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Rainbow cycles versus rainbow paths

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Abstract

An edge-colored graph F is rainbow if each edge of F has a unique color. The rainbow Turán number $\mathrm{ex}^*(n, F)$ of a graph F is the maximum possible number of edges in a properly edge-colored n-vertex graph with no rainbow copy of F. The study of rainbow Turán numbers was introduced by Keevash, Mubayi, Sudakov, and Verstraëte in 2007. Johnson and Rombach introduced the following rainbow-version of generalized Turán problems: for fixed graphs H and F, let $\mathrm{ex}^*(n, H, F)$ denote the maximum number of rainbow copies of H in an n-vertex properly edge-colored graph with no rainbow copy of F. In this paper we investigate the case $\mathrm{ex}^*(n, C_\ell, P_\ell)$ and give a general upper bound as well as exact results for $\ell = 3, 4, 5$. Along the way we establish a new best upper bound on $\mathrm{ex}^*(n, P_5)$. Our main motivation comes from an attempt to improve bounds on $\mathrm{ex}^*(n, P_\ell)$, which has been the subject of several recent papers.

1 Introduction

We say that an edge-colored graph is *rainbow* if no two edges receive the same color. We use the term *rainbow-F* to refer to a rainbow copy of the graph F. A properly edge-colored graph is *rainbow-F-free* if it contains no rainbow copy of F as a subgraph. The *rainbow Turán number* of a fixed graph F is the maximum possible number of edges in a properly edge-colored n-vertex rainbow-F-free graph G. We denote this maximum by $\mathrm{ex}^*(n, F)$. The study of rainbow Turán numbers was introduced by Keevash, Mubayi, Sudakov, and Verstraëte in [10].

Observe that $\mathrm{ex}(n, F) \le \mathrm{ex}^*(n, F)$, since any properly edge-colored F-free graph clearly contains no rainbow-F. In fact, it was proved in [10] that for any F,

$$\mathrm{ex}(n, F) \le \mathrm{ex}^*(n, F) \le \mathrm{ex}(n, F) + o(n^2).$$

However, for bipartite F, $\mathrm{ex}(n, F)$ and $\mathrm{ex}^*(n, F)$ are not asymptotic in general. For example, in [10] it was shown that asymptotically $\mathrm{ex}^*(n, C_6)$ is a constant factor larger than $\mathrm{ex}(n, C_6)$. Another interesting example concerns acyclic graphs. The

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maximum number of edges in an n-vertex graph containing no cycle is $n-1$. On the other hand, the maximum number of edges in a properly edge-colored n-vertex graph with no rainbow cycle is at least $n \log n$ (see [10]). Moreover, Das, Lee and Sudakov [2] showed that this maximum is at most $n^{1+\epsilon}$ for any $\epsilon > 0$ and n large enough. This upper bound was improved to $O(n(\log n)^4)$ by O. Janzer [7].

We denote by P_ℓ the path on ℓ edges, i.e., $\ell+1$ vertices. The behavior of $\mathrm{ex}^*(n, P_\ell)$ is known only for $\ell \leq 4$. For $\ell = 1$ and $\ell = 2$, the result $\mathrm{ex}^*(n, P_\ell) = \mathrm{ex}(n, P_\ell)$ is trivial, since any properly colored P_1 or P_2 is rainbow. For $\ell = 3$ and $\ell = 4$, Johnston, Palmer and Sarkar [8] showed:

Theorem 1.1 (Johnston-Palmer-Sarkar [8]). *If n is divisible by 4, then*

$$\mathrm{ex}^*(n, P_3) = \frac{3}{2}n.$$

If n is divisible by 8, then

$$\mathrm{ex}^*(n, P_4) = 2n.$$

The best-known general lower bound and upper bound are due to Johnston and Rombach [9] and Ergemlidze, Győri and Methuku [3], respectively:

Theorem 1.2 (Johnston-Rombach [9]; Ergemlidze-Győri-Methuku [3]). *For $\ell \geq 3$,*

$$\frac{\ell}{2}n + O(1) \leq \mathrm{ex}^*(n, P_\ell) \leq \left(\frac{9\ell+5}{7}\right)n.$$

Johnston and Rombach also considered a rainbow-version of generalized Turán problems popularized by Alon and Shikhelman [1]. For fixed graphs H and F, let $\mathrm{ex}^*(n, H, F)$ denote the maximum possible number of rainbow copies of H in an n-vertex properly edge-colored graph with no rainbow-F. In this paper we will be primarily concerned with determining the value of $\mathrm{ex}^*(n, C_\ell, P_\ell)$. Our motivation comes from the investigation of $\mathrm{ex}^*(n, P_\ell)$, but the study of $\mathrm{ex}^*(n, H, F)$ is a natural analogue of generalized Turán problems and rainbow Turán problems.

Let us mention that other generalizations have been investigated. Gerbner, Mészáros, Methuku and Palmer [4] considered the function $\mathrm{ex}(n, H, \text{rainbow-}F)$ which is the maximum number of copies of H in a properly edge-colored n-vertex graph with no rainbow-F. Most of their results address the case when $H = F$. The case $H = F = K_k$ was considered recently by Gowers and B. Janzer [5].

The construction achieving the lower bound for $\mathrm{ex}^*(n, P_4)$ contains many rainbow walks of length 4, but as there is no rainbow-P_4, each of these walks must be a cycle. In fact, this construction has the maximum number of rainbow-C_4 copies without a rainbow-P_4. A better understanding of $\mathrm{ex}^*(n, C_\ell, P_\ell)$ may help improve the bounds on $\mathrm{ex}^*(n, P_\ell)$.

The main results in this paper are summarized in the following theorem.

Theorem 1.3. *For $\ell \geq 3$,*

$$\frac{(\ell-1)!}{2}n \leq \mathrm{ex}^*(n, C_\ell, P_\ell) \leq (2\ell-3)^{\ell-2} \cdot \mathrm{ex}^*(n, P_\ell) \leq c(\ell)n$$

for some constant $c(\ell)$ depending on ℓ. Moreover, for $\ell = 3, 4, 5$ we have

$$\mathrm{ex}^*(n, C_\ell, P_\ell) = \frac{(\ell-1)!}{2}n$$

when n is divisible by $2^{\ell-1}$.

In Section 2 we give simple general bounds on $\mathrm{ex}^*(n, C_\ell, P_\ell)$ which gives the first part of Theorem 1.3. In Section 3 we give matching upper bounds when $\ell = 3, 4, 5$ and n is divisible by $2^{\ell-1}$. Note that this immediately implies tight asymptotic bounds for all n.

2 General bounds

We begin with the construction from [9] giving a lower bound on $\mathrm{ex}^*(n, P_\ell)$. Let $Q_{\ell-1}$ be the $\ell - 1$ dimensional cube, i.e., the graph whose vertex set is the set of 01-strings of length $\ell - 1$ and two vertices are joined by an edge if and only if their Hamming distance is exactly 1.

Now let us color the edges of $Q_{\ell-1}$ by the position in which their corresponding strings differ. For each vertex x of $Q_{\ell-1}$, let $\overline{x}$ be the *antipode* of x. That is, $\overline{x}$ is the unique vertex of Hamming distance $\ell - 1$ from x (i.e. all bits of x are swapped). Now add all edges $x\overline{x}$ to this graph and color these edges with a new color ℓ. Call these edges *diagonal* edges and denote the resulting edge-colored graph $D^*_{2^{\ell-1}}$. Note that the underlying (uncolored) graph of $D^*_{2^{\ell-1}}$ is often referred to as a *folded cube graph.*

It is easy to see that the edge-coloring above is proper. It was shown in [9] that $D^*_{2^{\ell-1}}$ contains no rainbow-P_ℓ. Let us give another argument here for completeness. Suppose that $D^*_{2^{\ell-1}}$ contains a rainbow path P of length ℓ. The path P must include an edge $x\overline{x}$ of color ℓ. Removing the edge $x\overline{x}$ from P leaves two subpaths P' and P'' (allowing for P'' to be the empty path when P ends with edge $x\overline{x}$). The subpath P' corresponds to bit changes to x and, as P is rainbow, P'' corresponds to the complement of these bit changes starting with $\overline{x}$. Therefore, P' and P'' share an end-vertex y (allowing for $y = \overline{x}$ when P'' is empty), i.e., P is a cycle, a contradiction.

Theorem 2.1. *For $\ell \geq 3$, we have $\frac{(\ell-1)!}{2}n \leq \mathrm{ex}^*(n, C_\ell, P_\ell)$ when n is divisible by $2^{\ell-1}$.*

Proof. Let G be a graph of $n/2^{\ell-1}$ vertex-disjoint copies of $D^*_{2^{\ell-1}}$. As each copy of $D^*_{2^{\ell-1}}$ has exactly ℓ edge colors, any rainbow-C_ℓ must contain an edge of color ℓ. Recall that edges of color ℓ are the diagonal edges.

Fix a diagonal edge $x\overline{x}$ and count the number of rainbow-C_ℓ copies containing $x\overline{x}$. This is precisely the number of length-$(\ell-1)$ rainbow paths between x and $\overline{x}$ colored from $\{1,2,\dots,\ell-1\}$. Each such rainbow-$P_{\ell-1}$ is obtained by a sequence of $\ell-1$ bit changes. There are $(\ell-1)!$ distinct sequences, each of which produces a distinct rainbow path between x and $\overline{x}$, so $x\overline{x}$ is included in $(\ell-1)!$ rainbow-C_ℓ copies. There are $2^{\ell-2}$ diagonal edges in each $D^*_{2^{\ell-1}}$ (one for each antipodal pair x, $\overline{x}$), and so a copy of $D^*_{2^{\ell-1}}$ contains a total of

$$(\ell-1)!\cdot 2^{\ell-2}$$

rainbow-C_ℓ copies. Thus the total number of rainbow-C_ℓ copies in G is

$$(\ell-1)!\cdot 2^{\ell-2}\cdot\frac{n}{2^{\ell-1}}=\frac{(\ell-1)!}{2}n.$$

□

We need the following simple lemma which will also be useful in Section 3.

Lemma 2.2. *Fix integers $k\geq\ell\geq 1$. If G is a properly k-edge-colored graph and xy is an edge of G, then xy is contained in at most $\frac{(k-1)!}{(k-\ell)!}$ rainbow-C_ℓ copies. In particular, if $k=\ell$, then xy is contained in at most $(\ell-1)!$ rainbow-C_ℓ copies.*

Proof. Note that the rainbow-C_ℓ copies containing an edge xy correspond to the rainbow paths of length $\ell-1$ with endpoints x and y which do not use the color on xy. For each rainbow path $P=xv_1v_2\cdots v_{\ell-2}y$, associate to P the ordered list of edge colors $(c(xv_1),\dots,c(v_{\ell-2}y))$. There are $\frac{(k-1)!}{(k-\ell)!}$ possible distinct lists, so we are done as long as no two distinct paths between x and y are associated to the same list. Suppose to the contrary that $P_1=xv_1v_2\cdots v_{\ell-2}y$ and $P_2=xw_1w_2\cdots w_{\ell-2}y$ are distinct rainbow paths with $(c(xv_1),c(v_1v_2),\dots,c(v_{\ell-2}y))=(c(xw_1),c(w_1w_2),\dots,c(w_{\ell-2}y))$. Since P_1 and P_2 are distinct, there is a smallest index i such that $v_i\neq w_i$; clearly, $i\geq 1$. But (making, if necessary, the identifications $x=v_0=w_0$), we have $c(v_{i-1}v_i)=c(w_{i-1}w_i)$. By the choice of i, $v_{i-1}=w_{i-1}$. This is a contradiction to the hypothesis that G is properly k-edge-colored, since now $v_{i-1}v_i$ and $v_{i-1}w_i$ are distinct edges incident to v_{i-1} which receive the same color. □

In a proper ℓ-edge-coloring each rainbow-C_ℓ contains an edge of color 1. In an n-vertex graph there are at most $\frac{n}{2}$ edges of color 1. Therefore, if we only use ℓ edge colors, Lemma 2.2 implies that there are at most $(\ell-1)!\cdot\frac{n}{2}$ rainbow-C_ℓ copies. This matches the lower bound given in Theorem 1.3. Thus, a proof that using a total of ℓ edge colors on the edges is optimal would determine $\mathrm{ex}^*(n,C_\ell,P_\ell)$. Unfortunately, such an argument appears to be difficult.

We now give an upper bound on $\mathrm{ex}^*(n,C_\ell,P_\ell)$ for general ℓ. The heart of the argument is the following simple lemma:

Lemma 2.3. *Let G be a properly edge-colored graph with no rainbow-P_ℓ. If $v_1v_2\cdots v_\ell v_1$ is a rainbow-C_ℓ in G, then $d(v_i)\leq 2\ell-3$ for $1\leq i\leq\ell$.*

Proof. Consider a vertex v_i on a rainbow cycle $C = v_1v_2\cdots v_\ell v_1$. Each edge v_ix where x is not on C must be colored with a color used on an edge of C (that is not incident to v_i) as otherwise we can construct a rainbow-P_ℓ. Thus, there are at most $\ell-2$ such edges v_ix. Moreover, v_i is adjacent to at most $\ell-1$ other vertices on C. Therefore, $d(v_i) \leq 2\ell-3$. □

In general a rainbow-C_ℓ cannot have many vertices of degree $2\ell-3$. In Section 3 we will make a deeper analysis of vertex degrees in the case when $\ell = 3, 4, 5$ to prove a stronger result.

Theorem 2.4. *For $\ell \geq 3$,*

$$\mathrm{ex}^*(n, C_\ell, P_\ell) \leq (2\ell-3)^{\ell-2} \cdot \mathrm{ex}^*(n, P_\ell) \leq c(\ell)n$$

for some constant $c(\ell)$ depending on ℓ.

Proof. Let G be a properly edge-colored graph on n vertices that does not contain a rainbow-P_ℓ. Fix an edge v_1v_2 and bound the number of rainbow-C_ℓ copies containing v_1v_2. We may assume that v_1v_2 is contained in at least one rainbow-C_ℓ.

Note that the number of rainbow-C_ℓ copies containing v_1v_2 is bounded above by the number of ways in which we can pick $\ell-1$ more edges to form a cycle $v_1v_2\cdots v_\ell v_1$. By Lemma 2.3, $d(v) \leq 2\ell-3$ for each v in a cycle with v_1v_2, so there are at most $2\ell-3$ ways in which to chose each vertex. Therefore, the number of rainbow-C_ℓ copies containing v_1v_2 is bounded above by $(2\ell-3)^{\ell-2}$. Therefore, the number of rainbow-C_ℓ copies is at most

$$(2\ell-3)^{\ell-2} \cdot \mathrm{ex}^*(n, P_\ell)$$

which is linear in n by Theorem 1.2. □

3 Asymptotic bounds

For small values of ℓ, we can determine $\mathrm{ex}^*(n, C_\ell, P_\ell)$ exactly when n is divisible by $2^{\ell-1}$. For the remaining values of n this gives tight asymptotic bounds.

Theorem 3.1. *If n is divisible by 4, then $\mathrm{ex}^*(n, C_3, P_3) = n$. Moreover, the unique graph attaining this maximum is the disjoint union of D_3^* copies, i.e., copies of the complete graph K_4 each with a proper 3-edge-coloring.*

Proof. Theorem 2.1 gives $n = \frac{(3-1)!}{2}n \leq \mathrm{ex}^*(n, C_3, P_3)$. For the upper bound, let G be an n-vertex graph with a proper edge-coloring with no rainbow-P_3. Note that every C_3 is rainbow, so it suffices to count the number of triangles. Thus, let us count the number of triangles containing a fixed edge xy. We may assume that xy is contained in at least one triangle, say $xyzx$. A triangle containing xy which is distinct from $xyzx$ is of the form $xyvx$, and so if xy is in two triangles, then $d(x) \geq 3$ and $d(y) \geq 3$ (since xv, yv, xz, yz, and xy are all edges in G). Observe that to avoid a rainbow-P_3, x, y and z all must have degree at most 3. Thus, if xy is in two triangles, then no other edges of G are incident to x or y. Therefore, xy is contained in at most two triangles.

For each edge e of G, let $f(e)$ be the number of triangles containing e. So the number of triangles in G is $\frac{1}{3}\sum_{e\in E(G)} f(e) \leq \frac{2}{3}e(G)$. Since G is rainbow-P_3-free, we have $e(G) \leq \mathrm{ex}^*(n, P_3) = \frac{3}{2}n$ by Theorem 1.1. Therefore, G contains at most $\frac{2}{3}\frac{3}{2}n = n$ (rainbow-)C_3 copies.

In order to achieve exactly n (rainbow-)C_3 copies each edge xy must be in exactly two triangles, say $xyzx$ and $xyvx$. It is easy to see that if any vertex x, y, z, v is adjacent to a vertex not in x, y, z, v, then we have a rainbow-P_3. Therefore, in order to have every edge in exactly two triangles, each component of G must be a K_4. Moreover, each K_4 component must be properly 3-edge-colored. □

Theorem 3.2. *If n is divisible by 8, then* $\mathrm{ex}^*(n, C_4, P_4) = 3n$.

Proof. Theorem 2.1 gives $3n = \frac{(4-1)!}{2}n \leq \mathrm{ex}^*(n, C_4, P_4)$. For the upper bound, let G be an n-vertex graph with a proper k-edge-coloring c with no rainbow-P_4. Fix an edge xy of G. Without loss of generality, $c(xy) = 1$. We wish to find an upper bound on the number of rainbow-C_4 copies containing edge xy. We may assume that xy is contained in a rainbow-C_4, say $xyzwx$, with edges colored $1, 2, 3, 4$, respectively.

If every rainbow-C_4 containing xy has its edges colored from $1, 2, 3, 4$, then it follows from Lemma 2.2 that xy is contained in at most 3! rainbow-C_4 copies. Now suppose that $xyuvx$ is a rainbow-C_4 containing xy and exactly one edge is of a color not in $\{1, 2, 3, 4\}$, say 5. Associate to this cycle the (ordered) list $L = (c(xy), c(yu), c(uv), c(vx))$ of its edge colors. We can obtain a different list L' by replacing the entry of color 5 in L by whichever element of $\{1, 2, 3, 4\}$ is not represented in L. It is easy to see that xy cannot be in rainbow-C_4 copies associated with both lists L and L'. Indeed, since the cycles associated to L and L' share three colors in the same order and the edge xy, the proper edge-coloring implies that they must be the same cycle. So if every rainbow-C_4 including xy has at most one edge not colored from $\{1, 2, 3, 4\}$, then the list of rainbow-C_4 copies containing xy can be put in bijective correspondence with a (possibly proper) subset of the list of all possible rainbow-C_4 copies colored from $\{1, 2, 3, 4\}$. Thus, if every rainbow-C_4 containing xy has at most one edge not colored from $\{1, 2, 3, 4\}$, then xy is in at most 3! rainbow-C_4 copies.

Now suppose (to the contrary) that $xyuvx$ is a rainbow-C_4 using two colors not in $\{1, 2, 3, 4\}$, say 5 and 6. If an edge of color 5 or 6 is incident to exactly one vertex of $xyzw$, then we have a rainbow-P_4. Therefore, without loss of generality, we have $u = w$ and $v = z$ and edge colors $c(uw) = 5$ and $c(xz) = 6$. Any additional edge incident to $xyzw$ forms a rainbow-P_4, so there are at most 3! rainbow-C_4 copies using edge xy.

We now count rainbow-C_4 copies in G by counting the rainbow-C_4 copies on each edge. Let $f(e)$ be the number of rainbow-C_4 copies on edge e of G. Then, as G is rainbow-P_4-free, we have $e(G) \leq \mathrm{ex}^*(n, P_4) = 2n$ by Theorem 1.1. Therefore, the number of rainbow-C_4 copies in G is

$$\frac{1}{4}\sum_{e\in E(G)} f(e) \leq \frac{1}{4}3!e(G) \leq \frac{3!}{4}2n = 3n$$

as desired. □

An unpublished result of Halfpap [6] states that the only rainbow-P_4-free graphs with $\mathrm{ex}^*(n, P_4)$ edges are 4-regular. This can be used to prove that the only rainbow-P_4-free graphs that have $\mathrm{ex}^*(n, C_4, P_4)$ edges are also 4-regular.

Finally, we determine $\mathrm{ex}^*(n, C_5, P_5)$. Note that the proofs of Theorems 3.1 and 3.2 relied on the bound on $\mathrm{ex}^*(n, P_\ell)$ given in Theorem 1.1. As $\mathrm{ex}^*(n, P_5)$ is not known exactly, we need a different approach. However, we start in the same way, by bounding the number of rainbow-C_5 copies on a fixed edge of a rainbow-P_5-free graph.

Throughout the proof of Lemma 3.3 and Theorem 3.4 we will be required to examine many similar cases. Frequently, we will state that it is easy to see that we have a particular edge-coloring. This will involve the inspection of several potential colorings of an individual edge in a given figure and discarding those that lead to either a coloring that is not proper or to a rainbow-P_5. It would be excessive to list every possible case, so we leave some of the details to the reader.

Lemma 3.3. *Let G be an n-vertex graph with a proper edge-coloring with no rainbow-P_5. Then each edge of G is contained in at most* 4! *rainbow-C_5 copies.*

Proof. Let us count the number of rainbow-C_5 copies in G containing edge v_1v_2. We may assume that v_1v_2 is in at least one rainbow-C_5, say $C = v_1v_2v_3v_4v_5v_1$, whose edges are colored (in order) $1, 2, 3, 4, 5$. Note that if every rainbow-C_5 containing v_1v_2 is colored from $\{1, 2, 3, 4, 5\}$, then, by Lemma 2.2, at most 4! rainbow-C_5 copies in G contain v_1v_2. An analogous argument to that in the proof of Theorem 3.2 shows that if every rainbow-C_5 containing v_1v_2 contains at most one edge not colored from $\{1, 2, 3, 4, 5\}$, then at most 4! rainbow-C_5 copies in G contain v_1v_2.

Now suppose that v_1v_2 is contained in a rainbow-C_5, say C', which contains at least two edges not colored from $\{1, 2, 3, 4, 5\}$. We claim that these two edges must be chords of C. We write $C' = v_1v_2xyzv_1$, allowing x, y, and z to equal v_3, v_4, or v_5. We know that two edges of C' are not colored from $\{1, 2, 3, 4, 5\}$; say their colors are 6 and 7. We note that to avoid a rainbow-P_5, the edges colored 6 and 7 must either be chords of C or share no vertices with C. So if neither the edge colored 6 nor the edge colored 7 is a chord of C, then without loss of generality, $c(xy) = 6$, $c(yz) = 7$, and x, y, and z are not equal to v_3,v_4, or v_5. The situation is then as in Figure 1.

It is clear that any choice of $c(v_1z)$ yields a rainbow-P_5. So either the edge colored 6 or the edge colored 7 is a chord. Without loss of generality, the edge colored 6 is a chord. Now suppose that the edge colored 7 is not. It is easy to see that one of the two cases pictured in Figure 2 must occur; dashed edges represent the two possible placements for the chord of color 6.

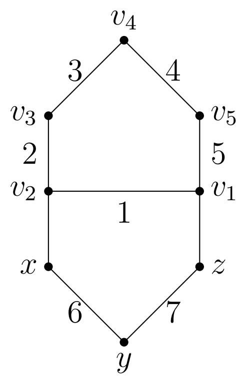

Figure 1

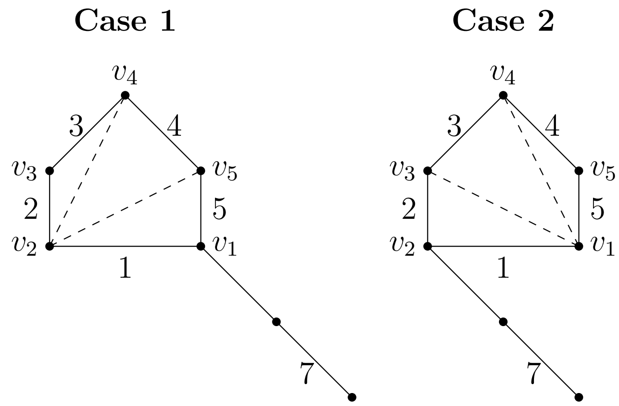

Figure 2

Note that the two cases are analogous by symmetry, so we only need to examine Case 1. Regardless of which choice we make for chord placement, an easy inspection shows that any coloring of the outgoing edge from v_1 results in a rainbow-P_5. Thus, we conclude that the edges colored 6 and 7 are chords of C.

We shall now show that v_1v_2 is contained in at most 4! rainbow-C_5 copies. There are $\binom{5}{2} = 10$ ways in which to place the chords within C; up to symmetry, six are distinct. They are pictured in Figure 3.

We need not consider Configuration 1 or 2, since it is clear that chords placed in these configurations cannot form a C_5 containing v_1v_2. In the remaining four configurations, we will show that v_1v_2 is contained in at most 4! rainbow-C_5 copies. In these arguments, we repeatedly use the following facts:

- v_1v_2 is contained in at most 3! rainbow-C_5 copies which contain only vertices from C (since there are 3! ways to permute v_3, v_4, and v_5).
- Given five vertices and three fixed edges among them, there are (at most) two ways in which to add another two edges to create a C_5.
- There is no edge with one vertex incident to C that is colored with a color not in $\{1, 2, 3, 4, 5\}$ as this results in a rainbow-P_5

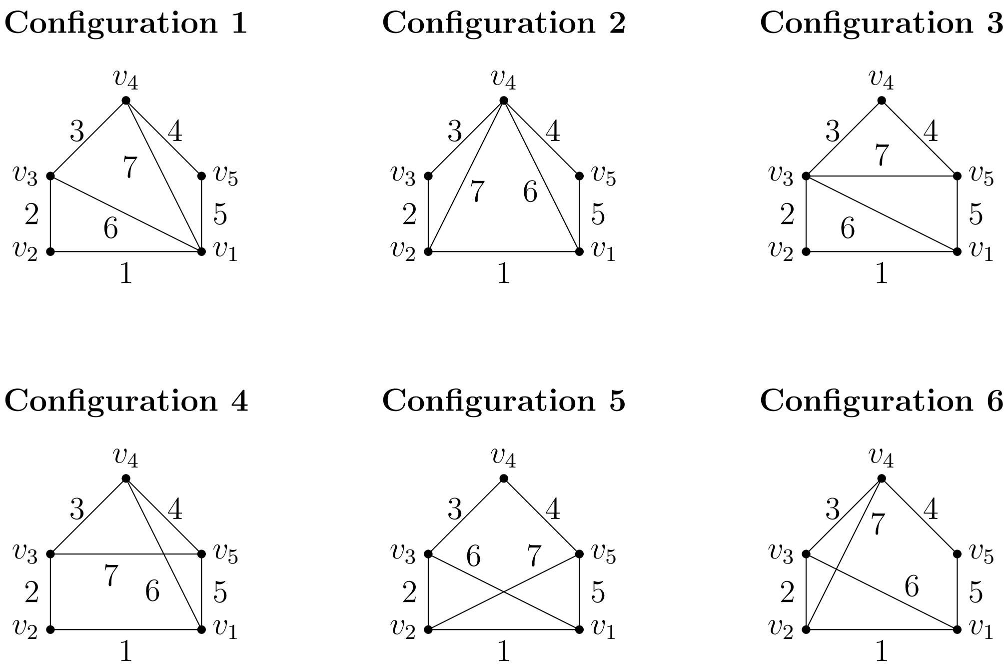

Figure 3

We first consider Configuration 3. If v_1v_2 is on a rainbow-C_5 containing both of the pictured chords, then this C_5 is of the form $v_2v_1v_3v_5uv_2$, where u is either equal to v_4 or to some vertex not on C. If $u \neq v_4$, then the situation is as pictured in Figure 4.

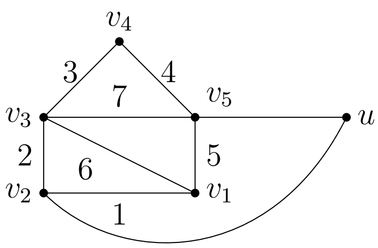

Figure 4

Because our coloring must be proper, $c(v_2u)$ is not 1 or 2. In order to avoid a rainbow-P_5, it is clear that $c(v_2u)$ must be in $\{1, 2, 3, 4, 5\}$. However, if $c(v_2u) \in \{3, 4, 5\}$, then either $uv_2v_1v_3v_5v_4$ or $uv_2v_1v_5v_3v_4$ is a rainbow-P_5. So we must have $u = v_4$. Thus, if the chords placed in Configuration 3 yield a rainbow-C_5 containing v_1v_2, then that rainbow-C_5 is $v_2v_1v_3v_5v_4v_2$, as drawn in Figure 5.

Note that to ensure that the coloring is proper and that $v_2v_1v_3v_5v_4v_2$ is a rainbow-C_5, we must have $c(v_2v_4) = 5$ or $c(v_2v_4)$ is a color not yet used, say 8. Now, inspect v_1, v_2, v_4, and v_5. It is easy (but somewhat tedious) to check that none of these vertices may be adjacent to any vertex u which is not on C; any color choice for such an edge will result in a rainbow-P_5 given the above configuration and regardless of

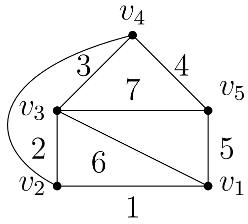

Figure 5

whether $c(v_2v_4)$ is chosen to equal 5 or 8. Thus, v_1v_2 lies in no cycle except those using only vertices from C. Hence, v_1v_2 is contained in at most $3! < 4!$ rainbow-C_5 copies.

In Configuration 4, we observe that v_2 and v_5 can only be adjacent to vertices on C, and that if v_4 is incident to an edge whose other endpoint is not on C, then that edge must be colored 1. So if a rainbow-C_5 contains v_1v_2 and uses vertices not on C, then it must include edges of the form v_1u and v_3w where u and w are not on C (although we allow $u = w$). We observe that $c(v_3w)$ cannot equal 5, so we must have $c(v_3w) = 4$ if the cycle is to be rainbow. This forces $c(v_1u) = 2$, since $c(v_1u)$ cannot equal 3. The number of rainbow-C_5 copies containing v_1u and v_3w is at most 2, since if $u \neq w$, then we have specified all five vertices of the cycle and three of its edges, so there are only two ways to add the remaining two edges. If $u = w$, then the fifth vertex of the cycle must be on C, and there are two choices for this vertex. So in Configuration 4, v_1v_2 is on at most $3! + 2 < 4!$ rainbow-C_5 copies.

In Configuration 5, we observe that v_1, v_2, and v_4 are adjacent only to vertices on C. Also, if v_3u is an edge with u not on C, then $c(v_3u) \in \{1, 4\}$, and if v_5w is an edge with w not on C, then $c(v_5w) \in \{1, 3\}$. So the only possible rainbow-C_5 copies containing v_1v_2 and vertices not on C contain edges v_3u and v_5w with $c(v_3u) = 4$ and $c(v_5w) = 3$. Note that, in order to have exactly five vertices, we must have $u = w$. We have now specified all five vertices and three edges of a cycle, so there are at most two ways to add edges to create a C_5. Hence, there are at most $3! + 2 < 4!$ containing v_1v_2 in Configuration 5.

In Configuration 6, we note that v_2, v_3, and v_5 are only adjacent to vertices on C. If v_1u is an edge with u not on C, then $c(v_1u) \in \{2, 4\}$, and if v_4w is an edge with w not on C, then $c(v_4w) \in \{2, 5\}$. Thus, the only possible rainbow-C_5 copies containing v_1v_2 and some vertex not on C use a pair of edges v_1u and v_4w. Since each of v_1, v_4 can have at most two neighbors not on C, and at most two cycles can be formed which include v_1v_2 and a fixed pair of edges v_1u and v_4w, the edge v_1v_2 is contained in at most $3! + 8 < 4!$ rainbow-C_5 copies.

Thus, if v_1v_2 is contained in a rainbow-C_5 which uses two colors not in $\{1, 2, 3, 4, 5\}$, then v_1v_2 is contained in strictly fewer than $4!$ rainbow-C_5 copies. □

We may immediately apply Lemma 3.3 to get $\mathrm{ex}^*(n, C_5, P_5) \leq \frac{4!}{5}\,\mathrm{ex}^*(n, P_5)$. If we could show that $\mathrm{ex}^*(n, P_5)$ was $\frac{5}{2}n$, then Lemma 3.3 would give the desired bound on $\mathrm{ex}^*(n, C_5, P_5)$. Unfortunately, this is not known. However, we can give a new upper bound on $\mathrm{ex}^*(n, P_5)$ which combined with Lemma 3.3 gives $\mathrm{ex}^*(n, C_5, P_5) \leq \frac{4!}{5} \cdot 4n = 19.2n$.

Theorem 3.4. $\mathrm{ex}^*(n, P_5) \leq 4n$.

Proof. Let G be an n-vertex graph with a proper edge-coloring and more than $4n$ edges. We will show that G contains a rainbow-P_5. The average degree of G is greater than 8. By removing vertices of degree at most 4, we can obtain a subgraph G' of G with minimum degree at least 5 and average degree greater than 8. In particular, G' has a vertex, say v, of degree at least 9.

Case 1: G' contains a rainbow-P_4 ending at v.

Let $P = vxyzw$ be a rainbow-P_4 ending at v. Since $d(v) \geq 9$, v must be adjacent to at least 5 vertices not on P. Since the coloring of G' is proper, none of these five edges receives the same color as vx. Three may receive the colors used for xy, yz, and zw, but two must receive colors not used in P. Either of these two edges will extend P to a rainbow-P_5.

Case 2: G' does not contain a rainbow-P_4 ending at v.

Using the fact that the minimum degree in G' is at least 5, we can greedily build a rainbow path of length 3 ending at v; moreover, since this path does not extend to a rainbow-P_4, then the situation must be as pictured in Figure 6.

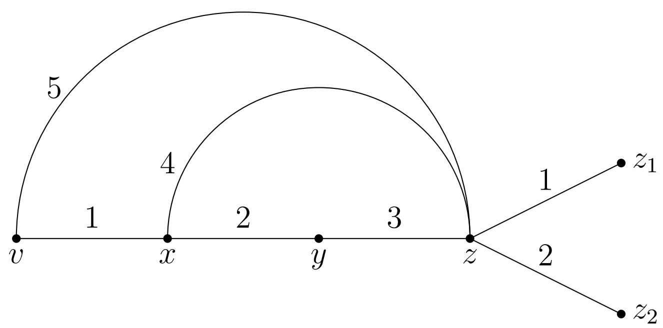

Figure 6

Consider the vertex y. Since $d(y) \geq 5$, y must be adjacent to at least two vertices not on $vxyz$. Call these y_1 and y_2 (we allow that y_1 and y_2 may not be distinct from z_1 and z_2). It is easy to see that if $c(yy_i)$ is not 2 or 4, then either $vzxyy_i$ or $vxzyy_i$ is a rainbow-P_4 ending in v, a contradiction. Moreover, $c(yy_i) \neq 2$ or the coloring is not proper. So both $c(yy_1)$ and $c(yy_2)$ must be 4, a contradiction. ☐

On the other hand, by Lemma 2.3, any vertex in a rainbow-C_5 in a rainbow-P_5-free graph has degree at most 7. We can count rainbow-C_5 copies on a fixed vertex as follows. For every vertex v which is on a rainbow-C_5, each rainbow-C_5 containing v begins with an edge incident to v. There are at most 7 choices of edge, and each is contained in at most 4! rainbow-C_5 copies by Lemma 3.3. Each rainbow-C_5 is counted ten times this way as each C_5 contains five vertices and each rainbow-C_5 is counted twice per vertex (because every rainbow-C_5 containing v in fact uses two edges incident to v, and is counted once by each). In this way we obtain the following slight improvement

$$\mathrm{ex}^*(n, C_5, P_5) \leq \frac{4!}{5} \cdot \frac{7}{2} n = 16.8n.$$

The bounds given above are clearly not the best possible; it is easy to show that if G is a rainbow-P_5-free graph containing a rainbow-C_5, say C, then not every vertex on C can have degree 7. Therefore, a more careful analysis of degree constraints for vertices on a rainbow-C_5 is needed.

The proof of our upper bound relies on Lemma 3.3 and another key step. We show that a rainbow-C_5 containing high-degree vertices must contain vertices of low degree. By appropriately pairing vertices of high degree and low degree we can show that the average degree over all vertices contained in a rainbow-C_5 is at most 5. Combining this observation with Lemma 3.3 will give the desired bound on $\mathrm{ex}^*(n, C_5, P_5)$.

Theorem 3.5. *If n is divisible by 16, then $\mathrm{ex}^*(n, C_5, P_5) = 12n$.*

Proof. Theorem 2.1 gives $12n = \frac{(5-1)!}{2} n \leq \mathrm{ex}^*(n, C_5, P_5)$. To prove the upper bound, consider an n-vertex graph G with a proper edge-coloring with no rainbow-P_5. Let V' be the set of vertices in G which are contained in at least one rainbow-C_5. By Lemma 3.3, any vertex $v \in V'$ is contained in at most $\frac{4!d(v)}{2}$ rainbow-C_5 copies, since each edge incident to v is in at most 4! rainbow-C_5 copies and each rainbow-C_5 containing v uses two edges incident to v. Thus, the total number of rainbow-C_5 copies in G is at most

$$\sum_{v \in V'} \frac{4!d(v)}{2 \cdot 5} = \frac{4!}{2 \cdot 5} \sum_{v \in V'} d(v).$$

If the average degree of vertices in V' is at most 5, then we immediately have

$$\frac{4!}{2 \cdot 5} \sum_{v \in V'} d(v) \leq \frac{4!}{2 \cdot 5} \cdot 5|V'| \leq \frac{4!}{2} n = 12n,$$

and we are done. In order to establish that the average degree in V' is at most 5, we will need the following technical claim.

Claim. *Let C be a rainbow-C_5 in G containing a vertex of degree at least 6 and let S be the set of vertices on C with degree at least 6. Then there is a set T of vertices in $V(C) \setminus S$ such that:*

(1) there is a matching between S and T such that if $u \in S$ and $v \in T$ are matched we have $d(u) + d(v) \leq 10$;

(2) if $v \in T$ is adjacent to $u \in V(G)$ and $d(u) \geq 6$, then $u \in S$.

Proof. We call a pair of sets S, T satisfying the above conditions an *S, T pair*.

Without loss of generality, $C = v_1v_2v_3v_4v_5v_1$ has edges colored (in order) $1, 2, 3, 4, 5$, and $d(v_1) > 5$. By Lemma 2.3, $d(v_1) \leq 7$, so we must either have $d(v_1) = 6$ or

$d(v_1) = 7$. We shall consider both cases. Frequently, in the cases below, we shall use the following simple observation: If any vertex v_i of C is incident to an edge which is not colored from $\{1, 2, 3, 4, 5\}$, then this edge is of the form v_iv_j for some vertex v_j of C. In particular, if $d(v_i) = 5 + k$, then v_i is incident to at least k chords of C whose colors are not in $\{1, 2, 3, 4, 5\}$. Recall that there is no edge of color not in $\{1, 2, 3, 4, 5\}$ with exactly one endpoint in C as otherwise we get a rainbow-P_5.

Case 1: $d(v_1) = 7$.

Since $d(v_1) = 7$, v_1 has three neighbors not on C, say u_1, u_2, and u_3, and both v_1v_3 and v_1v_4 are edges. Without loss of generality, we have $c(v_1u_1) = 2$, $c(v_1u_2) = 3$, $c(v_1u_3) = 4$, $c(v_1v_3) = 6$, and $c(v_1v_4) = 7$.

We first bound $d(v_2)$. It is easy to check that any edge v_2w with w not on C creates a rainbow-P_5. Also, if v_2v_4 is an edge, then (noting that $c(v_2v_4)$ is not equal to $1, 2, 3, 4$, or 7) either $u_2v_1v_3v_2v_4v_5$ or $u_2v_1v_5v_4v_2v_3$ is a rainbow-P_5. Hence, $d(v_2) \leq 3$. By symmetry, $d(v_5) \leq 3$.

We next bound $d(v_4)$. As noted above, v_4v_2 is not an edge. Also, if v_4w is an edge with w not on C, then $c(v_4w) \neq 1$, since otherwise $wv_4v_5v_1v_3v_2$ is rainbow. So v_4 has at most two neighbors not on C (since the edge incident to any such neighbor must be colored either 2 or 5), and at most three neighbors on C. Hence, $d(v_4) \leq 5$. By symmetry, $d(v_3) \leq 5$. Thus, in this case, $S = \{v_1\}$ and we put $T = \{v_2\}$ to form an S, T pair.

Case 2: $d(v_1) = 6$.

We distinguish three subcases.

Case 2.1: v_1 is adjacent to both v_3 and v_4, and both $c(v_1v_3)$ and $c(v_1v_4)$ are not in $\{1, 2, 3, 4, 5\}$.

Without loss of generality, $c(v_1v_3) = 6$ and $c(v_1v_4) = 7$. We observe that v_2 and v_5 are only adjacent to vertices on C, so have degrees at most 4. Moreover, suppose that one of v_3, v_4 has degree greater than 5. The two vertices are symmetric thus far, so suppose that $d(v_3) > 5$. We established in Case 1 that a vertex of degree 7 is never on a rainbow-C_5 containing any other vertex of degree greater than 5, so $d(v_3) = 6$. We note that if v_3u is an edge with u not on C, then $c(v_3u) \neq 5$ (as otherwise we get a rainbow-P_5). The picture then must be as in Figure 7.

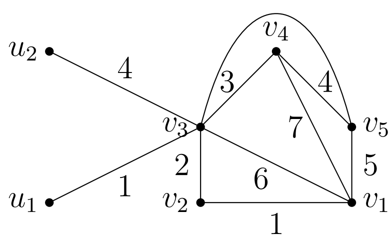

Figure 7

In order for the coloring to be proper, $c(v_3v_5)$ is either 7 or a color not yet used, say 8. With this fact, we may observe that v_4 is also adjacent only to vertices on C and thus $d(v_4) \leq 4$. Therefore, either $S = \{v_1\}$, and we put $T = \{v_2\}$, or $S = \{v_1, v_3\}$ and we put $T = \{v_2, v_4\}$ to obtain an S, T pair.

Case 2.2: v_1 is adjacent to both v_3 and v_4, and one of $c(v_1v_3), c(v_1v_4)$ is in $\{1, 2, 3, 4, 5\}$.

Without loss of generality, $c(v_1v_3)$ is not in $\{1, 2, 3, 4, 5\}$, say $c(v_1v_3) = 6$. So $c(v_1v_4) \in \{1, 2, 3, 4, 5\}$, which forces $c(v_1v_4) = 2$. The vertex v_1 is adjacent to two vertices not on C, say w_1 and w_2. Without loss of generality, $c(v_1w_1) = 3$ and $c(v_1w_2) = 4$. We draw this in Figure 8.

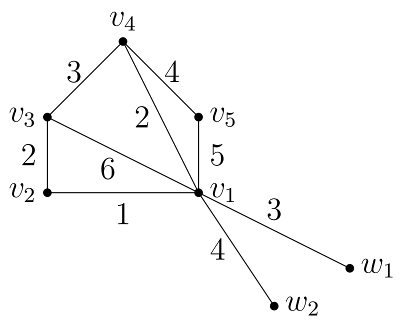

Figure 8

Observe that if the edge v_2v_4 is present, then $c(v_2v_4)$ is not in $\{1, 2, 3, 4\}$, and so either $w_1v_1v_5v_4v_2v_3$ or $w_1v_1v_3v_2v_4v_5$ is a rainbow-P_5. Thus, v_2v_4 is not an edge. Furthermore, if the edge v_2v_5 is present, then $c(v_2v_5)$ is not in $\{1, 2, 4, 5\}$, and cannot be 6 or 7, since then $v_4v_3v_2v_5v_1w_2$ is rainbow. So if v_2v_5 is an edge, then $c(v_2v_5) = 3$. Finally, if v_2u is an edge with u not on C, then $c(v_2u)$ must be in 4, else one of $uv_2v_3v_1v_5v_4$ or $uv_2v_1v_3v_4v_5$ is rainbow. Thus, $d(v_2) \leq 4$.

We have seen that v_2v_4 is not an edge. Observe that if v_4u is an edge with u not on C, then $c(v_4, u)$ cannot be in $\{2, 3, 4\}$, which forces $c(v_4u) = 5$, else either $uv_4v_5v_1v_2v_3$ or $uv_4v_5v_1v_3v_2$ is rainbow. So $d(v_4) \leq 4$.

Finally, suppose that $d(v_5) \geq 6$. So v_5 must have an incident edge of color not in $\{1, 2, 3, 4, 5\}$. This edge must have both endpoints in C. We have seen that if v_2v_5 is an edge it is color 3, so v_3v_5 must be this edge. The vertex v_3 is incident to an edge of color 6, so v_5v_3 must be a color not yet used, say 7. Now observe that $w_1v_1v_2v_3v_5v_4$ is rainbow, a contradiction. We illustrate this in Figure 9.

Therefore $d(v_5) \leq 5$, $d(v_2) \leq 4$, and $d(v_4) \leq 4$. We have $d(v_1) = 6$, and $d(v_3)$ may equal 6. Thus, either $S = \{v_1\}$ and we put $T = \{v_2\}$, or $S = \{v_1, v_3\}$ and we put $T = \{v_2, v_4\}$. It is clear that S and T satisfy Condition (1). We must check that T satisfies Condition (2).

It will suffice to show that neither v_2 nor v_4 is adjacent to a vertex u that is not on C and has degree $d(u) \geq 6$. Indeed, in both of the S, T pairs above, the only vertices which can appear in T are v_2 and v_4. Moreover, v_4 is only included in T

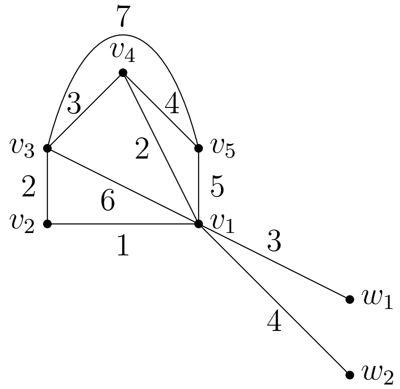

Figure 9

when $d(v_3) = 6$.

First, suppose first that v_2 is adjacent to a vertex u not on C with $d(u) \geq 6$. We have established already that $c(v_2u) = 4$. Furthermore, since $d(u) \geq 6$, u has at least one neighbor, say x, not on C. It is easy to see that $c(xu)$ must be 6. Therefore, u has only one neighbor not on C, and so u is adjacent to every vertex on C. In particular, uv_4 is an edge. This is pictured in Figure 10.

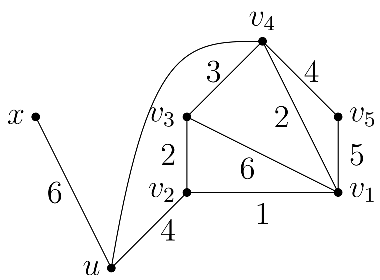

Figure 10

The edge uv_4 is not in colored from $\{3, 4, 6\}$, so we must have $c(uv_4) = 5$, else $v_2uv_4v_3v_1v_5$ is rainbow. But if $c(uv_4) = 5$, then $xuv_4v_3v_2v_1$ is rainbow. We conclude that v_2 is not adjacent to a vertex of degree at least 6 which is not on C.

Now, suppose that v_4 is adjacent to a vertex u such that u is not on C and $d(u) \geq 6$. As noted earlier in the case, $c(v_4u)$ must equal 5. Now, consider the neighbors of u. If uv_i is an edge for any vertex v_i on C, then we must have $c(uv_i) \in \{1, 2, 3, 4, 5\}$, else we immediately find a rainbow-P_5. Moreover, if ux is an edge, with x not on C, then we can check that we must have $c(ux) \in \{1, 3\}$ to avoid a rainbow-P_5. Thus, all edges incident to u must be colored from $\{1, 2, 3, 4, 5\}$ to avoid a rainbow-P_5. But this contradicts the assumption that $d(u) \geq 6$. We can therefore conclude that v_4 is not adjacent to any vertex u of degree greater than 5 which is not on C.

Case 2.3: v_1 is not adjacent to one of v_3, v_4.

Without loss of generality, v_1v_4 is not an edge. In order to achieve $d(v_1) = 6$,

v_1 must be adjacent to v_3 and have three neighbors not on C, say w_1, w_2, w_3, with $c(v_1w_1) = 2$, $c(v_1w_2) = 3$, and $c(v_1w_3) = 4$. This is illustrated in Figure 11.

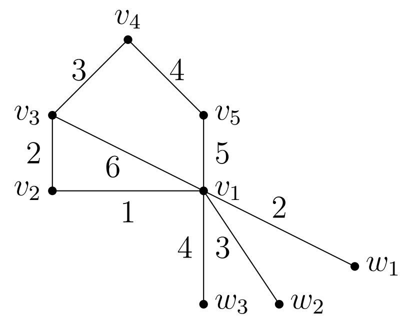

Figure 11

We first examine the degree of v_2. If v_2v_4 is an edge, then we must have $c(v_2v_4) = 6$, and if v_2v_5 is an edge, then we must have $c(v_2v_5) = 3$. If v_2u is an edge with u not on C (allowing $u = w_1, w_2$, or w_3), we must have $c(v_2u) = 4$ to avoid a rainbow-P_5. So $d(v_2) \leq 5$.

We next examine v_4. An edge v_4u with u not on C cannot be colored 3 or 4, and must not be colored 1 to avoid a rainbow-P_5. Moreover, if $c(v_4u) = 5$, then u must equal w_3 in order to avoid a rainbow-P_5. Now, note that if v_4w_3 is an edge with $c(v_4w_3) = 5$, then the addition of the chord v_2v_4 (which must be colored 6 by the above argument) creates a rainbow-P_5 $w_2v_1w_3v_4v_2v_3$. These observations together imply $d(v_4) \leq 4$ (v_4 having either two neighbors on C and at most two not on C, or three neighbors on C and at most one not on C).

Finally, we examine v_5. We have already seen that if v_2v_5 is present, then $c(v_2v_5) = 3$. If v_3v_5 is an edge, then we must have $c(v_3v_5)$ is 1 or a new color 7. If $c(v_3v_5) = 7$, then $w_3v_1v_2v_3v_5v_4$ is a rainbow P_5. Finally, if v_5u is an edge with u not on C, then $c(v_5u) \neq 2$. Thus, any edge incident to v_5 must be colored from $\{1, 3, 4, 5\}$, so $d(v_5) \leq 4$.

Summarizing, we have $d(v_1) = 6, d(v_2) \leq 5, d(v_3) \leq 6, d(v_4) \leq 4, d(v_5) \leq 4$. Therefore, if $S = \{v_1\}$, we put $T = \{v_5\}$ and if $S = \{v_1, v_3\}$, we put $T = \{v_4, v_5\}$. By an argument analogous to that in Case 2.2, this produces an S, T pair. □ □

With this claim established, we are now prepared to finish the proof. For each rainbow-C_5 in G containing a vertex of degree at least 6, we shall form an S, T pair as described in the claim. Recall that V' is the set of vertices of G contained in at least one rainbow-C_5. Let V'' be the set of vertices which are placed in S, T pairs. Then the average degree of the vertices in V' is

$$\frac{1}{|V'|}\sum_{v\in V'} d(v) = \frac{1}{|V'|}\left(\sum_{v\in V''} d(v) + \sum_{v\in V'\setminus V''} d(v)\right),$$

which we claim is at most 5. It suffices to show that $\sum_{v\in V''} d(v) \leq 5|V''|$, since we must have $\sum_{v \in V' \setminus V''} d(v) \leq 5|V' \setminus V''|$ because every vertex of degree greater than 5 in V' is in V''. Let us construct V'' with the following procedure. We pass through the rainbow-C_5 copies in G in arbitrary order. For each rainbow-C_5 copy we construct an S, T pair as given by the Claim. Then, we add from this S, T pair to V'' all vertices which are not already contained within V''.

We claim that at each step of the procedure, the average degree among vertices in V'' is at most 5. Clearly, this is true at the first step by Condition (1). Suppose the property holds for k steps, and consider the $(k+1)$st step. If at the $(k+1)$st step, no vertices from the chosen S, T pair yet lie in V'', then Condition (1) again implies that we add to V'' a set of vertices with average degree at most 5, and so the average degree condition on V'' is maintained. If one or more vertices from S but none from T already lie in V'', then we add to V'' a subset of the S, T pair which omits one or more vertices from S, and so clearly still has average degree at most 5 by Condition (1) again.

If a vertex $v \in T$ is already in V'', then it was placed in V'' at an earlier step because it was in an earlier S, T pair. By Condition (2), all neighbors of v of degree at least 6 were in the earlier S, T pair, and so are also already in V''. In particular, the vertex $u \in S$ which is matched to v in the S, T pair at step $k+1$ already lies in V''. Thus in this case, at step $k+1$, we either add no new vertices to V'', we add a single vertex of T to V'', or we add a matched pair u', v' with the property that $v' \in T$ is not already in V''. In the last case, as Condition (1) implies that the matched pair has average degree at most 5, the average degree condition on V'' is maintained. □

Remark. *As in the case of P_3 and P_4, the proof of Theorem 3.5 can be adapted to show that the only rainbow-P_5-free graphs attaining* $\mathrm{ex}^*(n, C_5, P_5)$ *are* 5*-regular. We exclude the details as they involve further analysis in the subcases in the proof.*

Acknowledgements

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