_{k-1} and $(1,0) \in Z_{k-1}$. The only elements that are not separated in Z_{k-1} are $(1,0)$ with $(z_0,1)$. Thus $f(Z_{k-1})$ is s-separated, and $(1,0) \in f(Z_{k-1})$. Furthermore, if the z-order of A was less than $k-1$ the same reasoning shows that $(1,0) \in \mathcal{Z}(A)$.

A similar argument shows the same for $\widehat{\mathcal{Z}}(\aleph)$. This has been summarized in the following lemma.

Lemma 2.9. *Let $A \in \mathcal{A}^{s,k}_{(1,0)}(n_0,n_1)$ and $\aleph \in \mathcal{A}^{s,k}_1(n_0+n_1)$. Then $f(\mathcal{Z}(A))$ is s-separated in $[n_0+n_1]$, $\widehat{\mathcal{Z}}(\aleph)$ is s-separated in $[n_0,n_1]$ and $(1,0) \in \mathcal{Z}(A) \cap \widehat{\mathcal{Z}}(\aleph)$.*

Proof. The result follows from the discussion preceding the lemma. □

Working from Definition 2.7, $(a_i,(i+1) \pmod 2) \in A$ for every i and

$$(z_i, i \pmod 2) = (a_{i-1} - d_i, i \pmod 2) \notin A$$

because A is s-separated and $d_i \leq s$. Notice that Z_i is obtained from Z_{i-1} by deleting $(a_i,(i+1) \pmod 2)$ and adding $(z_i, i \pmod 2) = (a_{i-1} - d_i, i \pmod 2)$. So at each step an element that belonged to A is taken out, and an element that was not in A is added. Hence the number of elements remains unchanged and $|Z_i| = |A| = k$. This works in a similar fashion for $\widehat{Z}_i$. Hence we obtain the following.

Lemma 2.10. *Let $A \in \mathcal{A}^{s,k}_{(1,0)}$ and $\aleph \in \mathcal{A}^{s,k}_1$. The sets $Z_i(A)$ and $\widehat{Z}_i(\aleph)$ have k-elements for every i.*

Proof. The proof follows from the discussion preceding the lemma. □

We are ready to provide the bijections.

Definition 2.11. Define $F : \mathcal{A}^{s,k}_{(1,0)}(n_0,n_1) \to [n_0+n_1]^s_k$ by

$$F(A) = f(\mathcal{Z}(A)).$$

Definition 2.12. Define $G : \mathcal{A}^{s,k}_1(n_0+n_1) \to [n_0,n_1]^s_k$ by

$$G(\aleph) = \widehat{\mathcal{Z}}(\aleph).$$

The remainder of this section is dedicated to proving that F is a bijection from $\mathcal{A}^{s,k}_{(1,0)}(n_0,n_1)$ onto $\mathcal{A}^{s,k}_1(n_0+n_1)$ (and that G is its inverse).

Lemma 2.13. *If $A \in \mathcal{A}^{s,k}_{(1,0)}(n_0,n_1)$ and $\aleph \in \mathcal{A}^{s,k}_1(n_0+n_1)$, then $F(A) \in \mathcal{A}^{s,k}_1(n_0+n_1)$ and $G(\aleph) \in \mathcal{A}^{s,k}_{(1,0)}(n_0,n_1)$.*

Proof. By Lemma 2.9, $1 \in F(A)$ and $(1,0) \in G(\aleph)$, and they both are s-separated. By Lemma 2.10, both $F(A)$ and $G(\aleph)$ contain k elements. Therefore $F(A) \in \mathcal{A}^{s,k}_1(n_0+n_1)$ and $G(\aleph) \in \mathcal{A}^{s,k}_{(1,0)}(n_0,n_1)$. □

Thus $F : \mathcal{A}^{s,k}_{(1,0)}(n_0, n_1) \to \mathcal{A}^{s,k}_{1}(n_0 + n_1)$ and $G : \mathcal{A}^{s,k}_{1}(n_0 + n_1) \to \mathcal{A}^{s,k}_{(1,0)}(n_0, n_1)$. We can now prove that they are inverses of one another, i.e. $G = F^{-1}$.

Lemma 2.14. *If F and G are defined as in Definitions 2.11 and 2.12, then $G = F^{-1}$.*

Proof. We will prove that $G(F(A)) = A$, and $F(G(\aleph)) = \aleph$ for every $A \in \mathcal{A}^{s,k}_{(1,0)}(n_0, n_1)$ and $\aleph \in \mathcal{A}^{s,k}_{1}(n_0, +n_1)$.

Let $A \in \mathcal{A}^{s,k}_{(1,0)}(n_0, n_1)$ and let $\aleph = F(A)$. Then $\bar{a}_0 = z_0$ and the first switch will be $\bar{z}_0 = a_0$ which means that $\bar{a}_i = z_i$, $\bar{d}_i = d_i$ and $\bar{z}_i = a_i$. Hence the z-order of A is the same as the $\widehat{z}$-order of $\aleph$, thus we have that

$$G(\aleph) = G(F(A)) = A.$$

Now let $\aleph \in \mathcal{A}^{s,k}_{1}(n_0 + n_1)$ and $A = G(\aleph)$. Applying the same reasoning as before, we have that

$$F(A) = F(G(\aleph)) = \aleph,$$

thus $G \circ F = id$ and $F \circ G = id$. □

Theorem 2.15. *Let $n_0 \geq sk + 1$ and $n_1 \geq sk$. Then*

$$\left|\mathcal{A}^{s,k}_{(1,0)}(n_0, n_1)\right| = \binom{n_0 + n_1 - sk - 1}{k - 1}.$$

Proof. From Lemma 2.14, $\left|\mathcal{A}^{s,k}_{(1,0)}(n_0, n_1)\right| = \left|\mathcal{A}^{s,k}_{1}(n_0 + n_1)\right|$. Thus, Lemma 2.3 yields

$$\left|\mathcal{A}^{s,k}_{(1,0)}(n_0, n_1)\right| = \binom{n_0 + n_1 - sk - 1}{k - 1}.$$

□

3 s-separated k-sets in p circles with a fixed element

In this section we show that the number of s-separated k-sets in p circles of sizes $n_0, \ldots, n_{p-1}$ having a fixed element is equal to the number of s-separated k-sets in one circle of size $n_0 + \cdots + n_{p-1}$ having a fixed element.

By $\mathcal{A}^{s,k}_{(1,0)}(n_0, \ldots, n_{p-1})$ we denote all the s-separated k-sets in $[n_0, \ldots, n_{p-1}]$ containing the element $(1, 0)$. Notice that any set in $\mathcal{A}^{s,k}_{(1,0)}(n_0, \ldots, n_{p-1})$ with j elements in the first $p - 1$ circles can be seen as a set in $\mathcal{A}^{s,j}_{(1,0)}(n_0, \ldots, n_{p-2}) \cup \left([n_{p-1}]^s_{k-j} \times \{p - 1\}\right)$. Thus, adding over all possible j we obtain

$$|\mathcal{A}^{s,k}_{(1,0)}(n_0, \ldots, n_{p-1})| = \sum_{j=1}^{k} |\mathcal{A}^{s,j}_{(1,0)}(n_0, \ldots, n_{p-2})| \times |[n_{p-1}]^s_{k-j}|. \tag{1}$$

We can use Equation (1) inductively to obtain the following.

Theorem 3.1. *Let $n_0 \geq sk+1$ and $n_i \geq sk$ for $i = 1, \dots, p-1$. Then*

$$\left|\mathcal{A}^{s,k}_{(1,0)}(n_0, \dots, n_{p-1})\right| = \binom{N-sk-1}{k-1},$$

where $N = \sum_{i=0}^{p-1} n_i$.

Proof. Notice that Lemma 2.3 states

$$|\mathcal{A}^{s,k}_1(n_0 + \cdots + n_{p-1})| = \binom{n_0 + \ldots + n_{p-1} - sk - 1}{k-1}.$$

We will show by induction on p, the number of circles, that

$$|\mathcal{A}^{s,k}_{(1,0)}(n_0, \dots, n_{p-1})| = |\mathcal{A}^{s,k}_1(n_0 + \cdots + n_{p-1})|,$$

and so the result will follow from Lemma 2.3. The case $p = 2$ is Theorem 2.15. Consider $p \geq 3$ and assume that

$$|\mathcal{A}^{s,k}_{(1,0)}(n_0, \dots, n_{p-2})| = |\mathcal{A}^{s,k}_1(n_0 + \cdots + n_{p-2})|.$$

Then, Equation (1) gives

$$|\mathcal{A}^{s,k}_{(1,0)}(n_0, \dots, n_{p-1})| = \sum_{j=1}^{k} |\mathcal{A}^{s,j}_1(n_0 + \cdots + n_{p-2})| \times |[n_{p-1}]^s_{k-j}|. \tag{2}$$

On the other hand, Equation (1) applied to a circle of size $n_0 + \ldots + n_{p-2}$ and a circle of size n_{p-1} implies

$$|\mathcal{A}^{s,k}_{(1,0)}(n_0 + \ldots + n_{p-2}, n_{p-1})| = \sum_{j=1}^{k} |\mathcal{A}^{s,j}_1(n_0 + \cdots + n_{p-2})| \times |[n_{p-1}]^s_{k-j}|. \tag{3}$$

Since the left side of Equation (2) equals the left side of Equation (3),

$$|\mathcal{A}^{s,k}_{(1,0)}(n_0, \dots, n_{p-1})| = |\mathcal{A}^{s,k}_{(1,0)}(n_0 + \ldots + n_{p-2}, n_{p-1})|.$$

By Theorem 2.15,

$$|\mathcal{A}^{s,k}_{(1,0)}(n_0 + \ldots + n_{p-2}, n_{p-1})| = \binom{N-sk-1}{k-1},$$

and by Lemma 2.3,

$$|\mathcal{A}^{s,k}_1(n_0 + \cdots + n_{p-1})| = \binom{N-sk-1}{k-1}.$$

Therefore

$$\left|\mathcal{A}^{s,k}_{(1,0)}(n_0, \dots, n_{p-1})\right| = \binom{N-sk-1}{k-1} = |\mathcal{A}^{s,k}_1(n_0 + \cdots + n_{p-1})|$$

and the proof follows by induction. ∎

Theorem 3.1 counts the number of sets in $[n_0, \ldots, n_{p-1}]_k^s$ containing $(1, 0)$. Notice that if $n_j \geq sk + 1$, then the elements can be relabeled to count the number of sets containing any element (a, j) in the j-th circle.

Corollary 3.2. *Let $n_j \geq sk+1$ and $n_i \geq sk$ for $i = 0, \ldots, p-1$. Let $(a, j) \in [n_j] \times \{j\}$ and*

$$\mathcal{A}_{(a,j)}^{s,k}(n_0, \ldots, n_{p-1}) = \left\{A \in [n_0, \ldots, n_{p-1}]_k^s : (a, j) \in A\right\}.$$

Then

$$\left|\mathcal{A}_{(a,j)}^{s,k}(n_0, \ldots, n_{p-1})\right| = \binom{N - sk - 1}{k - 1},$$

where $N = \sum_{i=1}^{p} n_i$.

Proof. The corollary follows from Theorem 3.1 after relabeling the elements and circles such that (a, j) is labeled $(1, 0)$. □

Notice that the equalities

$$|\mathcal{A}_{(1,0)}^{s,k}(n_0, n_1)| = \binom{n_0 + n_1 - sk - 1}{k - 1},$$

$$|\mathcal{A}_{(1,0)}^{s,k}(n_0)| = \binom{n_0 - sk - 1}{k - 1},$$

and

$$|[n_1]_k^s| = \frac{n}{n - sk}\binom{n - sk}{k},$$

together with Theorem 3.1 yield the following known equality, due to Rothe [22].

Corollary 3.3. *If $m \geq sk + 1$ and $n \geq sk$ then*

$$\sum_{j=1}^{k} \binom{n - sj - 1}{j - 1} \frac{m}{m - s(k - j)} \binom{m - s(k - j)}{k - j} = \binom{m + n - sk - 1}{k - 1}.$$

4 s-separated k-sets in p circles

In this section we use Corollary 3.2 to count the number of s-separated k-element sets in p circles of various sizes, provided all of the sizes are at least $sk + 1$. In order to do so, we add over every element of the circle the number of s-separated sets containing it, and then divide by the number of elements in each set.

Theorem 4.1. *Let $n_i \geq sk + 1$ for $i = 0, \ldots, p - 1$. Then*

$$|[n_0, \ldots, n_{p-1}]_k^s| = \frac{N}{k}\binom{N - sk - 1}{k - 1},$$

where $N = \sum_{i=0}^{p-1} n_i$.

Proof. As $n_i \geq sk+1$, Corollary 3.2 ensures that the number of s-separated k-sets containing a fixed element from the i-th circle is $\binom{N-sk-1}{k-1}$. Then, adding this number over all elements yields

$$N\binom{N-sk-1}{k-1}.$$

Notice that in doing this we counted each s-separated k-set k times, once per element in said set. Thus,

$$N\binom{N-sk-1}{k-1} = k\left|[n_0,\ldots,n_{p-1}]_k^s\right|,$$

and hence

$$\left|[n_0,\ldots,n_{p-1}]_k^s\right| = \frac{N}{k}\binom{N-sk-1}{k-1}.$$

□

Using that any s-separated k-set with j elements in one circle and $(k-j)$ elements in another is the union of an s-separated j-set in one circle with an s-separated $(k-j)$-set in the other we get the following:

$$\left|[n_0,n_1]_k^s\right| = \sum_{j=0}^{k}\left|[n_0]_j^s\right| \times \left|[n_1]_{k-j}^s\right|.$$

We can now follow the ideas from Corollary 3.3 and apply Theorem 4.1 to get the following.

Corollary 4.2.

$$\frac{n_0+n_1}{k}\binom{n_0+n_1-sk-1}{k-1} = \sum_{j=0}^{k}\frac{n_0}{n_0-sj}\binom{n_0-sj}{j}\frac{n_1}{n_1-s\left(k-j\right)}\binom{n_1-s\left(k-j\right)}{k-j}.$$

Corollaries 3.3 and 4.2 are famous in the literature and are due to Rothe [22]. Other nice bijective proofs of these formulas were given by Guo [10].

5 From the graph-theoretical point of view

In this section we restate our results in graph-theoretical notation. We start by giving some definitions.

The family of all independent sets of a graph G is denoted by $\mathcal{I}(G)$, the family of independent k-sets by $\mathcal{I}^k(G) := \{A \in \mathcal{I}(G) : |A| = k\}$ and the family of independent k-sets containing a given vertex v by $\mathcal{I}_v^k(G) := \left\{A \in \mathcal{I}^k(G) : v \in A\right\}$.

The s-th power of a graph G, denoted G^s, is obtained from G by adding edges between vertices at distance less than or equal to s. Thus, $V(G^s) = V(G)$, and $\{v,w\} \in E(G^s)$ if and only if the distance between v and w in G is less than or equal to s. In [4, 12, 13] the authors worked on proving an intersecting property on the subfamilies of $\mathcal{I}^k(G)$ where the graph G is the disjoint union of powers of graphs; said property is that the largest intersecting family of k-sets is $\mathcal{I}_v^k(G)$ for some vertex v. However, they do not give the sizes of those families as in [23].

We label the vertices as in 2.1 and note that an independent k-set of $C_{n_0}^s \cup \cdots \cup C_{n_{p-1}}^s$ corresponds to an s-separated k-set in $[n_0, \ldots, n_{p-1}]$. Theorem 3.1 yields the following result.

Corollary 5.1. *For every $i \in \{0, \ldots, p-1\}$ let $n_i \geq sk$ be an integer. If v is a vertex of $C_{n_j}^s$ and $n_j \geq sk+1$, then*

$$\left|\mathcal{I}_v^k\left(C_{n_0}^s \cup \cdots \cup C_{n_{p-1}}^s\right)\right| = \binom{\sum_{i=0}^{p-1} n_i - sk - 1}{k-1}.$$

Furthermore,

$$\left|\mathcal{I}^k\left(C_{n_0}^s \cup \cdots \cup C_{n_{p-1}}^s\right)\right| = \frac{\sum_{i=0}^{p-1} n_i}{k}\binom{\sum_{i=0}^{p-1} n_i - sk - 1}{k-1}.$$

Let $\alpha(G)$ denote the independence number of G, Corollary 5.1 gives the sizes of $\mathcal{I}_v^k\left(C_{n_0}^s \cup \cdots \cup C_{n_{p-1}}^s\right)$ and $\mathcal{I}^k\left(C_{n_0}^s \cup \cdots \cup C_{n_{p-1}}^s\right)$ when $1 \leq k \leq \min_i\left\{\alpha\left(C_{n_i}^s\right)\right\}$, that is, when $n_i \geq (s+1)k$ for each i.

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments which greatly improved the presentation of this manuscript. This work was partially supported by the Universidad Nacional de San Luis, grant PROIPRO 03-2216, and MATH AmSud, grant 18-MATH-01. The first author was supported by a doctoral scholarship from Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET).

References

[1] P. Borg, Extremal t-intersecting sub-families of hereditary families, *J. Lond. Math. Soc.(2)* **79** (2009), 167–185.

[2] P. Borg, Intersecting families, cross-intersecting families, and a proof of a conjecture of Feghali, Johnson and Thomas, *Discrete Math.* **341** (2018), 1331–1335.

[3] P. Borg, Stars on trees, *Discrete Math.* **340** (2017), 1046–1049.

[4] P. Borg and F. C. Holroyd, The Erdős-Ko-Rado properties of various graphs containing singletons, *Discrete Math.* **309** (2009), 2877–2885.

[5] W. J. C. Chen, D. G. L Wang and I. F. Zhang, Partitions of $\mathbb{Z}_n$ into arithmetic progressions, *European J. Combin.* **30** (2009), 764–773.

[6] B. S. El-Desouky, M. M. Gad and S. El-Eraqy, Generalization of Some Problems with s-Separation, *Applied Math.* 6 (2015), 1–6.

[7] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math.* **12** (1961), 313–320.

[8] C. Feghali, M. Johnson and D. Thomas, Erdős-Ko-Rado theorems for a family of trees, *Discrete Appl. Math.* **236** (2018), 464–471.

[9] V. J. W. Guo, A new proof of a theorem of Mansour and Sun, *European J. Combin.* **29** (2008), 1582–1584.

[10] V. J..W. Guo, Bijective proos of Gould's and Rothe's identities, i *Discrete Math.* **308** (2008), 1756–1759.

[11] V. J. W. Guo and J. Zeng, On arithmetic partitions of $\mathbb{Z}_n$, *European J. Combin.* **30** (2009), 1281–1288.

[12] A. J. W. Hilton, F. C. Holroyd and C. L. Spencer, King Arthur and his knights with two round tables, *Quart. J. Math.* **62** (2011), 625–635.

[13] A. J. W. Hilton and C. L. Spencer, A generalization of Talbot's theorem about King Arthur and his knights of the Round Table, *J. Combin. Theory Ser. A* **116** (2009), 1023–1033.

[14] F. C. Holroyd, Problem 338 (BCC 16.25), Erdős-Ko-Rado at the court of King Arthur, *Discrete Math.* **197/198** (1999), 812.

[15] F. C. Holroyd and A. Johnson, Problem 25, *Problems from the Sixteenth British Combinatorial Conference*, edited by P. J. Cameron, 1997.

[16] F. Holroyd, C. Spencer and J. Talbot, Compression and Erdős-Ko-Rado graphs, *Discrete Math.* **293** (2005), 155–164.

[17] F. Holroyd and J. Talbot, Graphs with the Erdős-Ko-Rado property, *Discrete Math.* **293** (2005), 165–176.

[18] G. Hurlbert and V. Kamat, Erdős-Ko-Rado theorems for chordal graphs and trees, *J. Combin. Theory Ser. A* **118** (2011), 1046–1049.

[19] I. Kaplansky, Solution of the problème des mènages, *Bull. Amer. Math. Soc.* **49** (1943), 784–785.

[20] J. Konvalina, On the number of combinations without unit separation, *J. Combin. Theory Ser. A* **31** (1981), 101–107.

[21] T. Mansour and Y. Sun, On the number of combinations without certain separations, *European J. Combin.* **29** (2008), 1200–1206.

[22] H. A. Rothe, Formulae de serierum reversione demonstratio unviversalis signis localibus combinatorio-analyticorum vicariis exhibita, Leipzig, 1793.

[23] J. Talbot, Intersecting families of separated sets, *J. Lond. Math. Soc.(2)* **68** (1) (2003), 37–51.

[24] R. Woodroofe, Erdős-Ko-Rado theorems for simplicial complexes, *J. Combin. Theory Ser. A* **118** (2011), 1218–1227.

(Received 19 Feb 2020; revised 4 Jan 2021)