On Row-Column Directed Block Designs

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Dedicated to the memory of Alan Rahilly, 1947 – 1992

ABSTRACT:

Given a BIBD(v,k,λ) we can construct a directed BIBD(v,k,λ^*) by writing each block of the BIBD twice, the second time in the reversed order. Here we prove the analagous expected but useful result that if a BIBD($v,b,r=ks,k,\lambda$) exists, then a RCDBIBD(v,k,λ^*) also exists. A nonexistence result is also given.

1. Introduction:

A balanced incomplete block design BIBD(v,b,r,k,λ) is a set of v elements (called points) together with a collection of k element subsets (called blocks) having the property that each pair of distinct points occurs together in exactly λ blocks. There are b blocks in all and each point occurs in r of them. For example, see Street and Street [8]. We will use BIBD(v,b,r,k,λ) and BIBD(v,k,λ) interchangeably. If the points in each block of a BIBD($v,k,2\lambda$) are arranged in such a way that each ordered pair occurs in λ blocks the design is a directed BIBD(v,b,r,k,λ^*). A block ($a_1, a_2, \ldots, a_k$) of a DBIBD is said to have $k(k-1)/2$ ordered pairs (a_i, a_j) $i = 1,2, \ldots, k-1$; $j = i+1, \ldots, k$. Directed designs have been studied by many authors. See, for example, Colbourn and Harms [3], Hamm, Lindner and Rodger [4], Seberry and Skillicorn [6], Street and Seberry [7], Street and Wilson [9] and recently by Bennett et al [2].

Consider the BIBD(4,3,2):

$$\begin{matrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{matrix}$$

where the blocks are written vertically as columns. In each block the points can be ordered to get a DBIBD($4,3,1^*$) as follows:

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \end{matrix}$$

We can consider the above arrangement as a 3 x 4 matrix, where the columns form a DBIBD($4,3,1^*$) and each point occurs exactly once in each row. We also observe that each ordered pair (a, b) of points occurs at least once in the rows. The above properties suggest the study of the following configuration:

Definition: A BIBD(v,b,r=sk,k,2λ) is said to be row-column directed if

(i) the BIBD is a DBIBD(v,k,λ*) and

(ii) the blocks form the columns of a kxb array and are arranged in such a way that:

(a) each point occurs s times in each row, and

(b) each ordered pair of points occurs at least $\lfloor ks^2/2 \rfloor$ times in the rows where

$$\frac{ks^2}{2}=\frac{k[\binom{b}{2}-v\binom{s}{2}]}{v(v-1)}$$

A row-column directed BIBD with the parameters v, b, r=ks, k and 2λ will be denoted by RCDBIBD(v,k,λ*). Sarvate and Hamm [5], after justifying condition (iia), settled the existence of RCDBIBD(v,k,λ*) for k=2 and for k=3 when v $\equiv$ 1(mod 6). They showed that no RCDBIBD(7,4,1*) exists and gave some construction techniques. In Section 3 we prove that if a BIBD(v,b,r=ks,k,λ) exists then a RCDBIBD(v,k,λ*) exists. A non existence result is given in Section 4.

2. Justification for the condition (iia):

In view of the condition (iib), a natural question may be whether it would be sufficient to allow each point to occur *almost* s times in each row. In this section we will show that in general if r $\not\equiv$ 0(mod 3), we can not get such a design with block size 3. Therefore condition (iia) is justified.

Consider a BIBD(v,b,r,3,2λ) which we wish to row-column direct. If k=3 then r = λ(v-1). Assume r $\equiv$ 1(mod 3) and let r=3s+1. If the BIBD can be directed according to conditions (i) and (iib) with relaxed condition (iia) then an arbitrary point *a* can occur (s+1) times in one of the rows and s times in the other two rows. There are λ(v-1)=r=3s+1 ordered pairs in the columns with *a* as the first entry. We get 2 such ordered pairs each time *a* appears in the first row, one such ordered pair when it appears in the second row and none when it is in the third row. The only way we could have 3s+1 ordered pairs with *a* as the first entry is to arrange *a* s times in the 1st row and s+1 times in the 2nd row. It is not possible for all points to appear (s+1) times in the 2nd row and s times in the first row. A similar argument works for the case where r $\equiv$ 2(mod3) and even if r$\equiv$0(mod3) with condition (ii) relaxed.

3. Construction of a RCDBIBD(v,k,λ*) from a BIBD(v,k,λ):

In this section we prove the following theorem:

<u>Theorem 3.1:</u> If a BIBD(v,b,r,k,λ) exists with r=ks then a RCDBIBD(v,k,λ*) exists.

To prove the above theorem we need two lemmas.

Lemma 3.2. If the blocks of a BIBD(v,b,r,k,λ) can be ordered so that each point occurs r/k times in each row of the corresponding matrix, X, then a RCDBIBD(v,k,λ^*) exists.

Proof: Construct a matrix Y by writing the columns of X in reverse order, reversing the points of the blocks as well. A straightforward verification of the fact that the augmented matrix X:Y gives a RCDBIBD(v,k,λ^*) is given below.

(i) X:Y is directed in the usual sense because each block in X is written in the reverse order in Y.

(ii) Each point occurs $2r/k$ times in each row of X:Y, because it occurs r/k times in each row of X.

(iii) Let a and b be any two points. Let x and y denote the number of times the pairs (a,b) and (b,a) have appeared respectively in the rows of X. Then the pairs (a,b) and (b,a) occur y and x times respectively in Y. The points a and b occur s times in each row of X and in each row of Y. Therefore the total number of times either of the pair (a,b) and (b,a) occur is the same: $x+y+ks^2$.

Lemma 3.3 (Agrawal [1]): In every binary equi-replicate design of constant block size k such that $bk=vr$ and $b=mv$, the treatments can be rearranged into blocks, so that every treatment occurs in a row m times.

Lemmas 3.2 and 3.3 together provide a proof of Theorem 3.1.

4. Nonexistence result

Theorem 4.1: If the number of unordered partitions of $(k-2)(v-1)/2$ with $(k-1)$ distinct nonnegative integers is less than k, then no symmetric (i.e., $k=r$) RCDBIBD(v, k, λ^*) exists, where $2\lambda<k$ and k is even.

Proof: Assume that a RCDBIBD(v,k,λ^*) exists where $k=r$. Since $s=1$, each ordered pair will occur $k/2$ times in the rows and each element i will appear as the first element in $(v-1)k/2$ ordered pairs from the rows. Without loss of generality, let $\{1,2,\cdots,k\}$ be the first block. Let t_{ij} denote the number of ordered pairs in row j which begin with i, for $i=1,\cdots,k$ and $j=1,\cdots,k$. Then we have the following situation:

(a) $t_{ii}=v-1$

(b) for $i=1,2,\cdots,k$,

$$\sum_{j=1,j\neq i}^{k} t_{ij} = \frac{(k-2)(v-1)}{2}$$

Now according to the assumption in the statement of the theorem, $x_1+x_2+\cdots+x_k = (k-2)(v-1)/2$ has fewer than k solutions with k-1 distinct integers. Therefore at least two of the points in $\{1,2,\cdots,k\}$ have occurred together in k blocks but $2\lambda<k$; hence a contradiction.

Example:

A RCDBIBD(7,4,1*) does not exist because $(k-1)(v-1)/2 = 6$ and $x+y+z = 6$ has 7 possible solutions, namely {6,0,0}, {4,1,1}, {3,3,0}, {2,2,2}, {5,1,0}, {4,2,0}, and {3,2,1}. For our purposes a solution can not have repeated values because a point can not occur more than once in the same block. Therefore the only feasible solutions are {5,1,0}, {4,2,0} and {3,2,1}, fewer than the required 4.

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